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DIAGONAL ELEMENTS OF ORTHOGONAL MATRICES

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1. All numbers mentioned below are assumed to be real; and $N(x_1, \cdots, x_n)$ will denote the number of negative numbers among $x_1, \cdots, x_n$. By a *point* we shall mean an n-dimensional vector. An orthogonal matrix will be called *proper* or *improper* according as its determinant is $+1$ or -1. All matrices will be understood to be of type $n \times n$.

The following interesting result was established by A. Horn ([1] Theorem 8).

THEOREM 1. *The numbers $d_1, \cdots, d_n$ are the diagonal elements of a proper orthogonal matrix if and only if $(d_1, \cdots, d_n)$ lies in the convex envelope of those points of the form $(\pm 1, \cdots, \pm 1)$ which have an even number of negative coordinates.*

The object of the present note is to derive from this result an *effective* criterion for deciding whether n given numbers can be the diagonal elements of a proper orthogonal matrix. We shall, in fact, establish

THEOREM 2. *Necessary and sufficient conditions for the numbers $d_1, \cdots, d_n$ to be the diagonal elements of a proper orthogonal matrix are*

$$|d_j| \leqq 1 \qquad (j = 1, \cdots, n) \tag{1}$$

and

$$\sum_{k=1}^{n} |d_k| \leqq n - 2 + 2\lambda \min_{1 \leqq j \leqq n} |d_j|, \tag{2}$$

where λ is defined as 1 *or* 0 *according as $N(d_1, \cdots, d_n)$ is even or odd.*

Horn ([1] Theorem 9) obtained Theorem 2 for the case when the d's are nonnegative, and therefore for the case when $N(d_1, \cdots, d_n)$ is even; furthermore, the *necessity* of conditions (1) and (2) for all cases is implicit in his argument ([1] p. 627). However, the proof we shall give here depends on different ideas.

We note two consequences of Theorem 2.

COROLLARY 1. *Necessary and sufficient conditions for the numbers $d_1, \cdots, d_n$ to be the diagonal elements of an improper orthogonal matrix are* (1) *and*

$$\sum_{k=1}^{n} |d_k| \leqq n - 2 + 2(1 - \lambda) \min_{1 \leqq j \leqq n} |d_j|,$$

where λ is defined as in Theorem 2.

COROLLARY 2. *Necessary and sufficient conditions for the numbers $d_1, \cdots, d_n$ to be the diagonal elements of a proper orthogonal matrix and also of an improper orthogonal matrix are* (1) *and* $\sum_{k=1}^{n} |d_k| \leqq n-2$.

It is clear that Corollary 2 follows at once from Theorem 2 and Corollary 1, while Corollary 1 follows from Theorem 2 by virtue of the fact that $d_1, \cdots, d_n$ are the diagonal elements of an improper orthogonal matrix if and only if $-d_1, \cdots, -d_n$ are the diagonal elements of a proper orthogonal matrix.

I wish to express my thanks to the referee for his useful comments.

2. We shall need some preliminary results.

LEMMA 1. *For any given numbers* $x_1, \cdots, x_n$ *we have*:

$$\text{(i)} \qquad \max \sum_{k=1}^{n} \delta_k x_k = \sum_{k=1}^{n} | x_k | ,$$

where the maximum is taken with respect to all sets of numbers $\delta_1, \cdots, \delta_n$ *such that*

$$\delta_1, \cdots, \delta_n = \pm 1, \qquad N(\delta_1, \cdots, \delta_n) \equiv N(x_1, \cdots, x_n) \pmod 2;$$

$$\text{(ii)} \qquad \max \sum_{k=1}^{n} \delta_k x_k = \sum_{k=1}^{n} | x_k | - 2 \min_{1 \leqq j \leqq n} | x_j | ,$$

where the maximum is taken with respect to all sets of numbers $\delta_1, \cdots, \delta_n$ *such that*

$$(3) \qquad \delta_1, \cdots, \delta_n = \pm 1, \qquad N(\delta_1, \cdots, \delta_n) \not\equiv N(x_1, \cdots, x_n) \pmod 2.$$

In the first place, we have trivially

$$\sum_{k=1}^{n} \delta_k x_k \leqq \sum_{k=1}^{n} | x_k | .$$

Also, taking δ_k as $+1$ or -1 according as $x_k \geqq 0$ or $x_k < 0$, we obtain

$$\sum_{k=1}^{n} \delta_k x_k = \sum_{k=1}^{n} | x_k | .$$

This proves (i). Next, write $\min_{1 \leqq j \leqq n} |x_j| = |x_s|$. If $\delta_1, \cdots, \delta_n$ satisfy (3), then $\delta_i x_i \leqq 0$ for some i. Hence

$$\sum_{k=1}^{n} \delta_k x_k \leqq \sum_{k=1}^{n} | x_k | - 2 | x_i | \leqq \sum_{k=1}^{n} | x_k | - 2 | x_s | .$$

Again, take δ_s as $+1$ or -1 according as $x_s < 0$ or $x_s \geqq 0$ and, for $k \neq s$, take δ_k as $+1$ or -1 according as $x_k \geqq 0$ or $x_k < 0$. Then (3) is satisfied, and we have

$$\sum_{k=1}^{n} \delta_k x_k = \sum_{k=1}^{n} | x_k | - 2 | x_s | .$$

Hence (ii) is proved.

LEMMA 2. *Let* $P=(x_1, \cdots, x_n)$ *and* $P_k=(x_{k1}, \cdots, x_{kn})$, $k=1, \cdots, m$, *be any points. Then* P *lies in the convex envelope of* $P_1, \cdots, P_m$ *if and only if, for any numbers* $a_1, \cdots, a_n$ *and some suffix* k *(which may depend on the* a*'s), we have*

$$a_1x_1 + \cdots + a_nx_n \leqq a_1x_{k1} + \cdots + a_nx_{kn}.$$

This result expresses the, intuitively obvious, fact that P lies in the convex envelope of $P_1, \cdots, P_m$ if and only if no hyperplane can separate P from all the points $P_1, \cdots, P_m$. For further details of this standard result, we refer to ([2] pp. 23–24).

As an immediate consequence of Theorem 1 and Lemma 2, we have

LEMMA 3. *The numbers* $d_1, \cdots, d_n$ *are the diagonal elements of a proper orthogonal matrix if and only if, for any numbers* $a_1, \cdots, a_n$, *we can find numbers* $\epsilon_1, \cdots, \epsilon_n$ *such that*

$$\epsilon_1, \cdots, \epsilon_n = \pm 1, \qquad N(\epsilon_1, \cdots, \epsilon_n) \equiv 0 \pmod 2 \tag{4}$$

and

$$\sum_{k=1}^{n} a_kd_k \leqq \sum_{k=1}^{n} a_k\epsilon_k. \tag{5}$$

3. We now come to the proof of Theorem 2. Assume, in the first place, that $d_1, \cdots, d_n$ are the diagonal elements of a proper orthogonal matrix. Then (1) is obviously satisfied. Take any numbers $a_1, \cdots, a_n$ such that

$$a_1, \cdots, a_n = \pm 1, \qquad N(a_1, \cdots, a_n) \equiv 1 \pmod 2. \tag{6}$$

Then, for suitable ϵ's satisfying (4), the relation (5) is valid. But $a_k\epsilon_k=-1$ for at least one value of k. Therefore $\sum_{k=1}^n a_kd_k \leqq n-2$, and so

$$\max \sum_{k=1}^{n} a_kd_k \leqq n-2, \tag{7}$$

where the maximum is taken with respect to all sets of numbers $a_1, \cdots, a_n$ satisfying (6). By Lemma 1, this maximum is equal to $\sum_{k=1}^n |d_k|$ if $N(d_1, \cdots, d_n)$ is odd and to $\sum_{k=1}^n |d_k| - 2\min_{1\leq j\leq n} |d_j|$ if $N(d_1, \cdots, d_n)$ is even. Hence, in view of (7), (2) is satisfied. This proves the necessity of (1) and (2).

Next, assume that (1) and (2) are given. Let $a_1, \cdots, a_n$ be any numbers. By (1), we clearly have

$$\sum_{k=1}^{n} a_kd_k \leqq \sum_{k=1}^{n} |a_k|. \tag{8}$$

Now suppose that $N(a_1, \cdots, a_n)$ is odd and $N(d_1, \cdots, d_n)$ even. Then $a_id_i \leqq 0$ for some i; and therefore

$$\sum_{k=1}^{n} a_kd_k \leqq \sum_{k=1}^{n} |a_k| \, |d_k| - 2\lambda |a_i| \, |d_i|. \tag{9}$$

This inequality obviously also holds when $N(a_1, \cdots, a_n)$ and $N(d_1, \cdots, d_n)$ are both odd. Writing

$$\min_{1 \leqq j \leqq n} |a_j| = |a_r|, \qquad \min_{1 \leqq j \leqq n} |d_j| = |d_s|$$

and using (9), (1), and (2), we infer that, whenever $N(a_1, \cdots, a_n)$ is odd,

$$\begin{aligned}
\sum_{k=1}^{n} a_k d_k &\leqq \sum_{k=1}^{n} |a_k| \, |d_k| - 2\lambda |a_r| \, |d_s| \\
&= \sum_{k=1}^{n} |a_k| - \sum_{k=1}^{n} (1 - |d_k|) |a_k| - 2\lambda |a_r| \, |d_s| \\
&\leqq \sum_{k=1}^{n} |a_k| - \sum_{k=1}^{n} (1 - |d_k|) |a_r| - 2\lambda |a_r| \, |d_s| \\
&= \sum_{k=1}^{n} |a_k| - |a_r| \left(n - \sum_{k=1}^{n} |d_k| + 2\lambda |d_s| \right) \\
&\leqq \sum_{k=1}^{n} |a_k| - 2 |a_r| .
\end{aligned}$$

Thus, in view of (8), we have for any values of $a_1, \cdots, a_n$:

$$(10) \qquad \sum_{k=1}^{n} a_k d_k \leqq \begin{cases} \displaystyle\sum_{k=1}^{n} |a_k| & \text{when } N(a_1, \cdots, a_n) \text{ is even,} \\ \displaystyle\sum_{k=1}^{n} |a_k| - 2|a_r| & \text{when } N(a_1, \cdots, a_n) \text{ is odd.} \end{cases}$$

But, by Lemma 1, it is possible to choose numbers $\epsilon_1, \cdots, \epsilon_n$, subject to the conditions (4), such that the right-hand side in (10) is equal to $\sum_{k=1}^{n} a_k \epsilon_k$. Thus, given any numbers $a_1, \cdots, a_n$, it is possible to satisfy (4) and (5) simultaneously; and it follows by Lemma 3 that $d_1, \cdots, d_n$ are the diagonal elements of a proper orthogonal matrix.

References

1. A. Horn, Doubly stochastic matrices and the diagonal of a rotation matrix, Amer. J. Math., vol. 76, 1954, pp. 620–630.

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NEAR-RINGS

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1. Introduction. The ring structure of the endomorphisms of a commutative group is well known. Most of the ring properties do not hold for the endomorphisms of a noncommutative group. Closure with respect to the additive operation fails. Even in the closed system generated additively by the endomorphisms, commutativity of addition and the right distributive law do not hold, nor are there necessarily additive inverses. However, the system generated additively by the endomorphisms and anti-endomorphisms of a group form a group with respect to addition, a semigroup with respect to operator multiplication, and multiplication is left distributive with respect to addition. This system is an example of a near-ring.

A *near-ring* is an ordered triple $\mathfrak{P}=(P, +, \cdot)$, where P is a nonempty set, and $+$, $\cdot$ are binary compositions on P (called addition and multiplication) such that (i) $(P, +)$ is a group, (ii) $(P, \cdot)$ is a semigroup, and (iii) multiplication is left distributive with respect to addition. A ring is of course a near-ring. The set of all transformations from a group to itself relative to point-wise addition and operator multiplication is also a near-ring. Other examples are presented in Section 2.

A *near-field*, a near-ring in which every nonzero element has a multiplicative inverse, is an example which is useful in the study of certain non-Desarguesian planes [**7**, **9**]. Near-rings appear to have application to the study of nonlinear operators, since the collection of all operators on a vector space forms a near-ring, and also appear to have application in characterizing endomorphisms of a group. Dickson [**6**] was the first to study finite near-fields. Zassenhaus [**13**] determined all finite near-fields while Kalscheuer [**8**] classified all near-fields over the reals. Neumann [**10**] considered near-fields connected with free groups. Blackett [**2**] considered a class of simple and semisimple near-rings, and Deskins [**5**] considered near-ring radicals. Other papers are also listed in the bibliography.

This paper discusses some of the elementary properties of near-rings. The notation and definitions are chosen so that ring theory is a special case. In Section 2 examples of near-rings are given which are used to illustrate the theory. The arithmetic of near-rings is discussed in Section 3. Certain special classes of near-rings pertinent to the development of the theory are presented in Section 4. In Section 5 it is shown that any near-ring is decomposable into a direct sum of special sub-near-rings, and the Peirce decomposition for rings is generalized. Extensions and embeddings are discussed in Section 6. In particular, the well-known connection between integral domains and fields is generalized to near-rings. The theory of ideals and homomorphisms is developed in Section 7 and the ideals of special classes of near-rings is characterized in Section 8. Section 9 indicates some problems and extensions of interest.

2. Examples. Let $\mathfrak{G}=(G, +)$ be a group. Let $\mathfrak{T}(\mathfrak{G})=(T(G), +, \cdot)$, where $T(G)$ is the set of all transformations on G, and $+$, are the binary compositions on $T(G)$ defined by $g(\alpha+\beta)=g\alpha+g\beta$, $g(\alpha\cdot\beta)=(g\alpha)\beta$ (operator multiplication), $g\in G$. $\mathfrak{T}(\mathfrak{G})$ is a near-ring. The identity of $(T(G), +)$ is the transformation ζ which carries every element of G into O. The additive inverse, $-\alpha$, of α is $\alpha\rho$, where ρ is the anti-isomorphism of $\mathfrak{G}$ which maps every element of G into its inverse. The identity transformation ϵ is the multiplicative identity of $T(G)$ and $-\epsilon=\rho$.

Let $\mathfrak{T}_R(\mathfrak{G})=(T_R(G), +, \cdot)$ be the smallest near-ring in $\mathfrak{T}(\mathfrak{G})$ containing the endomorphisms of $\mathfrak{G}$. All the anti-endomorphisms of $\mathfrak{G}$ lie in $T_R(G)$ since $-\alpha=\alpha\rho$ is an anti-isomorphism of $\mathfrak{G}$ if α is an endomorphism. Every element of $T_R(G)$ is a sum of elements $\alpha_1+\cdots+\alpha_n$ where either α_i or $-\alpha_i$ is an endomorphism, $i=1, \cdots, n$. Since the endomorphisms are right distributive, $T_R(G)$ has the property that every element is a sum of right or "anti"-right distributive elements.

Let $\mathfrak{T}_Z(\mathfrak{G})=(T_Z(G), +, \cdot)$, where $T_Z(G)=\{\xi_g\in T(G) \mid g'\xi_g=g;\ g, g'\in G\}$. $\mathfrak{T}_Z(\mathfrak{G})$ is a sub-near-ring of $\mathfrak{T}(\mathfrak{G})$ which has the property that $\alpha\xi=\xi$ for every $\alpha\in T(G)$, $\xi\in T_Z(G)$. It is the maximal sub-near-ring of $\mathfrak{T}(\mathfrak{G})$ with this property.

Let $\mathfrak{T}_C(\mathfrak{G})=(T_C(G), +, \cdot)$, where $T_C(G)=\{\alpha\in T(G) \mid 0\alpha=0\}$. $\mathfrak{T}_C(\mathfrak{G})$ is a sub-near-ring of $\mathfrak{T}(\mathfrak{G})$ with the property that $\zeta\alpha=\alpha\zeta$ for all $\alpha\in T_C(G)$. It is the maximal sub-near-ring of $\mathfrak{T}(\mathfrak{G})$ with this property.

If $\mathfrak{G}$ is a topological group, and $K(G)\subset T(G)$ the continuous transformations, then $(K(G), +, \cdot)$ is another sub-near-ring of $\mathfrak{T}(\mathfrak{G})$.

Other near-rings can be constructed from $\mathfrak{G}$. For example, define multiplication in G by $g\cdot h=h$, or $g\cdot h=0$, $g, h\in G$. Then $(G, +, \cdot)$ is a near-ring.

The analytic functions on the complex plane form a near-ring (with the right distributive law instead of the left distributive law when the usual functional notation is used) relative to point-wise addition and substitution, and another near-ring relative to point-wise multiplication and substitution.

Other examples of near-rings can be found in [**3, 4**].

3. The arithmetic of near-rings. The arithmetic of near-rings differs from the arithmetic of rings since the right distributive law and commutative law of addition are missing. For example, the zero of a near-ring may behave multiplicatively in a different fashion from the zero of a ring: $a0=0$, but $0a$ need not equal 0. Indeed $\zeta\xi=\xi$ for every $\xi\in T_Z(G)$. All the elements of $T_Z(G)$ behave multiplicatively on the right like zero (*i.e.* $\alpha\xi=\xi$, $\alpha\in T(G)$, $\xi\in T_Z(G)$).

Seemingly obvious properties of rings do not hold for near-rings. For example, even if a near-ring contains a multiplicative identity 1, its additive inverse -1 need not commute with all the elements. Multiplicative inverses, if they exist, may also behave in a bizarre manner. For example, commutative elements may have multiplicative inverses which do not commute. A study of the arithmetic of near-rings emphasizes the role played by the right distributive law and commutative addition in ring theory.

4. Special classes of near-rings. In this section restrictions on near-rings are introduced and some of the implications of these restrictions are indicated. A definition is needed. A *sub-near-ring* $\mathfrak{Q}$ of a near-ring $\mathfrak{P}=(P, +, \cdot)$ is a system $(Q, +, \cdot)$ such that $(Q, +)$ is a subgroup of $(P, +)$, and $(Q, \cdot)$ is a sub-semi-group of $(P, \cdot)$.

(a) *Rings.* Of course the most widely-studied near-rings are rings. A ring is a near-ring $\mathfrak{R}=(R, +, \cdot)$ with the two restrictions (i) $+$ is commutative, and (ii) the right distributive law holds. It is not sufficient to assume only (ii), as is seen by considering $(G, +, \cdot)$ in which $(G, +)$ is a noncommutative group and $g\cdot h=0$, $g, h\in G$.

If a near-ring $\mathfrak{P}$ *satisfies* (ii), *and* (iii) *every element of P is the product of two elements, then* $\mathfrak{P}$ *is a ring.* This result, first published by Olga Taussky, is easily proved by setting $a=a_1a_2$, $b=b_1b_2$ and considering the two expansions of $(a_1+b_1)(b_2+a_2)$. It follows that a near-ring with a multiplicative identity and satisfying (ii) is a ring. Thus the commutative law of addition for a ring with identity is redundant. There are other simple properties which guarantee (i) and (ii). For example, a near-ring $\mathfrak{P}$ having commutative multiplication and satisfying (iii) (*e.g. P* contains an identity) is a ring.

(b) *C-rings.* A *C-ring* is a near-ring $\mathfrak{C}$ with the property that $0c=0$ for every $c\in C$. A study of semi-simple C-rings was made by Blackett [2]. Rings and near-fields are C-rings. Other examples are $\mathfrak{T}_R(\mathfrak{G})$ and $\mathfrak{T}_C(\mathfrak{G})$ of Section 2. *A near-ring* $\mathfrak{P}$ *is a C-ring if and only if* $az=z$, $a\in P$ *implies that* $z=0$. For, if $\mathfrak{P}$ is a C-ring, assume that $az=z$ for every $a\in P$, then $0z=z$. But $0z=0$, so that $z=0$. On the other hand, if $az=z$, $a\in P$ implies that $z=0$, consider $0a$, $a\in P$. Then $b(0a)=(b0)a=0a$, $b\in P$, so that $0a=0$.

Let $\mathfrak{P}$ be a near-ring and $P_C=\{a\in P \mid 0a=0\}$. It is immediate that P_C determines a sub-near-ring of $\mathfrak{P}$ which is a maximal C-ring of P. Other characterizations of C-rings will be given in terms of ideals in Section 7.

(c) *Z-rings.* A *Z-ring* is a near-ring $\mathfrak{Z}$ with the property that $ab=b$ for every $a, b\in Z$. $\mathfrak{T}_Z(\mathfrak{G})$ defined in Section 2 is an example of a Z-ring. The following statements are immediate. Let $\mathfrak{G}=(G, +)$ be a group, and $\cdot$ the binary composition on G defined by $g\cdot h=h$, $g, h\in G$. Then the system $(G, +, \cdot)$ is a Z-ring, and conversely every Z-ring is characterized by a group $\mathfrak{G}$ and this multiplication. The system $(H, +, \cdot)$, where $\mathfrak{H}$ is a sub-group of $\mathfrak{G}$, is a Z-ring and a sub-near-ring of $(G, +, \cdot)$, and conversely.

Let $\mathfrak{P}$ be a near-ring and $P_Z=\{z\in P \mid az=z,\ a\in P\}$. The set P_Z determines a sub-near-ring of $\mathfrak{P}$ which is a Z-ring. Further, if $\mathfrak{Q}$ is a sub-near-ring of $\mathfrak{P}$ which is a Z-ring, then $\mathfrak{Q}\subset\mathfrak{P}_Z$. For, it is clear that $\mathfrak{P}_Z$ is a sub-near-ring of $\mathfrak{P}$ and a Z-ring. Let $\mathfrak{Q}$ be a sub-near-ring of $\mathfrak{P}$ which is a Z-ring. Then, since $0\in Q$, $aq=a(0q)=(a0)q=0q=q$, for $q\in Q$, $a\in P$. Hence $q\in P_Z$ by definition.

It will be shown in Section 5 that every near-ring is a direct sum of a C-ring and a Z-ring.

(d) *R-rings.* An element $a\in P$ is *anti-right distributive* if $(b+c)a=ca+ba$; $b, c\in P$. It follows at once that (i) an element a is right distributive if and only

if $-a$ is anti-right distributive, (ii) the product ab is right distributive if a, b are both right distributive or both anti-right distributive, and (iii) the product ab is anti-right distributive if one of a, b is right distributive and the other anti-right distributive. The arithmetic of right and anti-right distributive elements closely approximates the arithmetic in rings. For example, if $\mathfrak{P}$ is a C-ring and Q the set of right and anti-right distributive elements of P, then for $p \in P$, $q \in Q$, $-(pq) = (-p)q = p(-q)$. The proof follows from the observation that $0 = 0q = (p + (-p))q = pq + (-p)q$ or $(-p)q + pq$.

An element $a \in P$ is *weakly right distributive* if it is a finite sum of right and anti-right distributive elements. An *R-ring* is a near-ring in which every element is weakly right distributive. *The set of weakly right distributive elements of a near-ring $\mathfrak{P}$ form a sub-near-ring $\mathfrak{P}_R$.* For, the sum and difference of weakly right distributive elements is clearly weakly right distributive, and it is easy to check that the product is also.

The smallest near-ring of transformations on a group $\mathfrak{G}$ containing the endomorphisms of $\mathfrak{G}$ is an R-ring ($\mathfrak{T}_R(\mathfrak{G})$ in Section 2). It is universal in the sense that every R-ring can be embedded isomorphically into such a weakly right distributive near-ring of transformations (see Section 6).

If addition is commutative in an R-ring it follows trivially that it is a ring.

(e) *M-rings.* An *M-ring* is a near-ring with commutative multiplication. Since 0 commutes with every element of an M-ring it follows that an M-ring is a C-ring. A condition that an M-ring be a ring was given in (a).

Let $P_M = \{a \in P \mid ap = pa,\ p \in P\}$ be the *center* of P. If $c \in P_M$, $b \in P$, then $b(-c) = (-b)c = -(bc)$, and the elements of P_M are right distributive relative to P. (*i.e.* $(a_1 + a_2)c = a_1c + a_2c$; $a_1, a_2 \in P \cdot c \in P_M$). The statement follows from the identities $b(-c) = -(bc) = -(cb) = c(-b) = (-b)c$ and $(a_1 + a_2)c = c(a_1 + a_2) = ca_1 + ca_2 = a_1c + a_2c$. Note that if $c \in P_M$, then $-c$ is anti-right distributive over P.

The center of a near-ring may be empty. It does not follow that if P_M is nonempty that its elements determine a near-ring. For example in $\mathfrak{T}(\mathfrak{G})$, the identity ϵ is in the center but the additive zero ζ is not. Hence a near-ring is not determined. However, if $\mathfrak{P}_M = (P_M, +, \cdot)$ is a sub-near-ring of $\mathfrak{P}$, then $\mathfrak{P}_M$ is a C-ring and the elements of P are right distributive over P_M. (*i.e.* $(c_1 + c_2)a = c_1a + c_2a$; $c_1, c_2 \in P_M$, $a \in P$.) Further, if every element of P_M can be expressed as the product of two elements of P_M, then P_M is a ring. For, if P_M is a near-ring, $O \in P_M$, and hence P is a C-ring. The identity $(c_1 + c_2)a = a(c_1 + c_2) = ac_1 + ac_2 = c_1a + c_2a$; $c_1, c_2 \in P_M$, $a \in P$ shows that P is right distributive over P_M. The ring property follows from the previous remarks.

(f) *D-rings.* A *D-ring* is a near-ring $\mathfrak{D}$ such that (i) $ab = 0$; $a, b \in D$ implies that either $a = 0$ or $b = 0$, and (ii) for every $a \in D$ there are nonzero elements $a_l, a_r \in D$ such that a_la and aa_r are in the multiplicative center D_M. Near-fields are D-rings. Integral domains are examples of commutative rings which are D-rings, and an example of a noncommutative D-ring is the ring of integral quaternions. The D-rings are generalizations of integral domains. It is shown in

Section 6 that D-rings can be embedded isomorphically into near-fields.

The identities $(0a)a_r = 0(aa_r) = (aa_r)0 = 0$ imply that $0a = 0$, so that *a D-ring is a C-ring. The condition that $ab = 0$ in a D-ring implies $a = 0$ or $b = 0$ may be replaced by the left and right cancellation laws, i.e. $ab = ac$ or $ba = ca$ implies $b = c$ if $a \neq 0$.* Assume $ab = ac$, $a \neq 0$, then $a(b-c) = 0$ implies $b - c = 0$. If $ba = ca$, $a \neq 0$, then $(ba)a_r = (ca)a_r$ so that $aa_r(b-c) = 0$. But $aa_r \neq 0$ implies $b = c$. The converse is immediate. Just as in field theory, *either a D-ring $\mathfrak{D}$ has characteristic $p \neq 0$ (prime) and contains an isomorphic image of $GF(p)$, or $\mathfrak{D}$ has characteristic 0 and contains an isomorphic image of the ring of integers. In the latter case if $\mathfrak{D}$ is a near-field it contains an isomorphic image of the field of rationals.* This statement is proved in exactly the same way as the corresponding theorem for fields. In case D has finite cardinality the cancellation law implies the unique solution of the equations $ax = b$ and $ya = b$, $a \neq 0$, which in turn implies that *a finite D-ring is a near-field.*

(g) *A-rings.* An *A-ring* is a near-ring with commutative addition. A ring is of course an A-ring. The near-ring $\mathfrak{T}(\mathfrak{G})$, where $\mathfrak{G}$ is a commutative group, is an example of an A-ring which is not a ring. However, an A-ring which is an R-ring is a ring. It is immediate that *a near-ring with a multiplicative identity is an A-ring if and only if -1 is in the multiplicative center.*

5. A decomposition theorem. Let $\mathfrak{P} = (P, +, \cdot)$ be a near-ring. The near-ring $\mathfrak{P}$ is the direct sum $\mathfrak{Q} \oplus \mathfrak{R}$ of the sub-near-rings $\mathfrak{Q}$ and $\mathfrak{R}$ if every element of P can be expressed as a unique sum of elements $q + r$; $q \in Q$, $r \in R$. It is clear that, given any two near-rings $\mathfrak{Q}$, $\mathfrak{R}$, a near-ring $\mathfrak{P}$ can be formed which is the direct sum of $\mathfrak{Q}$ and $\mathfrak{R}$ by defining coordinate-wise addition and multiplication on $Q \times R$. The near-ring $\mathfrak{P}$ has similar properties to the corresponding direct sum of two rings.

An *idempotent* is an element $e \in P$ such that $e^2 = e$. The following theorem generalizes the Peirce decomposition for rings.

THEOREM. *Let e be an idempotent in P. Then every element $p \in P$ has two unique decompositions $p = ep + (-ep + p) = (p - ep) + ep$. Thus $\mathfrak{P} = \mathfrak{R} \oplus \mathfrak{S} = \mathfrak{S} \oplus \mathfrak{R}$, where $R = \{ep \mid p \in P\}$ and $S = \{s \in P \mid es = 0\}$.*

The system $\mathfrak{R} = (R, +, \cdot)$ and $\mathfrak{S} = (S, +, \cdot)$ are near-rings. The elements $p - ep$ and $-ep + p$ are in S. Suppose $p = r_1 + s_1 = r_2 + s_2$. Then $-r_2 + r_1 = s_2 - s_1$ must be in $T = R \cap S$. But the only element in T is 0, for suppose $t \in T$. Then $et = 0$ and $t = ep$ for some $p \in P$, and $0 = et = e(ep) = e^2 p = ep = t$. Thus $r_1 = r_2$ and $s_1 = s_2$. The uniqueness of the other representation is proved in the same way. Since $z^2 = z$ and $zp \in Z$ for $z \in P_Z$, $p \in P$, the following corollary is immediate.

COROLLARY. *Every element $z \in P_Z$ is an idempotent and determines the direct sum decompositions $\mathfrak{P} = \mathfrak{P}_Z \oplus \mathfrak{Q}_Z = \mathfrak{Q}_Z \oplus \mathfrak{P}_Z$ where $Q_Z = \{p \in P \mid zp = 0\}$.*

The special case $z = 0$ is of interest since it shows that every near-ring can be expressed as the direct sum of its maximal sub-Z-ring $\mathfrak{P}_Z$ (4c), and its maximal sub-C-ring $\mathfrak{P}_C$ (4b).

COROLLARY. *Every element* $p \in P$ *has unique decompositions* $p = p_1 + p_2 = p_2 + p_1'$ *where* $p_1, p_1' \in P_C$, *and* $p_2 \in P_Z$, *i.e.* $\mathfrak{P} = \mathfrak{P}_C \oplus \mathfrak{P}_Z = \mathfrak{P}_Z \oplus \mathfrak{P}_C$.

6. Extensions and embedding theorems. The well-known theorem on the embedding of a ring into the ring of endomorphisms of a commutative group can be generalized to near-rings. In fact *every near-ring can be embedded isomorphically into a near-ring of transformations* $\mathfrak{T}(\mathfrak{G})$ *on some group* $\mathfrak{G}$. *The elements in the isomorphic image can be taken as right multiplication operators on the additive group of the near-ring.* It follows that *every near-ring can be embedded isomorphically into a near-ring with identity.* These results are proved in [1]. In addition it is shown that *an R-ring (C-ring, Z-ring) can be embedded isomorphically into the R-ring (C-ring, Z-ring) of transformations* $\mathfrak{T}_R(\mathfrak{G})(\mathfrak{T}_C(\mathfrak{G}), \mathfrak{T}_Z(\mathfrak{G}))$ *on a group* $\mathfrak{G}$. *The elements of the image can again be taken as right multiplication operators.*

The theorem that an integral domain can be embedded in a field is now generalized to the embedding of D-rings into near-fields. It can be shown that it is a special case of a theorem proved for more general systems by Ore [12]. The theorem proved below includes as a special case the embedding of certain noncommutative integral domains into division rings. For example, the ring of integral quaternions, which is a D-ring, is embedded in the near-field of rational quaternions.

THEOREM. *A D-ring is isomorphic to a sub-near-ring of a near-field.*

LEMMA. *A D-ring* $\mathfrak{D}$ *can be embedded isomorphically in a D-ring* $\mathfrak{D}'$ *such that the nonzero elements of* D_M' *(its multiplicative center) form a group relative to multiplication.*

Consider the collection of pairs $(c, d) \in D_M \times D$ in which $c \neq 0$. This set may be partitioned into equivalence classes by defining $(c_1, d_1) \sim (c_2, d_2)$ whenever $c_1 d_2 = c_2 d_1$. It is easy to verify that $\sim$ is an equivalence relation. Let $[c, d]$ denote the equivalence class containing (c, d), and set $D' = \{ [c, d], c \in D_M, c \neq 0, d \in D \}$. Let $+$, $\cdot$ be the binary compositions on D' defined by $[c_1, d_1] + [c_2, d_2] = [c_1 c_2, c_2 d_1 + c_1 d_2]$ and $[c_1, d_1] \cdot [c_2, d_2] = [c_1 c_2, d_1 d_2]$. Straightforward calculations show that the system $\mathfrak{D}' = (D', +, \cdot)$ is a near-ring. It is a D-ring, for if $[c_1, d_1] \cdot [c_2, d_2] = [c, 0]$ (the zero of D') then $c_1 c_2 0 = c d_1 d_2$, and either d_1 or d_2 is zero. This implies that either $[c_1, d_1]$ or $[c_2, d_2]$ is the zero element of D'. Further $[c, d_l]$ and $[c, d_r]$ have the property that $[c, d_l] \cdot [c, d]$ and $[c, d] \cdot [c, d_r]$ are in D_M'.

Let μ be the mapping from D to D' defined by $d\mu = [c, cd]$, $c \in D_M$. The mapping is clearly well defined and a simple calculation shows that it is an isomorphism. Thus $\mathfrak{D}'$ contains an isomorphic image of $\mathfrak{D}$.

The nonzero elements of D_M' are of the form $[c_1, c_2]$, $c_1, c_2 \in D_M$, $c_1, c_2 \neq 0$. For if $[c, d] \in D_M'$, then for every $[c', d'] \in D'$, $[c, d] \cdot [c', d'] = [c', d'] \cdot [c, d]$ implies $cc'd'd = cc'dd'$, and $d'd = dd'$ for every $d' \in D$. Thus $d \in D_M$. The converse is clear. Finally, the nonzero elements of D_M' form a multiplicative group since $[c_2, c_1]$ $[c_1, c_2] = [c_1 c_2, c_1 c_2]$ the multiplicative identity of D'.

The proof of the theorem follows from the observation that $\mathfrak{D}'$ is in fact a near-field. For, let $\alpha=[c, d]$, $d\neq 0$, $\alpha_l=[d_ld, cd_l]$, $\alpha_r=[dd_r, cd_r]$. Then α_l and α_r are the left and right multiplicative inverses of α and $\alpha_r=(\alpha_l\alpha)\alpha_r=\alpha_l(\alpha\alpha_r)=\alpha_l$.

It is easily checked that the right distributive law holds in $\mathfrak{D}'$ if it holds in $\mathfrak{D}$. Hence a *D-ring which is a ring is isomorphic to a sub-(near)-ring of a division ring.*

7. Ideals and homomorphisms. In this section the concept of near-ring homomorphism is considered. The theory is analogous to the homomorphism theory for rings. However, the kernel of a homomorphism need not be a two-sided ideal in the usual sense. Neither need a two-sided ideal be the kernel of a near-ring homomorphism. Ideals will be defined (as in ring theory) so that the kernel of a homomorphism is an ideal and every ideal is the kernel of a homomorphism.

The *left coset* $a+S$, $a\in P$, modulo *a* set $S\subset P$ is the set $\{a+s \mid s\in S\}$. The *right coset* $S+a$ is defined analogously. A set S, or a near-ring $\mathfrak{S}$, is (*left*) *right invariant* in $\mathfrak{P}$ if $(Sa\subset S)aS\subset S$, $a\in P$. A sub-near-ring $\mathfrak{N}$ of $\mathfrak{P}$ is normal if $(N, +)$ is a normal sub-group of $(P, +)$. A (*left*) *right ideal* $\mathfrak{J}=(I, +, \cdot)$ of $\mathfrak{P}$ is a (left) right invariant normal sub-near-ring of $\mathfrak{P}$. A *two-sided ideal* is a right ideal which is also a left ideal. A *pseudo-right ideal* $\mathfrak{J}$ is a normal sub-near-ring of $\mathfrak{P}$ such that $ia-0a\in I$ for every $a\in P$, $i\in I$. A set S is *invariant* in $\mathfrak{P}$ if $(a+s_1)(b+s_2)-ab\in S$ for every a, $b\in P$ and s_1, $s_2\in S$. An *ideal* in $\mathfrak{P}$ is an invariant normal sub-near-ring of $\mathfrak{P}$. It should be noted that the normality of an ideal implies that the condition $(s_1+a)(s_2+B)-ab\in S$, a, $b\in P$, s_1, $s_2\in S$, is equivalent to the condition for invariance.

(1) *Every ideal* (*invariant sub-near-ring*) *is a left ideal* (*left invariant sub-near-ring*) *and pseudo-right ideal* (*pseudo-right invariant sub-near-ring*). (ii) *Every right ideal* (*right invariant sub-near-ring*) *is a pseudo-right ideal* (*pseudo-right invariant sub-near-ring*). The proof of (i) follows by taking special cases of the condition for invariance. Take $s_1=0$, $b=0$ to show that an ideal is left ideal and $a=0$, $s_2=0$ in the other case. Statement (ii) is trivial. Examples are now presented to show that, except for the relationships of this statement, the concepts of right invariance, left invariance, invariance and normality are independent. The near-ring $\mathfrak{Q}$ is a sub-near-ring of the near-ring $\mathfrak{P}$ having the properties stated and no others. The examples are easily verified.

(i) *Right invariant.* Let $\mathfrak{H}$ be a noncommutative nonnormal subgroup of a group $\mathfrak{G}$. Let $\mathfrak{P}$ be the sub-near-ring of $\mathfrak{T}_G(\mathfrak{G})$ such that $P=\{\alpha\in T_G(G) \mid g\alpha\in H\}$. Let $\mathfrak{Q}$ be the sub-near-ring of $\mathfrak{P}$ such that $Q=\{\beta_h\in P \mid h'\beta_h=h;\ h, h'\in H, h'\neq 0;\ 0\beta_h=0,\ g\beta_h=0,\ g\in H\}$.

(ii) *Left invariant.* Let $\mathfrak{H}$ be a nonnormal subgroup of $\mathfrak{G}$. Let $\mathfrak{P}=\mathfrak{T}_Z(\mathfrak{G})$. Let $\mathfrak{Q}$ be the sub-near-ring of $\mathfrak{P}$ such that $Q=\{\xi_h\in T_Z(G) \mid h\in H\}$. (*i.e.* $g\xi_h=h$; $h\in H$, $g\in G$.)

(iii) *Normal.* Let $\mathfrak{H}$ be a proper nontrivial subgroup of $\mathfrak{G}$. Let $\mathfrak{P}=\mathfrak{T}(\mathfrak{G})$ and $\mathfrak{Q}$ the sub-near-ring of $\mathfrak{P}$ such that $Q=\{\alpha\in T(G) \mid h\alpha=0,\ h\in H\}$. Another example is any subring of a ring which is neither a left or right ideal.

(iv) *Left invariant, invariant.* Let $\mathfrak{P}=(R\times S, \oplus, \odot)$ where $\mathfrak{R}=(R, +, \cdot)$ is a Z-ring and $\mathfrak{S}=(S, +, \cdot)$ is a near-ring with the multiplication $ab=0$, containing a nonnormal sub-near-ring $\mathfrak{T}$, and let $\oplus$, $\odot$ be defined by $(r_1, s_1)\oplus(r_2, s_2)=(r_1+r_2, s_1+s_2)$; $(r_1, s_1)\odot(r_2, s_2)=(r_2, 0)$. Let $\mathfrak{Q}$ be the sub-near-ring of $\mathfrak{P}$ such that $Q=\{(0, t)\mid t\in T\}$.

(v) *Right invariant, normal.* Let $\mathfrak{H}$ be a proper nontrivial subgroup of $\mathfrak{G}$. Let $\mathfrak{P}=\mathfrak{T}_C(\mathfrak{G})$ and let $\mathfrak{Q}$ be the sub-near-ring of $\mathfrak{P}$ such that $Q=\{\alpha\in T_C(G)\mid h\alpha=0, h\in H\}$. Another example is a right ideal (which is not an ideal) in a ring.

(vi) *Left invariant, normal.* Let $\mathfrak{H}$ be a proper, nontrivial subgroup of a commutative group $\mathfrak{G}$. Let $\mathfrak{P}=\mathfrak{T}(\mathfrak{G})$ and let $\mathfrak{Q}$ be the sub-near-ring of $\mathfrak{P}$ such that $Q=\{\xi_h\in T(G)\mid g\xi_h=h, g\in G, h\in H\}$. Another example is a left ideal (which is not a right ideal) in a ring.

(vii) *Right invariant, left invariant.* Let $\mathfrak{P}=\mathfrak{T}(\mathfrak{G})$, $\mathfrak{G}$ be a noncommutative group, and let $\mathfrak{Q}=\mathfrak{T}_Z(\mathfrak{G})$.

(viii) *Left invariant, invariant, normal.* Let $\mathfrak{H}$ be a normal subgroup of $\mathfrak{G}$. Let $\mathfrak{P}=\mathfrak{T}_Z(\mathfrak{G})$ and let $\mathfrak{Q}$ be the sub-near-ring of $\mathfrak{P}$ such that $Q=\{\xi_h\in P\mid g\xi_h=h, g\in G, h\in H\}$.

(ix) *Left invariant, right invariant, invariant.* Let $\mathfrak{P}$ be a near-ring with the multiplication $ab=0$, whose additive group is noncommutative and contains a normal subgroup $(Q, +)$. Let $\mathfrak{Q}$ be the sub-near-ring of $\mathfrak{P}$ determined by Q.

(x) *Right invariant, left invariant, normal.* Let $\mathfrak{G}$ be a commutative group. Let $\mathfrak{P}=\mathfrak{T}(\mathfrak{G})$ and $\mathfrak{Q}=\mathfrak{T}_Z(\mathfrak{G})$.

(xi) *Left invariant, right invariant, invariant, normal.* Let $\mathfrak{P}$ be a near-ring with the multiplication $ab=0$ or an R-ring, and let $\mathfrak{Q}$ be an ideal in $\mathfrak{P}$.

It was shown in (iv) that an invariant sub-near-ring need not be an ideal. However

THEOREM. *If $\mathfrak{P}$ is a near-ring with identity, then any invariant sub-near-ring is an ideal.*

The proof follows from the observation that $(a+i)(1+0)-a=a+i-a\in I$ for $a\in P$, $i\in I$. Relative to the definitions given, the homomorphism and ideal theory for rings carries over completely to near-rings. The very basic theorems of ring theory are now restated for near-rings.

THEOREM. *The cosets of $\mathfrak{P}$ modulo an ideal $\mathfrak{I}$ form a near-ring $\mathfrak{P}-\mathfrak{I}$ relative to the compositions $(a+I)+(b+I)=(a+b+I)$, $(a+I)\cdot(b+I)=(ab+I)$.*

The proof is similar to the corresponding proof for rings. Normality of $(I, +)$ replaces commutativity of addition and the invariance of $\mathfrak{I}$ is substituted for two-sided invariance. The near-ring $\mathfrak{P}-\mathfrak{I}$ is called the *difference near-ring* of $\mathfrak{P}$ modulo $\mathfrak{I}$.

A *homomorphism* of a near-ring $\mathfrak{P}$ into a near-ring $\mathfrak{Q}$ is a mapping of P into Q such that $(a+b)\theta=a\theta+b\theta$, and $(ab)\theta=(a\theta)\cdot(b\theta)$. An *isomorphism* (onto) is an (onto) 1-1 homomorphism. The near-ring $\mathfrak{P}$ is *isomorphic* to $\mathfrak{Q}(\mathfrak{P}\cong\mathfrak{Q})$ if there

exists an isomorphism from $\mathfrak{P}$ onto $\mathfrak{Q}$. The *kernel* of a homomorphism of $\mathfrak{P}$ to $\mathfrak{Q}$ is the inverse image of the identity of $(Q, +)$. The following theorems are examples of theorems which are true for rings and which carry over to near-rings. The proofs, which are essentially the same as in ring theory, are omitted.

If $\mathfrak{I}$ is an ideal in $\mathfrak{P}$, the natural map ν: $a\nu = a + I$, $a \in P$ is a homomorphism of $\mathfrak{P}$ onto $\mathfrak{P} - \mathfrak{I}$. The kernel of ν is $\mathfrak{I}$. Thus

THEOREM. *Every ideal is the kernel of a homomorphism (natural homomorphism). Conversely, the kernel of a homomorphism is an ideal.*

Let θ be a homomorphism of the near-ring $\mathfrak{P}$ into the near-ring $\mathfrak{Q}$ with kernel $\mathfrak{K}$ and image $\mathfrak{P}'$, and let $\mathfrak{I}$ be an ideal of $\mathfrak{P}$ contained in $\mathfrak{K}$. Then the map ϕ: $(a+I)\phi = a\theta$, $a \in P$, of $\mathfrak{P} - \mathfrak{I}$ onto $\mathfrak{P}'$ is a homomorphism. Further $\theta = \nu\phi$, and ϕ is an isomorphism if and only if $I = K$. Thus

THEOREM. *Every homomorphic image of a near-ring is isomorphic to a difference near-ring.*

8. Ideals in special classes of near-rings. A few theorems are now presented concerning ideals in special classes of near-rings. In the case of Z-rings the lattice structure of the ideals is completely determined.

(a) *Ideals in C-rings. A near-ring $\mathfrak{P}$ is a C-ring if and only if every ideal in $\mathfrak{P}$ is two-sided.* From Section 7, every ideal in $\mathfrak{P}$ is a left ideal and a pseudo-right ideal. In case $\mathfrak{P}$ is a C-ring a pseudo-right ideal is clearly a right ideal. Conversely, if every ideal in $\mathfrak{P}$ is two-sided, the ideal containing only the element 0 is two-sided. But then $0a = 0$ for every $a \in P$. It follows that *a near-ring is a C-ring if and only if the ideal $\{0\}$ is two-sided.*

(b) *Ideals in R-rings. Every ideal in a R-ring is two-sided and conversely every two-sided ideal in a R-ring is an ideal.* Since an R-ring is a C-ring, and every ideal in a C-ring is two-sided by the previous remarks, the first statement follows. To prove the converse, consider a two-sided ideal $\mathfrak{I}$ in $\mathfrak{P}$. Let p_1, $p \in P$, $i \in I$, and let p be either right distributive or anti-right distributive. Then the normality of $\mathfrak{I}$ insures that $(p_1+i)p = p_1p + i'$, $i' \in I$. Now consider any element $p_2 \in P$ and write $p_2 = p^1 + p^2 \cdots + p^n$ where p^i $(i = 1, \cdots, n)$ is right distributive or anti-right distributive. Then by the above remarks, and using the normality and right invariance of $I (p_1 + i_1)(p_2 + i_2) - p_1p_2 = (p_1 + i_1)p_2 + (p_1 + i_1)i_2 - p_1p_2 = (p_1 + i_1)(p^1 + \cdots + p^n) + i_3 - p_1p_2 = (p_1 + i_1)p^1 + (p_1 + i_1)p^2 + \cdots + (p_1 + i_1)p^n + i_3 - p_1p_2 = (p_1p^1 + i_1') + (p_1p^2 + i_2') + \cdots + (p_1p^n + i_n') + i_3 - p_1p_2 = p_1(p^1 + \cdots + p^n) + i_4 - p_1p_2 = i_5$ where $i_1, i_2, \cdots, i_5, i_1', i_2', \cdots, i_n'$ are in I. This proves that $\mathfrak{I}$ is an ideal.

(c) *Ideals in Z-rings. Let $\mathfrak{P}$ be a Z-ring and $(Q, +)$ a normal subgroup of $(P, +)$. Then $\mathfrak{Q} = (Q, +, \cdot)$ is an ideal of $\mathfrak{P}$. Conversely every ideal of $\mathfrak{P}$ is characterized by a normal subgroup of $(P, +)$.* The system $\mathfrak{Q}$ is a sub-near-ring of $\mathfrak{P}$ since $\mathfrak{Q}$ is closed under multiplication. The normality assures that $\mathfrak{Q}$ is an ideal, for $(p_1+q_1)(p_2+q_2) - p_1p_2 = p_2 + q_2 - p_2$ is in Q for p_1, $p_2 \in P$, q_1, $q_2 \in Q$. Conversely, if $\mathfrak{I}$ is an ideal, the group $(Q, +)$ is normal in $(P, +)$.

(d) *Annihilators and ideals.* Ideals in certain special near-rings may be constructed by various devices. For example, let $\mathfrak{P}$ be a near-ring, S a subset of P and $R(S)=\{a\in P \mid sa=0,\ s\in S\}$. (i) The system $\mathfrak{R}(S)=(R(S),\ +,\ \cdot)$ is a sub-near-ring of $\mathfrak{P}$ if $0\in S$ or if $\mathfrak{P}$ is a C-ring. (ii) If $\mathfrak{P}$ is a C-ring, then $\mathfrak{R}(S)$ is a right ideal. (iii) If S is right invariant and $\mathfrak{R}(S)$ a sub-near-ring of $\mathfrak{P}$, then $\mathfrak{R}(S)$ is an ideal. If $a,\ b\in R(S)$, then $a+b$ and $-a\in R(S)$, and if either $0\in S$ or $\mathfrak{P}$ is a C-ring then $ab\in R(S)$. This is easily checked. Statement (ii) follows from the observation that $s(ap)=(sa)p=0p=0$, and $s(-p+a+p)=-sp+0+sp=0$, for $p\in P$, $s\in S$, $a\in R(S)$. The proof of (iii) follows in a similar manner from a simple calculation.

The ideals constructed with annihilators can be generalized. Let $\mathfrak{P}$ be a near-ring and S, M subsets of P. Let $R(S,\ M)=\{a\in P \mid sa\in M,\ s\in S\}$. (i) If $(M,\ +)$ is a sub-group of $(P,\ +)$ such that $M\subset S$ or M is right invariant, then the system $\mathfrak{R}(S,\ \mathfrak{M})=(R(S,\ M),\ +,\ \cdot)$ is a sub-near-ring of $\mathfrak{P}$. (ii) Let $M\subset S$ and $(M,\ +)$ be a subgroup of $(P,\ +)$. Then (a) if $(M,\ +)$ is normal in $(P,\ +)$, $\mathfrak{R}(S,\ M)$ is normal in $\mathfrak{P}$, (b) if M is right invariant, $\mathfrak{R}(S,\ M)$ is right invariant, (c) if S is right invariant, $\mathfrak{R}(S,\ M)$ is left invariant, (d) if S is right invariant and $(M,\ +)$ is normal in $(P,\ +)$, $\mathfrak{R}(S,\ M)$ is a left ideal, and (e) if $(M,\ +)$ is right invariant and normal in $(P,\ +)$, $\mathfrak{R}(S,\ M)$ is a right ideal. (iii) Let $\mathfrak{P}$ be an R-ring, $(S,\ +)$ a subgroup of $(P,\ +)$ right invariant in P, and $(M,\ +)$, $M\subset S$ a normal subgroup of $(P,\ +)$ right invariant in $\mathfrak{P}$. Then $\mathfrak{R}(S,\ M)$ is an ideal. In (i) it is easy to check that $(R(S,\ M),\ +)$ is a group if $(M,\ +)$ is a group. The condition that $M\subset S$ or is right invariant assures that the multiplication is closed. Thus (i) is proved. Suppose that $(M,\ +)$ is normal in $(P,\ +)$. Then $sm\in M$ and $s(p+a-p)=sp+sa-sp\in M$ for $s\in S$, $p\in P$, $a\in R(S,\ M)$. This implies that $(R(S,\ M),\ +)$ is normal. The right invariance of M implies that $s(ap)=(sa)p\in M$, while the right invariance of S implies that $s(pa)=(sp)a\in M$ for $s\in S$, $p\in P$, $a\in R(S,\ M)$. Statements (d) and (e) follow. To prove (iii), let $s\in S$, $p_1,\ p_2\in P$, $a_1,\ a_2\in R(S,\ M)$. Let $p_2=p^1+\cdots+p^n$ where p^i $(i=1,\cdots,n)$ is right or anti-right distributive. Then $sa_1=m_1\in M$, $sp_1+sa_1=s_2\in S$, $s_2a_2=m_2\in M$, so that, using the normality of $(M,\ +)$ and the properties of the p^i: $s((p_1+a_1)(p_2+a_2)-p_1p_2)=(sp_1+sa_1)p_2+(sp_1+sa_1)a_2-sp_1p_2=(sp_1+m_1)(p^1+\cdots+p^n)+s_2a_2-sp_1p_2=(sp_1+m_1)p^1+(sp_2+m_1)p^2+\cdots+(sp_1+m_1)p^n+m_2-sp_1p_2=sp_1(p^1+\cdots+p^n)+m_3-sp_1p_2=m_4$, where $m_3,\ m_4\in M$. The normality of $\mathfrak{R}(S,\ M)$ follows from (iia). It should be noted that the condition that $\mathfrak{P}$ be an R-ring can be weakened to the condition that P be weakly right distributive over S.

9. Problems and extensions. This section presents a few problems involving near-rings.

Blackett [2], and Deskins [5], have considered the problem of semi-simplicity and the radical for certain classes of near-rings. An acceptable definition for the radical of a general near-ring is still needed. The characterization of ideals, left ideals, *etc.* in special classes of near-rings also presents a problem. For exam-

ple, what is the class of near-rings characterized by the property that every two-sided ideal is an ideal, and conversely? Rings, R-rings and simple C-rings lie in this class.

It has been shown [1] that the transformation near-rings $\mathfrak{T}(\mathfrak{G})$ and $\mathfrak{T}_C(\mathfrak{G})$ are simple, whereas $\mathfrak{T}_R(\mathfrak{G})$ is not always simple. If $\mathfrak{G}$ is simple, is $\mathfrak{T}_R(\mathfrak{G})$, and in general, what characterizes the class of groups $\mathfrak{G}$ such that $\mathfrak{T}_R(\mathfrak{G})$ is simple? Similar questions can be asked about transformation near-rings generated by other special transformations, *e.g.* inner automorphisms, automorphisms.

Every group can be taken as the additive group of a near-ring. What characterizes the class of near-rings with the property that, if the group operation is taken as the addition of a near-ring, then the only possible multiplications are $ab=0$ or b? More generally, given a class of near-rings (*e.g.*, rings, C-rings), what is a characterization of the corresponding additive groups? This is certainly not a simple problem in the case of rings. Similar problems can be formulated for the multiplicative semi-group of a near-ring.

A class of "near-vector" spaces has already had application [7]. The general theory of "near-vector" spaces could be developed to generalize vector spaces. A further generalization might be to "near-modules." The analogy to Lie and Jordan rings could also be developed for near-rings. Similarly, near-field embeddings and Galois theory are problems to be considered. The fact that any class of operators on a group generate a near-ring leads one to surmise that near-rings might have application to nonlinear operators. Banach and topological algebras might be generalized to topological near-rings.

Veblen and Wedderburn used near-fields to construct the first example of a non-Desarguesian finite plane. Further studies of non-Desarguesian planes involving near-fields have recently been made [7, 9]. In any Veblen-Wedderburn geometry based on a near-field the little theorem of Desargues holds for all pairs of triangles in perspective from the point (1, 0, 0). Is there a 1-1 correspondence between near-fields and geometries with this property [9]? For other geometric problems related to near-fields see [7].

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A TOPOLOGY FOR SEQUENCES OF INTEGERS I

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1. Introduction. In what follows we shall be concerned with the set S of all sequences which possess a natural density. The natural density of the sequence $A=\{a_n\}$ is defined by $\delta(A)=\lim_{n\to\infty} n^{-1}A(n)$ whenever the limit exists and where $A(n)$ is the number of elements of A not exceeding n. It is also assumed that A is a strictly-increasing sequence of positive integers and that all sequences are infinite.

In the following sections we shall define a metric in the set S which depends on the concept of density. We then deduce a number of the simpler topological properties of the space S and attempt to determine its structure.

2. A metric space. If $A=\{a_n\}$ and $B=\{b_n\}$, the distance between A and B will be defined by $\rho(A, B)=0$ if $A=B$, *i.e.*, if $a_n=b_n$ for all n and by $\rho(A, B)=k^{-1}+|\delta(A)-\delta(B)|$ otherwise, where k is the smallest value of n such that $a_n\neq b_n$. Any real-valued function having certain simple properties could be used in place of $\delta(A)$ in this definition and many of the subsequent results would remain valid or require only slight modification. In particular, the replacement of the natural density by either the lower or upper asymptotic density would yield an extension of this metric to the set of all sequences.

The metric $\rho(A, B)=k^{-1}$ is well known and has many interesting properties. A discussion of this metric and references to the literature can be found in the books of Ostmann [1]. These books also contain a vast amount of other material on sequences and related topics as well as a very comprehensive bibliography. Excellent summaries of some of the arithmetical properties of sequences have been given by Erdös and Niven [2] and by Niven [3].

THEOREM 1. *S is a metric space.*

Proof. It is obvious that $\rho(A, B)\geqq 0$ and that $\rho(A, B)=0$ if and only if $A=B$. It is also clear that $\rho(A, B)=\rho(B, A)$. Let $\rho(A, C)=k_1^{-1}+|\delta(A)-\delta(C)|$, $\rho(A, B)=k_2^{-1}+|\delta(A)-\delta(B)|$, $\rho(B, C)=k_3^{-1}+|\delta(B)-\delta(C)|$. If one of $\rho(A, B)$ and

$\rho(B, C)$ is zero, then $\rho(A, C)$ is equal to the other. Also, $k_1 \geqq \min \{k_2, k_3\}$ and it follows that $k_1^{-1} \leqq k_2^{-1} + k_3^{-1}$. The triangle law follows since

$$|\delta(A) - \delta(C)| \leqq |\delta(A) - \delta(B)| + |\delta(B) - \delta(C)|.$$

COROLLARY. *The metric of S is not equivalent to the one given by $\rho(A, B) = 0$ if $A = B$ and $\rho(A, B) = k^{-1}$ otherwise, where k has the same meaning as before.*

COROLLARY. *S is a bounded metric space.*

THEOREM 2. *S is not totally bounded.*

Proof. Suppose that $S = \bigcup_{k=1}^{n} E_k$, where each E_k has diameter less than one. Then all sequences in any given E_k must have the same first term, say n_k. If A is any sequence such that $a_1 \neq n_k$ for $1 \leqq k \leqq n$, then $A \notin S$ and this is a contradiction.

COROLLARY. *S is not compact.*

We prove directly that S is not compact as follows. Let $E_n = \{A \in S \mid a_1 \geqq n\}$. It is obvious that $\{E_n\}$ is a collection of closed sets having the finite intersection property. But $\bigcap_{n=1}^{\infty} E_n$ is empty and so S is not compact. This argument actually shows that S is not countably compact.

The following definition is used in the formulation of the next theorem: $d(A) = \text{g.l.b. } n^{-1}A(n)$. We shall also make repeated use of the obvious fact that $d(A) \leqq \delta(A)$.

THEOREM 3. *Let*

$$A_n \to A. \tag{1}$$

Then

$$\limsup_{n \to \infty} d(A_n) \leqq d(A). \tag{2}$$

If the double limit

$$\alpha = \lim_{n,k \to \infty} k^{-1}A_n(k) \tag{3}$$

exists, then (1) *implies*

$$d(A_n) \to d(A). \tag{4}$$

Proof. Suppose that $\rho(A_n, A) = k_n^{-1} + |\delta(A_n) - \delta(A)|$. For every $\epsilon > 0$ there is a k_0 such that $k_0^{-1}A(k_0) < d(A) + \epsilon$. Since by (1), $k_n \to \infty$, there is an N_0 such that $k_n > k_0$ for every $n > N_0$. Thus

$$d(A_n) \leqq k_0^{-1}A_n(k_0) = k_0^{-1}A(k_0) < d(A) + \epsilon$$

if $n > N_0$. This yields (2).

We now assume (3) and first prove

$$\alpha = \delta(A). \tag{5}$$

Now $\lim_{k\to\infty} k^{-1}A_n(k) = \delta(A_n)$ for all n and it follows that $\lim_{n,k\to\infty} k^{-1}A_n(k) = \delta(A)$ since the double limit exists and $\delta(A_n)\to\delta(A)$ [5]. Starting from (3) and (5) we now prove (4). Let $\epsilon > 0$. By (3) and (5) there is an N_1 and a K such that $k^{-1}A_n(k) > \delta(A) - \epsilon$ for every $n > N_1$, $k > K$. Since $k_n \to \infty$ by (1), we have $k_n > K$ for every $n > N_2 \geqq N_1$. Hence

$$k^{-1}A_n(k) = k^{-1}A(k) \geqq d(A)$$

for all $n > N_2$ and all $k \leqq K$ and

$$k^{-1}A_n(k) > \delta(A) - \epsilon \geqq d(A) - \epsilon$$

for all $n > N_2$ and all $k > K$. Thus $k^{-1}A_n(k) > d(A) - \epsilon$ for all $n > N_2$ and all k. Therefore $d(A_n) \geqq d(A) - \epsilon$ for all $n > N_2$. Combining this with (2) we obtain (4).

It is easy to construct examples to show that condition (1) is not sufficient to yield (4) in the preceding theorem. As an application of the first part of the theorem, consider the set $F = \{A \in S \mid d(A) = \delta(A)\}$. Let $\{A_n\}$ be a sequence of elements in F and suppose that $A_n \to A$. Then $\delta(A) = \lim \delta(A_n) = \lim d(A_n) \leqq d(A)$. Hence $A \in F$ and F is a closed subset of S.

3. Topological properties of S. The proofs of several of the following results depend on the simple fact that there exists a sequence A such that $\delta(A) = \alpha$ for $0 \leqq \alpha \leqq 1$.

THEOREM 4. *S is separable.*

Proof. For every rational number r choose a fixed sequence $\{a_{r,n}\}$ with the natural density r, $0 \leqq r \leqq 1$. Let $S_{r,p}$ denote the class of sequences of the form $\{a_1, \cdots, a_t, a_{r,p+1}, a_{r,p+2}, \cdots\}$. Obviously every sequence in $S_{r,p}$ has the natural density r. Each class $S_{r,p}$ consists of a finite number of sequences. The number of sequences contained in each $\bigcup_p S_{r,p}$ is countable. As the set of rational numbers is countable, $\bigcup_{r,p} S_{r,p}$ is a countable union of countable sets and hence also countable. It remains to show that $\bigcup_{r,p} S_{r,p}$ is everywhere dense in S. Let $A = \{a_n\} \in S$. To every $\epsilon > 0$ there corresponds an r such that $|\delta(A) - r| < \epsilon/2$. Choose $k > 2/\epsilon$. If p is large enough, $S_{r,p}$ will contain the sequence $B = \{a_1, \cdots, a_k, a_{r,p+1}, a_{r,p+2}, \cdots\}$. Then

$$\rho(A, B) \leqq k^{-1} + |\delta(A) - \delta(B)| < (\epsilon/2) + (\epsilon/2) = \epsilon.$$

It can easily be shown that the set of all sequences which differ from a finite union of mutually disjoint arithmetic progressions in a finite number of terms is a countable everywhere-dense subset of S.

THEOREM 5. *S is nowhere locally compact.*

Proof. Let $A = \{a_n\} \in S$ and $N = \{B \in S \mid \rho(A, B) < \epsilon\}$, where $0 < \epsilon < 1$. Choose $B = \{b_n\} \in S$ such that $|\delta(A) - \delta(B)| = \epsilon/2$ and if n_k is the smallest value of n such that $a_k < b_n$ put $A_k = \{a_1, \cdots, a_k, b_{n_k}, b_{n_k+1}, \cdots\}$. Thus $\delta(A_k) = \delta(B)$ and $\{A_k\}$ is a Cauchy sequence. Since

$$\rho(A_k, A) \leqq k^{-1} + |\delta(A_k) - \delta(A)| = k^{-1} + \epsilon/2,$$

we have $A_k \in N$ for every $k > 2/\epsilon$. Since $\rho(A_k, A) > |\delta(A_k) - \delta(A)| = \epsilon/2$, A is not a limit point of $\{A_k\}$. As $\{A_k\}$ cannot have limit points different from A, $\{A_k\}$ has no limit points. Now if E is any open set containing A, then $N \subset E \subset \overline{E}$ if ϵ is small enough. Hence $\overline{E}$ is not sequentially compact and therefore not compact.

COROLLARY. *S is neither complete nor sequentially compact.*

THEOREM 6. *S is totally disconnected.*

Proof. Let E be any subset of S containing at least two elements. Now let E_1 denote the set of all sequences in E which agree with $A^* \in E$ in the first k or more places and let $E_2 = E - E_1$. It is clear that $A^* \in E_1$ for all k and that E_2 is nonempty for k sufficiently large. Also, $\rho(A_1, A_2) \geqq k^{-1}$, where $A_1 \in E_1$, $A_2 \in E_2$, and it follows that $\rho(E_1, E_2) \geqq k^{-1}$. But this implies that $\rho(\overline{E}_1, \overline{E}_2) \geqq k^{-1}$. Thus $E = E_1 \cup E_2$, where E_1 and E_2 are nonempty and satisfy the Hausdorff-Lennes condition. Hence E is not connected and S is totally disconnected.

COROLLARY. *Given any positive integer n, there exist sets E_1, E_2 such that $S = E_1 \cup E_2$ and $\rho(E_1, E_2) = n^{-1}$.*

THEOREM 7. *If $S = E_1 \cup E_2$, then $\rho(E_1, E_2) = 0$ or n^{-1}, where n is a positive integer.*

Proof. Suppose that $S = E_1 \cup E_2$ and let $d = \rho(E_1, E_2) > 0$. Then, given $\epsilon_n > 0$, there exists $A_n^1 \in E_1$ and $A_n^2 \in E_2$ such that

$$\rho(A_n^1, A_n^2) = k_n^{-1} + |\delta(A_n^1) - \delta(A_n^2)| < d + \epsilon_n.$$

Now let $\lambda_n = \frac{1}{2}|\delta(A_n^1) - \delta(A_n^2)|$ and choose a sequence A_n which agrees with A_n^1 and A_n^2 in the first $k_n - 1$ places but is different from both in the k_nth place and such that $\delta(A_n) = \frac{1}{2}\{\delta(A_n^1) + \delta(A_n^2)\}$. Then $\rho(A_n, A_n^1) = \rho(A_n, A_n^2) = \rho(A_n^1, A_n^2) - \lambda_n < d + \epsilon_n - \lambda_n$. Now let $\epsilon_n \to 0$ and suppose that zero is not a limit point of $\{\lambda_n\}$. Then $\rho(A_n, A_n^1) = \rho(A_n, A_n^2) < d$ for a sufficiently large value of n and it follows that $A_n \notin E_1 \cup E_2$ for this n and this is a contradiction. Now suppose that zero is a limit point of $\{\lambda_n\}$. Then there exists a subsequence $\{\lambda_{n_i}\}$ such that $\lambda_{n_i} \to 0$ and it follows that $k_{n_i}^{-1} \to d$. But this can happen only if $d = 0$ or if all but a finite number of $k_{n_i}^{-1}$ are equal to d.

Another problem that arises in connection with the space S is that of determining its dimension. As a consequence of Theorem 4 most of the well-known results of dimension theory could be used for this purpose. Also, in view of Theorem 5, it does not follow from Theorem 6 that S is zero-dimensional although it may seem that this should be the correct result. It will follow from the results of the next section that there exists a continuous one-one mapping of S onto a zero-dimensional subset of the unit interval.

4. Structure of S. Let $a=0\cdot\alpha_1\alpha_2\cdots\alpha_n\cdots$, where $\alpha_n=1$ if $n\in A$ and $\alpha_n=0$ otherwise. Also, let R denote the set of all real numbers (in the binary scale) in the interval $(0, 1]$ for which

$$(*)\qquad \lim_{n\to\infty} n^{-1}\sum_{i=1}^{n}\alpha_i$$

exists. Let the topology in R be determined by the metric $\rho(a, b)=0$ if $a=b$ and

$$\rho(a, b) = k^{-1} + \lim_{n\to\infty} n^{-1}\left|\sum_{i=1}^{n}(\alpha_i-\beta_i)\right|$$

otherwise, where $k=\max\left\{\sum_{i=1}^{m}\alpha_i, \sum_{i=1}^{m}\beta_i\right\}$ and m is the smallest value of n such that $\alpha_n\neq\beta_n$. It is clear that $\delta(A)$ exists if and only if the limit $(*)$ exists and in this case the two are equal. Since all sequences are infinite, the mapping $\pi: A\to a$ establishes a one-one correspondence between S and R and it is easy to see that π is an isometry. It follows from the remark following Theorem 4 that the rational numbers are everywhere dense in R. It is also clear that π is a continuous mapping of S into the unit interval and that R is a zero-dimensional subset of the unit interval.

Since the limit $(*)$ is equal to $\frac{1}{2}$ if and only if a is simply normal in the binary scale, it follows that R includes all such numbers and that all sequences corresponding to such numbers have natural density $\frac{1}{2}$. As a consequence of the fact that almost all numbers are normal [6, 7], we can conclude that $mR=1$ and that every sequence has a natural density $\frac{1}{2}$ with probability 1. The following results characterize to a certain extent the structure of the open and closed sets of S.

THEOREM 8. *All open and closed sets of R are measurable subsets of the unit interval.*

Proof. Let $S_k(A_0)$ be the set of all $A\in S$ which agree with A_0 in the first $k-1$ or more places; let

$$S_k(A_0, \gamma) = \left\{A\in S \,\middle|\, |\delta(A)-\delta(A_0)| < \gamma - k^{-1}\right\}$$

and

$$S(A_0, \gamma) = \left\{A\in S \,\middle|\, \rho(A, A_0) < \gamma\right\}.$$

Then

$$S(A_0, \gamma) = \bigcup_{k>1/\gamma}\left\{S_k(A_0)\cap S_k(A_0, \gamma)\right\}.$$

Now $\pi\{S_k(A_0)\}$ is an interval and $\pi\{S_k(A_0, \gamma)\}$ has measure zero or one. It follows that all the open spheres in R are measurable and, since R is separable, so are all open sets. Also, all closed sets of R are measurable since they are complements of open sets with respect to R and $mR=1$.

COROLLARY. *A subset E of the unit interval is measurable if $E\cap R$ is an open or closed subset of R.*

The proof of the next theorem is obtained by introducing suitable refinements in the proof of the preceding one and is therefore omitted.

THEOREM 9. *If E is an open (closed) set in R, then there exists an open (closed) subset E^* of the unit interval such that $m(E\cup E^*-E\cap E^*)=0$.*

5. Multiplication of sequences. Finally, we shall consider briefly a certain binary operation on sequences which we shall call multiplication. If $A=\{a_n\}$, $B=\{b_n\}$, then define the product $AB=\{b_{a_n}\}$. This definition was introduced by Niven [**3**], who also proved that $\delta(AB)$ exists and that $\delta(AB)=\delta(A)\delta(B)$ whenever $\delta(A)$ and $\delta(B)$ exist. This result makes it possible to show that S is a topological structure of a type which has been studied rather extensively by Wallace [**4**] and others.

THEOREM 10. *S is a mob.*

Proof. It is obvious that the set of all sequences is a semigroup (with an identity and right cancellation). But S is closed under multiplication (by Niven's theorem) and is therefore a semigroup also. Since all metric spaces are Hausdorff spaces, all that remains is to show that multiplication is continuous in the topology of S. Let $\rho(A_n, A)\to 0$, $\rho(B_n, B)\to 0$, where $A=\{a_i\}$, $A_n=\{a_{n,i}\}$, $B=\{b_i\}$ and $B_n=\{b_{n,i}\}$. Then

$$\begin{aligned}\rho(A_nB_n, AB) &= k_n^{-1}+\left|\delta(A_nB_n)-\delta(AB)\right| \\ &= k_n^{-1}+\left|\delta(A_n)\delta(B_n)-\delta(A)\delta(B)\right|\to 0,\end{aligned}$$

where k_n is the smallest value of k such that $b_{n,a_{n,k}}\neq b_{a_k}$.

It has also been shown by Niven [**3**] that $d(AB)\geqq d(A)d(B)$. Suppose that $d(A)=\delta(A)$, $d(B)=\delta(B)$. Then $\delta(AB)=\delta(A)\delta(B)=d(A)d(B)\leqq d(AB)$ and consequently $d(AB)=\delta(AB)$, *i.e.*, the set $F=\{A\in S\mid d(A)=\delta(A)\}$ is closed under multiplication.

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ON THE NUMERICAL SOLUTION OF DIRICHLET-TYPE PROBLEMS

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1. Introduction. A study of the solution of

$$\frac{\partial^2 u}{\partial z^2}+\frac{\partial^2 u}{\partial \rho^2}-(K/\rho)\frac{\partial u}{\partial \rho}\equiv u_{zz}+u_{\rho\rho}-(K/\rho)u_\rho=0,\ \rho\neq 0 \tag{1.1}$$

for K a fixed, real, nonzero constant is of considerable interest. Applications occur in potential theory for $K=-1$, in fluid dynamics and theory of heat flow for $K=1$, and in certain problems of stress for $K=3,\ -3,\ -5$ (see [1]).

The problem to be considered is a Dirichlet-type problem. In the $z\rho$-plane, let G be a closed, bounded, simply-connected plane region whose interior is denoted by R and whose boundary curve is denoted by S. Let G not contain any point where $\rho=0$. Let $g(z,\rho)$ be defined and continuous on S. The problem, then, is to produce a function $u(z,\rho)$ such that for fixed K

(a) $$u(z,\rho)\equiv g(z,\rho) \text{ on } S$$

and

(b) $$u(z,\rho) \text{ satisfies equation (1.1) in } G.$$

Under quite general conditions, there exists a unique solution and only such cases will be considered [2], [3]. However, the analytical determination of $u(z,\rho)$ is quite another story from that of its existence and usually offers what are, at present, insurmountable problems. The approach here, then, will be from a numerical analysis point of view and will be general enough to apply for arbitrary, but fixed, nonzero K and will utilize the rectangular type grid.

2. The numerical method. Let h and d be fixed positive constants and let (z_0,ρ_0) be an arbitrary, but fixed, point of G. Denote by G_h the set of all points of the form $(z_0+mh,\ \rho_0+nd)$ contained in G, where m and n are integers. Two points (z_1,ρ_1) and (z_2,ρ_2) of G_h are called adjacent if and only if the straight line segment joining them is contained in G and

(a) $$(z_1-z_2)^2+(\rho_1-\rho_2)^2=h^2 \text{ and } \rho_1=\rho_2$$

or

(b) $$(z_1-z_2)^2+(\rho_1-\rho_2)^2=d^2 \text{ and } z_1=z_2.$$

The interior of G_h, denoted by R_h, is the set of all points of G_h which have four adjacent points in G_h. The boundary of G_h, denoted by S_h, is the set defined by $G_h=R_h\cup S_h$, $R_h\cap S_h=\theta$.

Suppose G_h consists of n points. Number these in a one-to-one fashion with the integers $1, \cdots, n$. Denote the coordinates of the point numbered k by (z_k, ρ_k) and the unknown function u at (z_k, ρ_k) by $u(z_k, \rho_k) \equiv u_k$, for $k = 1, \cdots, n$.

Let (z_i, ρ_i) be an arbitrary point of S_h, the lattice boundary. Approximate u_i by $g(z', \rho')$, where (z', ρ') is the nearest point of S to (z_i, ρ_i). If (z', ρ') is not unique, choose any one of the set of nearest points and use it. The problem of finding numerical approximations to $u(z, \rho)$ on S_h is, though crudely done, adequate for present purposes.

We require then at each point (z_i, ρ_i) of R_h, that u satisfy

$$\begin{aligned}(2.1)\qquad & -2(1 + p^{-2})u(z_i, \rho_i) + p^{-2}[u(z_i + h, \rho_i) + u(z_i - h, \rho_i)] \\ & + \left(1 - \frac{dK}{2\rho}\right)u(z_i, \rho_i + d) + \left(1 + \frac{dK}{2\rho}\right)u(z_i, \rho_i - d) = 0\end{aligned}$$

where $h = pd$. Application of equation (2.1) to each point of R_h yields a system of linear algebraic equations which, when solved, yields the remaining numerical approximations of the analytical solution.

3. Derivation of difference analogue (2.1). We seek to approximate a solution $u(z, \rho)$ which is of class C^4 and use the notation

$$(3.1)\qquad i!j!A_{i,j} \equiv u_{i,j}; \qquad u_{i,j} \equiv \frac{\partial^{i+j}u}{\partial z^i \partial \rho^j} \cdot$$

Let (z, ρ) be an arbitrary point of R_h. The aim will be to develop a "5-point" analogue, so consider (z, ρ) and its four adjacent points. Let (z, ρ), $(z+h, \rho)$, $(z, \rho+d)$, $(z-h, \rho)$, $(z, \rho-d)$ be denoted, respectively, by 0, 1, 2, 3, 4. Hence, $(z_0, \rho_0) = (z, \rho)$, $(z_1, \rho_1) = (z+h, \rho)$, $(z_2, \rho_2) = (z, \rho+d)$, $(z_3, \rho_3) = (z-h, \rho)$, $(z_4, \rho_4) = (z, \rho-d)$. Also let

$$(3.2)\qquad h = pd, h > 0, d > 0, p > 0.$$

To produce a difference equation which approximates (1.1), let

$$(3.3)\qquad L(u) \equiv \sum_0^4 \alpha_i u_i = 0.$$

By use of equation (3.2), the definition of $A_{i,j}$ in (3.1), and the Taylor expansions of u_1, u_2, u_3, u_4 one may rewrite equation (3.3) as

$$\begin{aligned}L(u) \equiv\ & A_{0,0}(\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) + d[A_{1,0}p(\alpha_1 - \alpha_3) + A_{0,1}(\alpha_2 - \alpha_4)] \\ (3.4)\qquad & + d^2[A_{2,0}p^2(\alpha_1 + \alpha_3) + A_{0,2}(\alpha_2 + \alpha_4)] \\ & + d^3[A_{3,0}p^3(\alpha_1 - \alpha_3) + A_{0,3}(\alpha_2 - \alpha_4)] + O(d^4) = 0.\end{aligned}$$

Equation (1.1) may be rewritten as: $A_{2,0} = KA_{0,1}/(2\rho) - A_{0,2}$. Substitution of this latter expression for $A_{2,0}$ in equation (3.4) yields:

$$
\begin{aligned}
L(u) \equiv\; & A_{0,0}(\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4) \\
& + d\left\{A_{1,0}p(\alpha_1 - \alpha_3) + A_{0,1}\left[\alpha_2 - \alpha_4 + (\alpha_1 + \alpha_3)\frac{Kp^2 d}{2\rho}\right]\right\} \\
& + d^2 A_{0,2}[\alpha_2 + \alpha_4 - p^2(\alpha_1 + \alpha_3)] \\
& + d^3[A_{3,0}p^3(\alpha_1 - \alpha_3) + A_{0,3}(\alpha_2 - \alpha_4)] + O(d^4) = 0.
\end{aligned} \tag{3.5}
$$

Let

$$
\begin{aligned}
\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 &= \epsilon_1 = O(d^4), \\
\alpha_2 - \alpha_4 + (\alpha_1 + \alpha_3)\frac{Kp^2 d}{2\rho} &= \epsilon_2 = O(d^3), \\
\alpha_1 - \alpha_3 &= \epsilon_3 = O(d^3), \\
\alpha_2 + \alpha_4 - p^2(\alpha_1 + \alpha_3) &= \epsilon_4 = O(d^2), \\
\alpha_2 - \alpha_4 &= \epsilon_5 = O(d).
\end{aligned}
$$

The solution of this linear system is

$$
\begin{aligned}
\alpha_1 &= \epsilon_3/2 + \rho\epsilon_2/(Kp^2 d) - \rho\epsilon_5/(Kp^2 d), \\
\alpha_2 &= \epsilon_4/2 + \epsilon_5/2 + \rho\epsilon_2/(Kd) - \rho\epsilon_5/(Kd), \\
\alpha_3 &= -\epsilon_3/2 + \rho\epsilon_2/(Kp^2 d) - \rho\epsilon_5/(Kp^2 d), \\
\alpha_4 &= \epsilon_4/2 - \epsilon_5/2 + \rho\epsilon_2/(Kd) - \rho\epsilon_5/(Kd), \\
\alpha_0 &= \epsilon_1 - 2\rho\epsilon_2/(Kp^2 d) + 2\rho\epsilon_5/(Kp^2 d) - \epsilon_4 - 2\rho\epsilon_2/(Kd) + 2\rho\epsilon_5/(Kd).
\end{aligned}
$$

Substitution of these values into equation (3.3) yields the general difference analogue

$$
\begin{aligned}
& [\epsilon_1 - \epsilon_4 - 2\rho\epsilon_2/(Kp^2 d) + 2\rho\epsilon_5/(Kp^2 d) - 2\rho\epsilon_2/(Kd) + 2\rho\epsilon_5/(Kd)]u_0 \\
& \quad + [\epsilon_3/2 + \rho\epsilon_2/(Kp^2 d) - \rho\epsilon_5/(Kp^2 d)]u_1 \\
& \quad + [\epsilon_4/2 + \epsilon_5/2 + \rho\epsilon_2/(Kd) - \rho\epsilon_5/(Kd)]u_2 \\
& \quad + [-\epsilon_3/2 + \rho\epsilon_2/(Kp^2 d) - \rho\epsilon_5/(Kp^2 d)]u_3 \\
& \quad + [\epsilon_4/2 - \epsilon_5/2 + \rho\epsilon_2/(Kd) - \rho\epsilon_5/(Kd)]u_4 = 0.
\end{aligned} \tag{3.6}
$$

In order to obtain equation (2.1) from equation (3.6), let $\epsilon_1 = \epsilon_2 = \epsilon_3 = \epsilon_4 = 0$, $\epsilon_5 = -dK$, divide through by ρ, and note that $u_0 = u(z, \rho)$, $u_1 = u(z+h, \rho)$, $u_2 = u(z, \rho+d)$, $u_3 = u(z-h, \rho)$, $u_4 = u(z, \rho-d)$.

It is also now quite apparent, from equation (3.6), why the discussion here does not yield the well-known results for $K=0$ as a special case.

4. Theorems on the numerical method of Section 2. In this section let $\bar{\rho} = GLB_{\rho\in G}|\rho|$. By definition of G, $\bar{\rho} > 0$. Let $u(z, \rho)$ be the solution of the Dirichlet type problem being considered, and let $U(z, \rho)$ be the solution of the numerical method described in Section 2. This latter assumption means that if

(z, ρ) is a point of R_h and $U_0 = U(z, \rho)$, $U_1 = U(z+h, \rho)$, $U_2 = U(z, \rho+d)$, $U_3 = U(z-h, \rho)$, $U_4 = U(z, \rho-d)$, then

$$(4.1) \quad -2(1 + p^{-2})U_0 + p^{-2}(U_1 + U_3) + \left(1 - \frac{dK}{2\rho}\right)U_2 + \left(1 + \frac{dK}{2\rho}\right)U_4 = 0.$$

THEOREM 1. *The solution of the system of linear equations which results by application of the method of Section* 2 *is unique if* $0 < d < 2\bar{\rho}/|K|$.

Proof. It is sufficient to show that the determinant of the system of linear equations is not zero and this is done by demonstrating that the only solution of the homogeneous system which results by considering $g(z, \rho) \equiv 0$ on S is the zero vector. Suppose then there exists a nontrivial solution for the homogeneous system. For some point of R_h, $U \neq 0$. Suppose $U > 0$. Let the largest value occur at some point, say, (z_0, ρ_0). Then

$$(4.2) \quad 2(1 + p^{-2})U_0 = p^{-2}(U_1 + U_3) + \left(1 - \frac{dK}{2\rho}\right)U_2 + \left(1 + \frac{dK}{2\rho}\right)$$

and

$$(4.3) \qquad M = U_0 \geqq U_i, \qquad i = 1, 2, 3, 4.$$

Case 1. Suppose $U_1 = U_2 = U_3 = U_4$. Then (4.2) becomes

$$(4.4) \qquad 2(1 + p^{-2})U_0 = 2(1 + p^{-2})U_1.$$

Hence $U_0 = U_1 = U_2 = U_3 = U_4 = M$.

Case 2. Suppose not all of U_1, U_2, U_3, U_4 are equal. Then one is a maximum If U_1 is maximum, let $U_2 = U_1 - k_2$, $U_3 = U_1 - k_3$, $U_4 = U_1 - k_4$, where k_2, k_3, k_4 are nonnegative and at least one is positive. Then (4.2) becomes

$$(4.5) \quad 2(1 + p^{-2})U_0 = 2(1 + p^{-2})U_1 - k_3p^{-2} - k_2\left(1 - \frac{dK}{2\rho}\right) - k_4\left(1 + \frac{dK}{2\rho}\right).$$

However, since $d < 2\bar{\rho}/|K|$, it follows that $(1 - dK/2\rho)$ and $(1 + dK/2\rho)$ are positive. Hence from (4.5) it follows that $2(1+p^{-2})U_0 < 2(1+p^{-2})U_1$ or, equivalently, that $U_0 < U_1$, which contradicts (4.3). Hence $U_0 = U_1 = U_2 = U_3 = U_4 = M$.

If U_2 or U_3 or U_4 is selected as the maximum, analogous reasoning yields the same result. So, Case 2 is not possible.

Continuing in an analogous fashion, one may show in a finite number of steps that U at a boundary point has the value M. But this contradicts the fact that U at every point of S_h is zero, by the method of approximation for points of S_h. Hence the theorem is proved.

Note that Theorem 1 establishes a practical, sufficient condition which assures a unique numerical solution. In all that follows, then, it is assumed that $d < 2\bar{\rho}/|K|$.

We proceed now by generalizing methods, ideas, and results of Gerschgorin [4].

DEFINITION. Let

$$L[u(z,\rho)] \equiv d^{-2}[-2(1+p^{-2})u(z,\rho)+p^{-2}u(z+h,\rho)+p^{-2}u(z-h,\rho) + \left(1-\frac{dK}{2\rho}\right)u(z,\rho+d)+\left(1+\frac{dK}{2\rho}\right)u(z,\rho-d)\Big].$$

LEMMA 1. *If* $L[v]\leqq 0$ *on* R_h *and* $v\geqq 0$ *on* S_h, *then* $v\geqq 0$ *on* R_h.

The proof follows immediately as in [3] or [5].

LEMMA 2. *If* $-|L[v_1]|\geqq L[v_2]$ *in* R_h *and* $|v_1|\leqq v_2$ *on* S_h, *then* $|v_1|\leqq v_2$ *on* R_h.

The proof follows immediately as in [3] or [5].

LEMMA 3. *If* $|L[v]|\leqq A$ *on* R_h, $|v|\leqq B$ *on* S_h *and* r *is the radius of any circle of the kind described below, which contains* G, *then* $|v|\leqq Ar^2/4+B$, *on* R_h.

Proof. By assumption, G is closed, bounded, simply-connected, and contains no point of the form $(z, 0)$. Hence, it follows that for all (z,ρ) in G, ρ has the same sign and also has a finite maximum and a finite minimum in G. Select, now, a circle $(z-a)^2+(\rho-b)^2=r^2$ which contains G, such that b satisfies the inequality

$$1-\frac{K}{2\rho}(\rho-b)\geqq 1 \tag{4.6}$$

for all ρ in G. At least one such circle exists since G is bounded, since $\rho\neq 0$, and since b may be readily selected as follows:

(a) choose $b\geqq\rho^*$; if ρ^* and K are of the same sign, where $\rho^*=\max_{\rho\in G}\rho$,
(b) choose $b\leqq\bar{\rho}$; if $\bar{\rho}$ and K are of different sign, where, $\bar{\rho}=\min_{\rho\in G}\rho$.

Now let

$$w(z,\rho)=\left[\frac{Ar^2}{4}\left\{1-\frac{(z-a)^2+(\rho-b)^2}{r^2}\right\}+B\right].$$

Direct calculation yields $L[w(z,\rho)]=-A[1-(K/2\rho)(\rho-b)]$. However, since $A\geqq 0$, by assumption, and by (4.6), it follows that $L[w(z,\rho)]\leqq -A$. Since, then, $|L[v]|\leqq A$ on R_h, by assumption, it follows that, on R_h, $-|L[v]|\geqq L[w]$. Moreover, since $|v|\leqq B$ on S_h and $w\geqq B$ on S_h, $w\geqq|v|$. Hence $-|L[v]|\geqq L[w]$ on R_h and $w\geqq|v|$ on S_h. By Lemma 2, $w\geqq|v|$ on R_h, or $|v|\leqq w\leqq Ar^2/4+B$, on R_h, which proves the lemma.

THEOREM 2. *If* $u(z,\rho)$ *is of class* C^4, $\rho\neq 0$ *in* G, u *denotes the solution of the Dirichlet type problem described in Section* 1, U *denotes the solution of the linear*

system which results by application of the method of Section 2, *and* $d<2\bar{\rho}/|K|$, *then, on* R_h,

$$|U-u| \leqq (r^2/4)[M_4h^2/12 + M_4d^2/12 + M_3d^2|K|/(6\bar{\rho}) + M_4d^3|K|/(24\bar{\rho})] + M_1(h+d), \tag{4.7}$$

where

$$M_k = \max_{i,j}\left[\max_{(z,\rho)\in G} |u_{i,j}|\right], \qquad i+j=k;\ i, j \textit{ nonnegative integers},$$

$\bar{\rho}=GLB_{\rho\in G}|\rho|$, *and r is the radius of any circle of type described in the proof of Lemma* 3.

Proof. Let $P=L[u]-(u_{zz}+u_{\rho\rho}-(K/\rho)u_\rho)$. Substitution of finite Taylor series expansions for $u(z+h,\rho)$, $u(z-h,\rho)$, $u(z,\rho+d)$, $u(z,\rho-d)$ in $L[u]$, yields

$$P = \frac{h^2}{4!}u_{4,0}(\beta_1,\rho) + \frac{h^2}{4!}u_{4,0}(\beta_2,\rho) + \frac{d^2}{4!}u_{0,4}(z,\gamma_1) + \frac{d^2}{4!}u_{0,4}(z,\gamma_2) - \frac{d^2Ku_{0,3}(z,\rho)}{6\rho} + \frac{Kd^3u_{0,4}(z,\gamma_2)}{(2\rho)4!} - \frac{Kd^3u_{0,4}(z,\gamma_1)}{(2\rho)4!}.$$

Hence $|P| \leqq M_4h^2/12+M_4d^2/12+M_3d^2|K|/(6\bar{\rho})+|K|M_4d^3/(4!\bar{\rho})$.

Now, since $u_{zz}+u_{\rho\rho}-(K/\rho)u_\rho=0$, then

$$\begin{aligned}|L[u]-(u_{zz}+u_{\rho\rho}-(K/\rho)u_\rho)| &= |L[u]| = |P| \\ &\leqq M_4h^2/12 + M_4d^2/12 + M_3d^2|K|/(6\bar{\rho}) + M_4d^3|K|/(24\bar{\rho}).\end{aligned} \tag{4.8}$$

Also, for any point of S_h, U was selected as the value of $g(z',\rho')$ at some nearest point (z',ρ') on S, and $g(z',\rho')=u(z',\rho')$ on S. Hence for any point (z,ρ) of S_h

$$|U(z,\rho)-u(z,\rho)| = |g(z',\rho')-u(z,\rho)| = |u(z',\rho')-u(z,\rho)| \tag{4.9}$$

where (z',ρ') is a point of S and

$$(z-z')^2+(\rho-\rho')^2 \leqq h^2 \quad \text{or} \quad (z-z')^2+(\rho-\rho')^2 \leqq d^2.$$

Therefore

$$\begin{aligned}|U(z,\rho)-u(z,\rho)| &= |u(z',\rho')-u(z,\rho')+u(z,\rho')-u(z,\rho)| \\ &\leqq |u_z(\delta_1,\rho')(z-z')| + |u_\rho(z,\delta_2)(\rho-\rho')| \\ &\leqq M_1h + M_1d = M_1(h+d).\end{aligned} \tag{4.10}$$

It must also be noted that $L[U]=0$, by equation (4.1). Hence

$$|L[u-U]| = |L[u]-L[U]| = |L[u]|. \tag{4.11}$$

Applying Lemma 3 to equations (4.8), (4.10) and (4.11), one finds that on R_h $|U-u| \leqq (r^2/4)[M_4h^2/12 + M_4d^2/12 + M_3d^2|K|/(6\bar{\rho}) + M_4d^3|K|/(24\bar{\rho})] + M_1(h+d)$, which proves the theorem.

THEOREM 3. *Under the conditions of Theorem 2, the numerical solution U converges to the analytical solution u as $(h^2+d^2)\to 0$.*

The proof follows directly from inequality (4.7) and from the method described for approximating $u(z, \rho)$ on S_h.

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MATHEMATICAL NOTES

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ON THE TURAN INEQUALITY FOR CERTAIN POLYNOMIALS

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1. Let us define for the sequence of functions $\{f_n(x)\}$ the expression

$$\Delta_n(f) = f_n^2(x) - f_{n+1}(x)f_{n-1}(x) \qquad (n \geqq 1).$$

Szegö [4] proved that $\Delta_n(P) \geqq 0$ $(-1 \leqq x \leqq 1;\, n \geqq 1)$, where $P_n(x)$ is the Legendre polynomial, and indicated that the same inequality holds for the Hermite, Laguerre, and ultraspherical polynomials. The latter implies that the derivatives of the Legendre polynomials also satisfy the Turan inequality [2].

We shall prove here that the derivatives of a certain class of polynomials satisfy the Turan inequality. This class includes the Hermite and the Tchebycheff polynomials as special cases. Only the knowledge of the recurrence relation shall be required. We shall also indicate that the derivatives of the Laguerre polynomials have this property.

2. Let us prove first the following:

LEMMA. *If $\{p_n(x)\}$ are the polynomials defined by*

$$(2.1) \qquad p_{n+1}(x) = (A_n x + B_n)p_n(x) - C_n p_{n-1}(x) \qquad (n \geqq 1),$$

where $p_0(x)=1$, $p_1(x)=A_0x+B_0$, $C_n>0$ *and* $A_n>0$, *then*

$$\lambda_{n+1,k}(x) = p_{n+1}^{(k)}(x)p_n^{(k-1)}(x) - p_{n+1}^{(k-1)}(x)p_n^{(k)}(x) > 0$$
$$(k = 1, 2, 3, \cdots, n+1; n \geqq 1),$$

where $p_n^{(k)}(x)$ *indicates the kth derivative of* $p_n(x)$.

Proof. If we differentiate (2.1) k times, we can see easily that

$$\frac{p_{n+1}^{(k)}(x)p_n^{(k)}(y) - p_{n+1}^{(k)}(y)p_n^{(k)}(x)}{x-y}$$
$$= A_np_n^{(k)}(x)p_n^{(k)}(y) + kA_n\frac{p_n^{(k-1)}(x)p_n^{(k)}(y) - p_n^{(k-1)}(y)p_n^{(k)}(x)}{x-y}$$
$$+ C_n\frac{p_n^{(k)}(x)p_{n-1}^{(k)}(y) - p_n^{(k)}(y)p_{n-1}^{(k)}(x)}{x-y}.$$

Taking the limit as $x \to y$, we get

$$\lambda_{n+1,k+1}(x) = A_n[p_n^{(k)}(x)]^2 + C_n\lambda_{n,k+1}(x) + kA_n\Gamma_{n,k}(x), \tag{2.2}$$

where $\Gamma_{n,k}(x) = [p_n^{(k)}(x)]^2 - p_n^{(k+1)}(x)p_n^{(k-1)}(x)$.

Now for any sequence of polynomials $\{f_n(x)\}$ whose zeros are all real and simple we have

$$[f_n'(x)]^2 - f_n(x)f_n''(x) > 0 \qquad (-\infty < x < \infty; n \geqq 1).$$

But since $p_n(x)$ has all its zeros real ([3], p. 44) then the zeros of $p_n^{(k-1)}(x)$ are all real and consequently

$$\Gamma_{n,k}(x) > 0 \qquad (-\infty < x < \infty; k = 1, \cdots, n; n \geqq 1). \tag{2.3}$$

We also note that $\lambda_{k-1,k}(x) = 0$, and

$$\lambda_{k,k}(x) = p_k^{(k)}(x)p_{k-1}^{(k-1)}(x) = k!(k-1)!\prod_0^{k-1} A_r \prod_0^{k-2} A_s > 0. \tag{2.4}$$

Thus it follows by induction from (2.2), (2.3), and (2.4) that

$$\lambda_{n,k}(x) > 0 \qquad (-\infty < x < \infty; k = 1, \cdots, n, n \geqq 1).$$

This completes the proof of the lemma.

3. Now consider the polynomials defined by

$$\begin{aligned} p_{n+1}(x) &= xA_np_n(x) - C_np_{n-1}(x) && (n \geqq 1), \\ p_0(x) &= 1, \qquad p_1(x) = A_0x, && A_n \neq 0. \end{aligned} \tag{3.1}$$

If we differentiate (3.1) k times, we get, leaving out simple calculations,

$$\begin{aligned}\Delta_n(p^{(k)}) &= C_{n-1}\Delta_{n-1}(p^{(k)}) + kA_{n-1}\lambda_{n,k}(x) + (C_n - C_{n-1})[p_{n-1}^{(k)}(x)]^2 \\ &\quad + (A_{n-1} - A_n)[kp_n^{(k-1)}(x) + xp_n^{(k)}(x)]p_{n-1}^{(k)}(x) \\ &= C_{n-1}\Delta_{n-1}(p^{(k)}) + (C_n - C_{n-1})[p_{n-1}^{(k)}(x)]^2 + kA_{n-1}\lambda_{n,k}(x) \\ &\quad + \left(\frac{A_{n-1}}{A_n} - 1\right)p_{n-1}^{(k)}(x)[p_{n+1}^{(k)}(x) + C_n p_{n-1}^{(k)}(x)] \\ &= C_{n-1}\Delta_{n-1}(p^{(k)}) + \left(\frac{A_{n-1}C_n - A_nC_{n-1}}{A_n}\right)[p_{n-1}^{(k)}(x)]^2 \\ &\quad + kA_{n-1}\lambda_{n,k}(x) + \left(1 - \frac{A_{n-1}}{A_n}\right)\Delta_n(p^{(k)}) + \left(\frac{A_{n-1}}{A_n} - 1\right)[p_n^{(k)}(x)]^2.\end{aligned}$$

Hence

$$\begin{aligned}\frac{A_{n-1}}{A_n}\Delta_n(p^{(k)}) &= C_{n-1}\Delta_{n-1}(p^{(k)}) + kA_{n-1}\lambda_{n,k}(x) \\ &\quad + \left(\frac{A_{n-1}C_n - A_nC_{n-1}}{A_n}\right)[p_{n-1}^{(k)}(x)]^2 + \left(\frac{A_{n-1}}{A_n} - 1\right)[p_n^{(k)}(x)]^2.\end{aligned} \tag{3.2}$$

THEOREM. *Let* $p_n(x)$ *be defined by* (3.1) *with the added conditions that* $C_n > 0$, $A_n > 0$, $C_nA_{n-1} - C_{n-1}A_n \geqq 0$, $A_n \leqq A_{n-1}$. *Then*

$$\Delta_n(p^{(k)}) \geqq \left\{k!\prod_{r=0}^{k-1} A_r\right\}^2 \cdot \{C_kC_{k-1}\cdots C_{n-1}\}\{A_n/A_k\} \tag{3.3}$$

$(k=0, 1, \cdots, n;\ n \geqq 1)$.

Proof. We begin by noting that

$$\begin{aligned}\Delta_k(p^{(k)}) = [p_k^{(k)}(x)]^2 &= \left\{k!\prod_{r=0}^{k-1} A_r\right\}^2 > 0 \qquad (k \geqq 1), \\ &= 1 \qquad (k = 1).\end{aligned}$$

It follows from (3.2) that if $G_{n,k} = \text{g.l.b. } \Delta_n(p^{(k)})$ then

$$G_{n,k} \geqq \frac{A_nC_{n-1}}{A_{n-1}} G_{n-1,k}.$$

Iteration of this shows that

$$G_{n,k} \geqq \{C_{n-1}C_{n-2}\cdots C_k\}\cdot\{A_n/A_k\}\cdot G_{k,k}.$$

Since $G_{k,k} = \{k!\prod_{r=0}^{k-1} A_r\}^2$ we obtain our desired result.

One interesting special case of this theorem is when $A_n = 1$. This case includes:

(i) the Hermite polynomials: $C_n = n$,

(ii) a class of generalized Hermite polynomials studied by Toscano [6]: $C_n = n+\beta$ where $\beta > -1$,

(iii) a class of polynomials studied by Carlitz [1] in connection with the special functional equation: $C_n = n^2$,

(iv) the Tchebycheff polynomials: $C_n = 1/2$.

The inequality (3) becomes for Case (i),

$$\Delta_n(H^{(k)}) \geqq (n-1)!(k!)k.$$

The corresponding inequality for other special cases can be easily computed.

We also remark here that although the derivatives of the Legendre polynomials satisfy the Turan inequality, the theorem we established above does not include this case.

4. The Laguerre polynomials are defined by the recurrence relation

$$(n+1)L_{n+1}^{(\alpha)}(x) = (-x+2n+\alpha+1)L_n^{(\alpha)}(x) - (n+\alpha+1)L_{n-1}^{(\alpha)}(x)$$

$$L_0^{(\alpha)}(x) = 1, \; L_1^{(\alpha)}(x) = (1+\alpha) - x.$$

They also satisfy

$$\frac{d}{dx} L_n^{(\alpha)}(x) = - L_{n-1}^{(\alpha+1)}(x)$$

from which we see that

$$\Delta_n\left(\frac{d^k}{dx^k} L^{(\alpha)}\right) = \Delta_{n-k}(L^{(\alpha+k)}).$$

By [5],

$$\Delta_n(L^{(\alpha)}) \geqq 0 \qquad (\alpha > -1; -\infty < x < \infty).$$

Hence we finally get

$$\Delta_n\left(\frac{d^k}{dx^k} L^{(\alpha)}\right) \geqq 0 \quad (\alpha > -k-1; -\infty < x < \infty).$$

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EXPECTATIONS FOR SUMS OF POWERS

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A sequence of independent random variables with a uniform distribution is chosen from the interval (0, 1). The process is continued until the sum of the nth powers of the chosen numbers exceeds 1. One problem that arises is to determine the expected number of such choices.

This problen has been solved by W. Weissblum [1] for $n=1$ in which case the expected number is e. In this note, we solve the general problem in terms of a definite integral and find, thereby, the asymptotic behavior as $n\to\infty$.

Heuristically, it might be reasoned, the "average" number is 1/2 and so the expectation should be around 2^n. This is not the case and in fact it turns out that the solution which we designate by E_n satisfies $E_n \sim cn$ where c is given below. This is not really surprising since the "average" nth power number is $\int_0^1 x^n dx = 1/(n+1)$.

E_n will be given by [2]

$$E_n = 1 + f_1 + f_2 + \cdots,$$

where f_i is the probability of failure up to and including the ith trial. Geometrically, f_i will be given by the volume enclosed by

$$x_1^n + x_2^n + \cdots + x_i^n \leqq 1, \qquad x_r \geqq 0 \ (r = 1, 2, \cdots i).$$

This volume is determined immediately by an application of Dirichlet's integral ([3], p. 258) and is given by $f_i = [\Gamma(1+1/n)^i]/[\Gamma(1+i/n)]$. It follows that

$$E_n = \sum_{i=0}^{\infty} \frac{\Gamma(1+1/n)^i}{\Gamma(1+i/n)}.$$

For $n=1$, we obtain Weissblum's result $E_1 = e$.

Now using the standard contour integral ([3], p. 245)

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi i}\int_c e^s s^{-z} ds$$

for $\Gamma(1+i/n)^{-1}$, shifting contours, extracting a residue, and rewriting as a real integral, we obtain for $x>0$ that

$$\sum_{i=0}^{\infty} \frac{x^i}{\Gamma(1+i/n)} = ne^{x^n} - \frac{x\sin(\pi/n)}{\pi/n}\int_0^{\infty} \frac{e^{-t^n}dt}{t^2+x^2-2tx\cos(\pi/n)}.$$

E_n is equal to the above expression for $x=\Gamma(1+1/n)$.

We now find E_n, asymptotically.

Using the relation $x=\Gamma(1+1/n)=e^{-\gamma/n+O(1/n^2)}$ in the above integral, we find that

* On leave of absence from Polytechnic Institute of Brooklyn.

$$\int_0^\infty \frac{e^{-t^n}dt}{t^2 + x^2 - 2tx\cos(\pi/n)} \sim \int_{-\infty}^\infty \frac{e^{-e^u}du}{(u+\gamma)^2 + \pi^2},$$

while $x^n \to e^{-\gamma}$. Consequently, $E_n \sim cn$, where

$$c = e^{e^{-\gamma}} - \int_{-\infty}^\infty \frac{e^{-e^u}du}{(u+\gamma)^2 + \pi^2}.$$

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3. E. T. Whittaker, G. N. Watson, A Course in Modern Analysis, Cambridge, 1946.

CLASSROOM NOTES

EDITED BY C. O. OAKLEY, Haverford College

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TAYLOR'S THEOREM AND NEWTON'S METHOD

F. D. PARKER, University of Alaska

Most texts in calculus study Newton's method long before the study of Taylor's series and Taylor's theorem. While studying the latter it may be worth while to stop to refine Newton's method and to show the relation between the two.

If we use only the first two terms of a Taylor series of a function $f(x)$ expanded about a point x_1 near a zero of $f(x)$, then $f(x) = f(x_1) + f'(x_1)(x - x_1)$. Setting $f(x)$ equal to zero we get an approximation to the root $x = x_1 - f(x_1)/f'(x_1)$, which is familiar to the student as Newton's method. If now we take three terms of the Taylor series, we get an approximation

$$x = x_1 - \frac{f'(x_1) \pm \{[f'(x_1)]^2 - 2f(x_1)f''(x_1)\}^{1/2}}{f''(x_1)}$$

provided, of course, $f''(x_1) \neq 0$. While Newton's method fits a line to the graph of the function, this refinement fits a parabola with the same slope and curvature to the graph of the function. The choice of the sign is dictated by the problem. Further refinements are possible of course, but not practical.

In solving $x^4 + 2x^3 + x^2 - 6x - 12 = 0$ with a first approximation $x = 2$, Newton's method yields 1.7778, the suggested refinement yields 1.7265, whereas the correct root is $\sqrt{3} = 1.7321$.

EQUATIONS WITH TRIGONOMETRIC VALUES AS ROOTS

KENNETH W. WEGNER, Carleton College

The purpose of this note is to make readily available for classroom use the 64 irreducible polynomial equations with integral coefficients and of degree two through seven whose roots are of the form $\pm\sin y$, $\pm\cos y$, $\pm\tan y$, $\pm\cot y$, $\pm\sec y$, or $\pm\csc y$, where y is a rational number of degrees.

Of these 64 equations, 14 are quadratic, 8 are cubic, 18 are quartic, 4 are of degree five, and 20 are of degree six. There is none of degree seven. The 6 quadratics involving 30°, 45° and 60° will not be listed. The remaining 58 equations are the following 22 along with others obtained from these 22 by changing the roots to their negatives, to their reciprocals, and to their negative reciprocals:

	Equation	Roots
(1)	$4x^2+2x-1=0$	$\sin 18°$, $-\sin 54°$
(2)	$x^2-4x+1=0$	$\tan 15°$, $\tan 75°$
(3)	$x^2+2x-1=0$	$\tan 45°/2$, $-\tan 135°/2$
(4)	$8x^3-6x+1=0.$	$\sin 10°$, $\sin 50°$, $-\sin 70°$
(5)	$8x^3+4x^2-4x-1=0$	$\sin 90°/7$, $-\sin 270°/7$, $\sin 450°/7$
(6)	$5x^4-10x^2+1=0$	$\pm\tan 18°$, $\pm\tan 54°$
(7)	$16x^4-16x^2+1=0$	$\pm\sin 15°$, $\pm\sin 75°$
(8)	$16x^4+8x^3-16x^2-8x+1=0$	$\sin 6°$, $-\sin 42°$, $-\sin 66°$, $\sin 78°$
(9)	$x^4-4x^3-14x^2-4x+1=0$	$\tan 9°$, $-\tan 27°$, $-\tan 63°$, $\tan 81°$
(10)	$16x^4-20x^2+5=0$	$\pm\sin 36°$, $\pm\sin 72°$
(11)	$8x^4-8x^2+1=0$	$\pm\sin 45°/2$, $\pm\sin 135°/2$
(12)	$x^4+8x^3+2x^2-8x+1=0$	$\tan 15°/2$, $\tan 75°/2$, $-\tan 105°/2$, $-\tan 165°/2$
(13)	$x^4+4x^3-6x^2-4x+1=0$	$\tan 45°/4$, $-\tan 135°/4$, $\tan 225°/4$, $-\tan 315°/4$
(14)	$32x^5-16x^4-32x^3+12x^2+6x-1=0$	$\sin 90°/11$, $-\sin 270°/11$, $\sin 450°/11$, $-\sin 630°/11$, $\sin 810°/11$
(15)	$3x^6-27x^4+33x^2-1=0$	$\pm\tan 10°$, $\pm\tan 50°$, $\pm\tan 70°$
(16)	$64x^6-96x^4+36x^2-3=0$	$\pm\sin 20°$, $\pm\sin 40°$, $\pm\sin 80°$
(17)	$64x^6-112x^4+56x^2-7=0$	$\pm\sin 180°/7$, $\pm\sin 360°/7$, $\pm\sin 540°/7$
(18)	$7x^6-35x^4+21x^2-1=0$	$\pm\tan 90°/7$, $\pm\tan 270°/7$, $\pm\tan 450°/7$
(19)	$x^6-12x^5+3x^4+40x^3+3x^2-12x+1=0$	$\tan 5°$, $\tan 25°$, $-\tan 35°$, $-\tan 55°$, $\tan 65°$, $\tan 85°$
(20)	$x^6-8x^5-13x^4+48x^3-13x^2-8x+1=0$	$\tan 45°/7$, $-\tan 135°/7$, $\tan 225°/7$, $\tan 405°/7$, $-\tan 495°/7$, $\tan 585°/7$
(21)	$64x^6-32x^5-96x^4+48x^3+32x^2-16x+1=0$	$\sin 30°/7$, $\sin 150°/7$, $-\sin 330°/7$, $\sin 390°/7$, $\sin 510°/7$, $-\sin 570°/7$
(22)	$64x^6+32x^5-80x^4-32x^3+24x^2+6x-1=0$	$\sin 90°/13$, $-\sin 270°/13$, $\sin 450°/13$, $-\sin 630°/13$, $\sin 810°/13$, $-\sin 990°/13$

These equations complete the list started in "Trigonometric Values that are Algebraic Numbers," Kenneth W. Wegner, *The Mathematics Teacher*, December, 1957. In that article derivations and suggested uses of equations with sines and cosines as roots were given. Here derivations of the tangent equations will be illustrated with (19).

In the identity $(3\tan^2 y-1)\tan 3y=\tan^3 y-3\tan y$, substitute $y=5°$, $65°$, and $125°$, obtaining $(3x^2-1)(2-\sqrt{3})=x^3-3x$ with roots $\tan 5°$, $\tan 65°$, and

$-\tan 55°$. Similarly, $(3x^2-1)(2+\sqrt{3})=x^3-3x$ has roots $\tan 25°$, $\tan 85°$, and $-\tan 35°$. Equation (19) is obtained by combining these two equations.

That the 64 equations referred to above are irreducible can be verified by theorems on degrees of algebraic numbers as given on page 37 of *Irrational Numbers* by Ivan Niven (No. 11 of The Carus Mathematical Monographs). That the 64 equations are the only ones of the type described can be established by a method suggested by D. H. Lehmer (this MONTHLY, vol. 40, 1933, p. 165).

A NOTE ON AN ADDITIVE PROPERTY OF NATURAL NUMBERS

J. VAN YZEREN, Technological University, Eindhoven, Holland

Some years ago A. Moessner drew attention to the following property of natural numbers.

Consider a set of number sequences beginning with the sequence of natural numbers. Skipping every nth number we form the sequence of partial sums. In this sequence skip every $(n-1)$th number and again form the sequence of partial sums. Again every $(n-2)$th number is skipped and so on. Then the nth sequence will be $1, 2^n, 3^n, 4^n, \cdots$.

This property is trivial for $n=2$. If $n>2$ it may be noticed that, whereas the first and the nth sequences are arithmetical progressions (of the first and nth order), the intermediate sequences are not.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
$n=3$	1	3		7	12		19	27		37	48		61	75	
	1			8			27			64			125		

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	1	3	6		11	17	24		33	43	54		67	81	96
$n=4$	1	4			15	32			65	108			175	256	
	1				16				81				256		

Several rather intricate proofs and generalizations of Moessner's statement have been given (see References). However it may be useful to show that his result can be looked upon as a straightforward consequence of Horner's algorithm applied to the polynomial x^n.

According to Horner's algorithm the coefficients of, *e.g.*, $a_0x^3+a_1x^2+a_2x+a_3$, arranged as a polynomial in $x-1$, are found in the following way.

$$\begin{array}{cccc}
a_0 & a_1 & a_2 & a_3 \\
 & a_0 & a_0+a_1 & a_0+a_1+a_2 \\
a_0 & a_0+a_1 & a_0+a_1+a_2 & \underline{a_0+a_1+a_2+a_3} \\
 & a_0 & 2a_0+a_1 & \\
a_0 & 2a_0+a_1 & \underline{3a_0+2a_1+a_2} & \\
 & a_0 & & \\
\underline{a_0} & \underline{3a_0+a_1} & &
\end{array}$$

From the special polynomial $x^3=x^3+0x^2+0x+0$ we arrive at $(x-1)^3+3(x-1)^2+3(x-1)+1$, from this at $(x-2)^3+6(x-2)^2+12(x-2)+8$, proceeding stepwise by means of the Horner blocks

1	0	0	0
	1	1	1
1	1	1	1
	1	2	
1	2	3	
	1		
1	3		

1	3	3	1
	1	4	7
1	4	7	8
	1	5	
1	5	12	
	1		
1	6		

1	6	12	8
	1	7	19
1	7	19	27
	1	8	
1	8	27	
	1		
1	9		

Clearly, this sequence can be prolonged ad infinitum, as the bottom figures of each block appear in the first row of the next one.

From this block sequence Moessner's sequences ($n=3$) can be derived in the following way. First suppress all *copied* figures (appearing in the copied first rows, but also in all even rows). Next write the blocks columnwise under each other. Then the left of the two arrays underneath is found; the right one is the analogue derived from the polynomial x^4,

x^3	$+\ 0x^2$	$+\ 0x$	$+\ 0$
1	0	0	0
1	1	1	1
1	2	3	
1	3		
1	4	7	8
1	5	12	
1	6		
1	7	19	27
1	8	27	
1	9		
1	10	37	64

x^4	$+\ 0x^3$	$+\ 0x^2$	$+\ 0x$	$+\ 0$
1	0	0	0	0
1	1	1	1	1
1	2	3	4	
1	3	6		
1	4			
1	5	11	15	16
1	6	17	32	
1	7	24		
1	8			
1	9	33	65	81

Apparently, these arrays contain Moessner's sequences ($n=3$ and $n=4$) written down vertically. According to Horner's algorithm the numbers of the last column are the constant terms of the polynomial x^n when expressed in powers of $x-i$. Hence they are the numbers i^n. This is the statement of Moessner.

References

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2. Oskar Perron, Beweis des Moessnerschen Satzes, Sitz. Bayer. Akad. Wiss., 1952, Nr. 4, pp. 31–34.

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5. ———, Z. math. u. naturw. Unterricht, 1953, VI, p. 26–28.

6. ———, Eine Verallgemeinerung des Moessnerschen Satzes, Compositio Mathematica, 1954, vol. 12, pp. 263–270.

MATHEMATICAL EDUCATION NOTES

Edited by JOHN R. MAYOR, American Association for the Advancement of Science and the University of Maryland, and JOHN A. BROWN, University of Delaware

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CURRICULUM STUDIES IN MATHEMATICS

Because of the great effort now being given to improvement of instruction in mathematics, even mathematicians sometimes have difficulty in keeping up with what is going on. In response to numerous requests, there are listed twelve curriculum studies in mathematics, with addresses from which further information can be obtained. In the cases in which only names and addresses are given, those directing the studies have been invited to contribute longer statements for publication in this Department. Two of these appear in this issue. The others have already appeared or will be included in later issues.

The editors apologize for any omissions in this list and will be pleased to have their attention called to studies which should be included in a supplementary list.

Advanced Placement Program. The Advanced Placement Program is a continuation of the School and College Study of Admission with Advanced Standing. The Program provides descriptions of college-level courses to be given in schools and prepares examinations based on these courses. Colleges, in turn, consider for credit and advanced placement students who have taken these courses and examinations. The program is thus an instrument of cooperation which extends the educational opportunities available to able and ambitious students by coordinating effectively their work in school and college. Teachers who are setting up college-level courses should read the course descriptions in the booklet, *Advanced Placement Program Syllabus*, which may be obtained by writing to the Advanced Placement Program, College Entrance Examination Board, 425 West 117th Street, New York 27, New York.

American Society for Engineering Education. A report on the teaching of mathematics for engineers of the joint committee of the American Society for Engineering Education and the Mathematical Association of America. *Journal of Engineering Education*, vol. 45, 1955, and this MONTHLY, vol. 62, 1955, pp. 385–392.

Ball State Teachers College. For the past three years, an experimental tenth grade geometry course has been offered at the laboratory high school of Ball State Teachers College. The postulate set that is used is a modified version of the Hilbert postulates. Plans for this year call for continuation of this course in the laboratory school and several high schools near the College. Work has also been started on a ninth grade algebra course. For information, write to Professor Charles F. Brumfiel, Ball State Teachers College, Muncie, Indiana.

University of Chicago. Since the founding of the College at the University of Chicago, the mathematics department of the College has been concerned with the development of a modern and basic course in mathematics for general education. The course now in use, which has gone through frequent revision, involves many of the ideas with which other of the curriculum groups are now working. The experience at the University of Chicago has been of great value and will continue to be of value to all efforts to modernize the mathematics curriculum. For information write to Professor A. L. Putnam, Box 23, Eckhart Hall, University of Chicago, Chicago 37, Illinois.

Commission on Mathematics. Dr. Robert E. K. Rourke, Executive Director, Commission on Mathematics, 425 West 117th Street, New York 27, New York.

Committee on the Undergraduate Program. For about five years the Mathematical Association of America has sponsored a Committee on the Undergraduate Program. The Committee has considered ways (1) to bring to freshmen the calculus, (2) to introduce to them set notions and probability theory, (3) and to do this without imposing unrealistic demands on students with two and one-half years of mathematics in high school. The Committee has designed two types of sophomore courses. One of these is for physical science and engineering majors, and the other is intended to answer increasing demands from the biological and social sciences. Copies of Collected Reports of the C.U.P. may be obtained from the Mathematical Association of America, University of Buffalo, Buffalo, New York. The Committee was discharged on September 1, 1958, but the Association is making plans to continue this work.*

University of Illinois Committee on School Mathematics. Dr. Max Beberman, UICSM, University High School, Urbana, Illinois.

University of Maryland Mathematics Project (Junior High School). Dr. John R. Mayor, Director of Education, American Association for the Advancement of Science, 1515 Massachusetts Avenue, N.W. Washington 5, D.C.

National Council of Teachers of Mathematics Secondary School Curriculum Committee. The Secondary School Curriculum Committee of the National Council of Teachers of Mathematics has been organized with ten subcommittees, each of which is expected to issue a preliminary report in the spring of 1959. The Committee in giving its attention to the mathematics program for *all* students, grades 7 through 12, is in a strategic position to assist all of the current curriculum investigations, to interpret for teachers and schools recommendations of other groups, and to bring about acceptance by schools and teachers of sound proposals of all other groups. For information, write to Mr. Frank B. Allen, Lyons Township High School, La Grange, Illinois.

Oklahoma State Committee for the Improvement of Mathematics Instruction. Dr. James H. Zant, Oklahoma State University, Stillwater, Oklahoma.

School Mathematics Study Group. Dr. E. G. Begle, School Mathematics Study Group, Drawer 2502A, Yale Station, Yale University, New Haven, Connecticut.

Social Science Research Council Committee on Mathematics in Social Science Research. This Committee was appointed by the Council in 1958 as the successor to its former Committee on Mathematical Training of Social Scientists. The former Committee planned and conducted five summer institutes; it prepared a statement recommending policies for the mathematical training of social scientists (ITEMS, June 1955, p. 13), and aided in the preparation of other materials. For information and reprints write to Social Science Research Council, 230 Park Avenue, New York 17, New York.

MATHEMATICS CURRICULUM STUDIES IN OKLAHOMA

JAMES H. ZANT, Oklahoma State University

Curriculum revision in mathematics has been approached in Oklahoma with the belief that the development of an overall mathematics program from the kindergarten through grade 12 is desirable. The program should be designed to insure for all students mathematical competency and an appreciation of the role of mathematics from a modern aspect, to enable them to meet the challenge

* A conference on the work of the CUP was held in Washington, D. C. November 15–16, 1958. A report of the conference will appear in an early issue of the MONTHLY.

of living adequately in a highly-complex and technological age. The stimulus for the program as it has developed came from the Oklahoma Curriculum Improvement Commission of the State Department of Education with the active interest and financial support of the Frontiers of Science Foundation of Oklahoma, Inc. and from public-minded school administrators from over the state.

In the summer of 1957 a one-month Mathematics Workshop was held at the Oklahoma State University. Participants consisted of a group of able teachers of mathematics in the Oklahoma schools, who were assisted by a staff of competent mathematicians and mathematics educators from within the state and from the outside. This group developed a 94 page bulletin* for the purpose of helping all teachers in their efforts to improve instruction in mathematics from grades one through twelve.

It was recognized that, though the general education of young men and women will be neglected unless they have a thorough understanding of the principles of mathematics, perhaps no area in the curriculum has been less subject to modification and change. Curriculum reorganization, which involves the introduction of new and unfamiliar concepts, must be done on a very broad base if results are to be achieved with any degree of rapidity. This group of teachers and mathematicians used this broad approach in a first attempt to open the way to bring the mathematics curriculum up to date and to provide teachers at all levels with material that they can use to improve their own competence. It was hoped that through their efforts there would be introduced into programs of study new approaches to old subject matter and, perhaps more important, this would encourage the introduction of new subject matter of great importance in the modern world but relatively lacking in the traditional approach to mathematics.

A total of 15,000 copies of the above-mentioned bulletin were printed and distributed widely over the state by the State Department of Education. It was soon apparent, however, that without adequate leadership the efforts of the Mathematics Workshop would not be truly effective over the state as a whole. Hence a committee appointed by the Oklahoma Curriculum Improvement Commission, called the State Steering Committee for Mathematics, began discussing future plans. In order to consolidate state leadership and to initiate a permanent program for improving the state mathematics curriculum, the Commission, following the suggestion of the Steering Committee and again with the active financial support of the Frontiers of Science Foundation of Oklahoma, Inc. and the State Department of Education, sponsored a Leadership Conference on the Improvement of Mathematics Teaching in the Schools of Oklahoma, March 19–22, 1958, at Oklahoma City.

The participants in the Leadership Conference were from the colleges and universities, public and private, engaged in teacher training in the state and included a small number, probably too few, of teachers and administrators

* The Improvement of the Teaching of Mathematics. The Oklahoma State Department of Education, Oliver Hodge, Superintendent. $1.00.

from the public schools. Fourteen of the seventeen colleges and universities were represented, in most cases by two staff members, one from mathematics and one from education. There were 35 participants. The activities of the conference consisted of lectures on various phases of mathematics and mathematics education by the consultants and leaders in the state. Discussions sharpened and clarified points made by the various speakers and at the end the group arrived at definite conclusions and a plan for action in the state.* The "plan for action" included a recommendation to the Oklahoma Curriculum Improvement Commission that the Commission appoint a *State Committee for the Improvement of Mathematics Instruction* including a State Supervisor for Mathematics Education and approximately 20 members consisting of college or university mathematicians, professional education staff members, and elementary and secondary school teachers and supervisors. Further developments of a program in improving mathematics teaching in Oklahoma will be the responsibility of the State Committee working through the Oklahoma Curriculum Improvement Commission.

The State Committee has been appointed with James H. Zant as the Chairman and the program for the future activities in this area is now being completed. It is expected that this will include plans for organizing study groups and subcommittees in various areas; definite plans for writing teaching materials, first in the form of teaching units and eventually in the form of courses for grades and areas; making a definite effort to get information about the need and methods of a new program in mathematics before the teachers and administrators of the state; plans for retraining teachers now in service; and for the reorganization of pre-service training of our elementary and secondary mathematics teachers.

THE UNIVERSITY OF MARYLAND MATHEMATICS PROJECT

M. L. Keedy, University of Maryland

The University of Maryland Mathematics Project (Junior High School), made possible by a grant from the Carnegie Corporation of New York, is now beginning the second of its three years. The study is being directed by John R. Mayor, with the author as his associate. An advisory committee, which assists in the formulation of policy, represents the areas of mathematics, science, engineering, psychology and education within the university, the U. S. Office of Education, the Maryland State Department of Education, and the public school systems near the University. The four major school systems in the Washington, D. C., area—Prince Georges and Montgomery Counties, Maryland; Arlington County, Virginia; and the District of Columbia—are cooperating in the study, with some twenty-five junior high school mathematics teachers participating. These teachers meet weekly at the University of Maryland, to in-

* A 126-page report consisting of the papers read at the Conference and the final recommendations to the Commission was multilithed for distribution within the state.

crease their knowledge of modern mathematical concepts and of recent advances in the theories of learning. They also assist with preparation and revision of teaching materials, conduct interviews of children, and teach newly-prepared materials in their own classes.

The primary goal of the project is to determine maturity levels at which certain mathematical concepts can be appropriately taught, and to prepare materials for a teaching sequence in grades seven and eight which is mathematically and psychologically sound, and appropriate to modern-day needs. Results of the study are being made available to curriculum planning groups across the country, and increasing cooperation with such groups and with other persons engaged in experimental work in mathematics curriculum is being developed.

During the summer of 1958 the staff prepared the first portion of an experimental seventh grade course, on the basis of experience gained in the first year of the study, in addition to conducting a four-week National Science Foundation Institute in conjunction. The 44 Institute participants, representing a wide geographic area, studied mathematics and studied and assisted with preparation of the experimental course materials, in addition to observing an experimental class of seventh grade children.

In the academic year 1958–59 the experimental seventh grade course is being taught in about 25 schools in the greater Washington area and in some others as well. A psychologist has joined the staff for the purpose of directing an evaluation of the experimental course. The teachers involved will continue in a weekly seminar at the University of Maryland, assisting with revision of the experimental course and beginning to develop materials for an eighth grade course, to be taught the following year.

In the experimental seventh grade course, arithmetic and algebra are not distinguished, but are merged by considering *properties of numbers*. The concept of a *mathematical system* is developed and used thereafter as applicable. Unifying concepts are given stress, vocabulary is simplified where possible, and linguistic precision is emphasized where mathematical content is involved. Applied problems of types ordinarily taught in grade seven are included in the exercises, but the first emphasis is on mathematical understanding. The course is designed to include procedures in which students reason both inductively and deductively, although the deduction is largely of an informal nature.

REACTIONS TO THE BOWLING GREEN CONFERENCE*

W. NORMAN SMITH, University of Wyoming

To report on a meeting of the size and complexity of the Bowling Green Conference accurately, briefly and objectively is impossible. The best I can do

* This is the third statement by a mathematician who participated in the 1958 Annual Conference of the National Commission for Teacher Education and Professional Standards. Other statements by F. A. Ficken and Paul C. Rosenbloom appeared in this Department in November, 1958.

is to list a few random impressions, colored by the discussion group of which I was a member, as well as by my personal views.

The most striking, as well as the most encouraging, impression was that of the apparent recognition by delegates from the entire educational spectrum of the importance of subject-matter preparation in the training of teachers. I was assured by one delegate that this was a radical shift—that five years ago any mention of subject matter would have been squelched by the shibboleth, "You don't teach mathematics, you teach boys and girls."

Just how the prospective teacher was to find the time for more intensive work was a moot point. There was a general feeling that five years of college work should be a minimum, but many of the delegates felt that this was impractical as an immediate goal. There seemed to be no willingness on the part of the professional educators to reduce the number of required hours in education, nor did the teachers themselves suggest that this should be done, although all agreed that a certain amount of pruning in the educational vineyard was desirable.

Another encouraging aspect was the realization that the education of teachers was a joint project requiring close cooperation between the liberal arts departments and the professional educators. It was, however, obvious that not all of the heat of conflict between these two groups had been dissipated.

I am confident that the Bowling Green Conference will lend impetus to a movement already underway in Wyoming for closer cooperation between the teachers in the state, the College of Education, and the College of Arts and Sciences.

NATIONAL DEFENSE EDUCATION ACT OF 1958

This Act, passed in the closing days of the 85th Congress, authorizes the expenditure of funds through the Office of Education of the Department of Health, Education, and Welfare under the headings: Loans to Students; Financial Assistance for Strengthening Science, Mathematics, and Modern Foreign Language Instruction; Fellowships; Guidance, Counseling, and Testing; Language Development; Research in Television and Motion Pictures; Area Vocational Education Programs.

Mathematicians should become familiar with the plans for implementation of this Act in their states, especially in relation to the title on Financial Assistance for Strengthening Science, Mathematics, and Foreign Language Instruction.

Inquiries might be addressed to state departments of education or to Dr. Kenneth E. Brown, U. S. Office of Education.

CONTINENTAL CLASSROOM IN PHYSICS*

CONTINENTAL CLASSROOM—the TV course for college credit in Atomic Age Physics launched October 6 over the National Broadcasting Company nationwide network—is proving to be a "hit." Over 250 colleges and universities across the country are offering the course for credit. In addition, at least 20,000 engineers, technicians, homemakers,

* Submitted upon request by The American Association of Colleges for Teacher Education, Office of the National Coordinator of Atomic Age Physics. For further information: Edwin P. Adkins or Hope C. Corso, In Care of National Broadcasting Company, 30 Rockefeller Plaza, New York 20, New York.

gifted high school students, career-Army personnel and others are viewing the program to update themselves in modern-day science. Teaching the course is Dr. Harvey E. White, professor of physics at the University of California and consultant to the Atomic Energy Commission. Guest lecturers include noted physicists and scientists from leading U. S. institutions of higher education.

Supervising the over-all effort—for the American Association of Colleges for Teacher Education—is Dr. Edwin P. Adkins, on leave from New York State University College for Teachers in Albany where he is director of education. Financial support for the program has been contributed by the Ford Foundation, the Fund for the Advancement of Education, and by industry: Bell Telephone System, the California Oil Company, General Foods Fund, International Business Machines Corporation, Pittsburgh Plate Glass Foundation and United States Steel.

The second semester on Nuclear Physics, commencing February 11, promises to attract an even larger number of viewers than this semester's offering on Basic Physics. The program—telecast at 6:30 a.m., local time—will end June 5.

CURRENT ITEMS

The complete report on *Science and Math* in the *NASSP Bulletin*, Sept. 1958, pp. 5–12, is available in reprint form on request as long as the supply lasts. For a free copy of *The Place of Science and Mathematics in the Comprehensive Secondary-School Program*, write to *The Spotlight*, 1201 Sixteenth Street, N.W., Washington 6, D.C.

Science and mathematics receive special emphasis in a plan to give four different diplomas which has recently been approved by the Indianapolis Board of Education. The four diplomas do not simply reflect the kind of study undertaken, for they also take into account how well the students do.

A National survey of science and mathematics teachers in American public high schools has been started by the U. S. Office of Education and will be ready for distribution around the first of the year. This survey is the most recent of similar surveys published in 1955 and 1957.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1346. *Proposed by J. M. Gandhi, Belgaum, India*

Prove that if p is a prime then $\binom{2p}{p} \equiv 2$, mod p.

E 1347. *Proposed by V. F. Ivanoff, San Carlos, California*

Prove that $\sum_{i=0}^{n} \binom{n}{i} F_{n-i} = F_{2n}$, where $F_0, F_1, \cdots, F_n$ is any set of $n+1$ consecutive Fibonacci numbers.

E. 1348. *Proposed by M. S. Klamkin and Raphael Miller, AVCO Research and Development, Lawrence, Mass.*

Find the locus of the centroids of all equilateral triangles inscribed in an ellipse.

E 1349. *Proposed by P. L. Chessin, University of Maryland*

Consider the $n\times n$ matrix $[a_{ij}]$ where $a_{11}=\cos\theta$, $a_{ii}=2\cos\theta$ $(i=2,\cdots,n)$, $a_{i,i+1}=a_{i+1,i}=1$ $(i=1,\cdots,n-1)$, and all other elements are zero. Show that $|a_{ij}|=\cos n\theta$.

E 1350. *Proposed by N. A. Court, University of Oklahoma*

(a) The tangents to the ninepoint circle of a triangle T at the midpoints of the sides of T form a triangle homothetic to the orthic triangle of T. (b) The homothetic center of the two triangles is a point on the Euler line of T.

SOLUTIONS

Existence of a Unit Element

E 1316 [1958, 365]. *Proposed by R. C. Buck, University of Wisconsin*

Let S be a set of elements with an associative multiplication. Suppose that S has a special element u with the property that u is a left and a right divisor of every element in S. Does S have to possess a unit?

Solution by J. V. Whittaker, University of British Columbia. For any $x\in S$, let $yu=x=uz$ and $su=u=ut$. Then $sx=s(uz)=(su)z=uz=x$ and, similarly, $xt=x$. Thus $st=s=t$ is a unit of S.

Also solved by R. G. Albert, M. D. Anderson and Jerry Bebernes (jointly), Lawrence Arnold, A. Bager, Peter Beisswanger, L. P. Belluce, M. P. Berenson, J. L. Brenner, J. L. Brown, Jr., P. L. Chessin, R. M. Conkling, H. M. Farkas, N. J. Fine, Fred Galvin, W. V. Gamzon, Virginia S. Hanly, J. Hooley, A. F. Kaupe, Jr., Irving Kay, J. M. Kingston, Joe Lipman, R. T. J. Mahoney, D. C. B. Marsh, J. S. Moore, Jr., F. D. Parker, D. S. Passman, W. J. Pervin, John Rainwater, Theodore Reiss, Azriel Rosenfeld, Jack Silver, Paul Slepian, J. W. Smith, Anthony Trampus, and the proposer.

A Tetrahedron and a Sphere

E 1317 [1958, 365]. *Proposed by N. A. Court, University of Oklahoma*

Let $(T)=DABC$ be a tetrahedron and let (M) be a sphere having its center M on the axis of the circle ABC. If one and only one point is taken in each pair of points determined by (M) on the edges of (T) issued from the vertex D, show that the $2^3=8$ planes thus determined are cut by the plane ABC along four pairs of isotomic transversals of the triangle ABC.

Solution by the Proposer. 1. Let

(a) $$U,\ U';\ V,\ V';\ W,\ W$$

be the traces of (M) on the edges DA, DB, DC, respectively. The points V, V',

W, W' determine a complete quadrangle (q) inscribed in the circle (M_a) in which plane DBC cuts (M). The four pairs of points

$$(I) \qquad P,\ P';\ X,\ X';\ B,\ C;\ S,\ S'$$

in which line BC cuts the three pairs of opposite sides VW, $V'W'$; VW', $V'W$; VV', WW' of (q) and the circle (M_a), belong to the same involution, by Desargues' theorem [see, for inst., L. Cremona, *Projective Geometry*, p. 143, art. 183].

2. The points of intersection S, S' of (M_a) with line BC are also the traces on BC of the sphere (M). These points therefore lie also in the circle (M_d) in which (M) is cut by plane ABC. Now (M_d) is concentric with circle ABC, since by assumption M lies on the axis of circle ABC. Hence the two segments SS' and BC have the same midpoint, say A'.

The two pairs of points B, C; S, S' determine the symmetrical involution having A' and the point at infinity of BC as the double points, or, what is the same thing, the involution of isotomic points on side BC of triangle ABC. On the other hand, these two pairs of points also belong to the involution (I) considered in part 1. Hence (I) is identical with the isotomic involution. Consequently, the two pairs of lines VW, $V'W'$; VW', $V'W$ determine on BC two pairs of isotmic points (*cf.* Nathan Altshiller-Court, *College Geometry*, 2nd ed., p. 158, ex. 7). We similarly treat the faces DCA, DAB of (T).

3. Consider any two of the eight planes determined by the points (a) and involving all six points, say UVW and $U'V'W'$. The three pairs of lines VW, $V'W'$; WU, $W'U'$; UV, $U'V'$ determine on the three transversals BC, CA, AB, respectively, three pairs of isotomic points P, P'; Q, Q'; R, R' (part 2). The triads of points PQR, $P'Q'R'$ lie in the planes UVW, $U'V'W'$, respectively, and all six lie in the plane ABC. Hence the points of each triad are collinear, and the two lines PQR, $P'Q'R'$ are two isotomic transversals of triangle ABC. We may similarly treat the other three pairs of analogous planes. Hence the proposition.

The reader may consider the cases where the sphere (M) passes through the vertex D or through the circle ABC and formulate the corresponding results.

A Theorem on Permutations

E 1318 [1958, 366]. *Proposed by Gordon Raisbeck, Bell Telephone Laboratories, Inc.*

A. A. Mullin has proved (this MONTHLY [1957, p. 669]) that

$$\Phi(n) = \sum_{r=0}^{n} {}_nP_r = \sum_{r=0}^{n} \frac{n!}{(n-r)!} \sim (n!)e$$

for large n, where $\sim$ denotes asymptotic equality. Prove that

$$\Phi(n) = [(n!)e], \qquad n \geqq 1,$$

where $[x]$ denotes the largest integer not greater than x.

I. *Solution by W. J. Blundon, Memorial University of Newfoundland.* We have

$$\begin{aligned}(n!)e - \Phi(n) &= n!\{1/(n+1)! + 1/(n+2)! + 1/(n+3)! + \cdots\} \\ &= 1/(n+1) + 1/(n+1)(n+2) + 1/(n+1)(n+2)(n+3) + \cdots.\end{aligned}$$

This expression is clearly positive, but is less than

$$(n+1)^{-1} + (n+1)^{-2} + (n+1)^{-3} + \cdots = 1/n \leqq 1.$$

Since $\Phi(n)$ is an integer, it follows that $\Phi(n) = [(n!)e]$.

II. *Solution by J. D. E. Konhauser, State College, Pa.* Since $\Phi(n)$ is the total number of permutations of n objects, this problem is equivalent to the solved problem E 1186 [1956, 343].

Also solved by R. G. Albert, Lawrence Arnold, Peter Beisswanger, J. L. Brenner, J. L. Brown, Jr., P. L. Chessin, E. L. Ellis and D. L. Muench (jointly), William Faris, N. J. Fine, Fred Galvin, Lawrence Glasser, L. D. Goldberg, Michael Goldberg, Emil Grosswald, J. H. Hodges, Vern Hoggatt, Norbert Kaufman, A. F. Kaupe, Jr., John Kelley and W. E. Lawrence (jointly), Morton Kupperman, Joe Lipman, R. T. J. Mahoney, D. C. B. Marsh, Clifford Marshall, L. C. Marshall, Leo Moser, T. F. Mulcrone, S.J., A. A. Mullin, J. B. Muskat, F. D. Parker, D. S. Passman, C. F. Pinzka, D. A. Robinson, Azriel Rosenfeld, R. E. Shafer, H. K. Shepard, D. L. Smith, D. P. Thompson, L. K. Williams, David Zeitlin, and the proposer.

A Quiz Contestant

E 1319 [1958, 366]. *Proposed by S. W. Golomb, California Institute of Technology*

A quiz contestant selects a category containing n questions, k of which are too difficult for him. The questions are selected from the category at random, and the contestant continues answering until he misses a question. What is the probability that he will miss on the ath question?

Solution by C. F. Pinzka, University of Cincinnati. The probability that the contestant answers the first $a-1$ questions correctly is

$$\binom{n-k}{a-1} \Big/ \binom{n}{a-1}$$

and the probability that he then misses the ath question is $k/(n-a+1)$. The probability that these events occur in succession is

$$\frac{\binom{n-k}{a-1}}{\binom{n}{a-1}} \cdot \frac{k}{n-a+1} = \frac{\binom{n-a}{k-1}}{\binom{n}{k}} = \frac{(n-k)!(n-a)!k}{n!(n-k-a+1)!}.$$

Also solved by D. S. Adorno, R. G. Albert, Peter Beisswanger, A. P. Boblétt, Julian Braun, D. A. Breault, J. L. Brown, Jr., W. J. Cahill, P. L. Chessin, A. G. Clark, R. M. Conkling, E. L. Ellis and D. L. Muench (jointly), William Faris, N. J. Fine, Fred Galvin, L. D. Goldberg, Michael Goldberg, Edwin Goldfarb and R. W. Simister (jointly), A. G. Grace, Jr., R. E. Greenwood, J. H. Hodges, A. R. Hyde, Elaine Johnson, Irwin Kabus, Norbert Kaufman, A. F. Kaupe, Jr., D. A. Kearns, J. D. E. Konhauser, Sam Kravitz, W. H. Kruskal, W. E. Lawrence, Joe Lipman, Peter McManus, D. C. B. Marsh, Helen Marston, Leo Moser, T. H. Mott, Jr., J. B. Muskat, C. S. Ogilvy, C. A. Reiher, Azriel Rosenfeld, R. E. Shafer, Jack Silver, Paul Slepian, R. H. Wilson, Jr., David Zeitlin, and the proposer.

The problem was located in W. Feller, *Theory of Probability*, vol. 1, prob. 12, p. 59 (1st ed.) or prob. 22, p. 60 (2nd ed.).

Composite Values of a Polynomial

E 1320 [1958, 366]. *Proposed by G. S. Stoller, Polytechnic Institute of Brooklyn*

(a) Let $P(x)$ be any polynomial in x with integral coefficients. Prove that there exists an infinite number of integers t such that $P(t)$, $P(t+1)$, $\cdots$, $P(t+m)$ are all composite for any given positive integer m.

(b) Let $P(x)=x^2+1$. Find a value of t satisfying part (a) for $m=5$.

Solution by Virginia S. Hanly, North American Aviation, Columbus, Ohio. (a) Let k be arbitrary such that $|P(k)|>1$ and $x\geqq k$ implies $|P(x)|$ monotone increasing. Let u be the least common multiple of the numbers $|P(k)|$, $|P(k+1)|$, $\cdots$, $|P(k+m)|$. We set $t=u+k$. Consider $P(t+i)$ for $i=0, 1, 2, \cdots, m$. We have $P(t+i)\equiv P(k+i)$ mod u. Thus $P(k+i)$ is a proper divisor of $P(t+i)$ so that $P(t+i)$ is composite.

(b) In the case of the function $P(x)=x^2+1$ with $m=5$ we may choose $k=1$ to obtain the value l.c.m. $(2, 5, 10, 17, 26, 37)+1=81{,}771$ for t.

Also solved by R. G. Albert, W. J. Blundon, N. J. Fine, Fred Galvin, Sidney Kravitz, Joe Lipman, D. C. B. Marsh, Leo Moser, D. S. Passman, Jeff Scargle, R. E. Shafer, Jack Silver, W. A. Veech, and the proposer.

None of the methods developed for determining t in part (a) led to the least value of t, namely $t=27$, for part (b).

Moser pointed out that the result is an immediate consequence of the following theorem of H. Heilbronn (Über die Verteilung der Primzahlen in Polynomen, *Mathematische Annalen* 104 (1931), pp. 794–799). If $f(x)$ is a nonconstant integer-valued polynomial, then there exists a constant c such that for every m the number of primes in the set $f(1), f(2), \cdots, f(m)$ is less than $cm/\log m$.

AN INTERESTING PYTHAGOREAN TRIANGLE

Victor Thébault notes an interesting Pythagorean triangle in which the two perpendicular sides are integers having the same digits in reverse order, *viz.*, 88209 and 90288, the hypothenuse being 126225.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4822. *Proposed by J. deGroot, University of Amsterdam, Netherlands*

Prove that the additive group R of rational numbers is (up to isomorphisms) the only group satisfying the following conditions: (1) R is abelian, (2) R is infinite, and (3) every endomorphism (that is a homomorphic mapping of R in itself) is either an automorphism or a mapping on the null element.

4823. *Proposed by G. Matthews, St. Dunstan's College, Catford, England*

Let X be a lower-semi-matrix whose elements x_{ij} $(i, j=1, 2, 3, \cdots)$ are independent variables if $j\leqq i$ and zero if $j>i$, and let D be the differential operator matrix defined by $D_{ij}=\partial/\partial x_{ij}$ if $j<i$, $D_{ij}=0$ if $j>i$ and $D_{ii}=\sum_{k=1}^{i}\partial/\partial x_{kk}$. Prove that $D(X^2)=2X$, and hence by induction that $D(X^n)=nX^{n-1}$, where n is any positive integer greater than 2.

4824. *Proposed by D. J. Newman, AVCO Research and Development, Wilmington, Mass.*

Let $ABCD$ be a rectangle, $AB=1$, $BC=2$. Suppose that it is conformally mapped onto the upper half plane. If $A\to a$, $B\to b$, $C\to c$, $D\to d$, exhibit an elementary relation between a, b, c, d.

4825. *Proposed by J. J. Schäffer, University of Uruguay*

Prove that in any n-dimensional (real or complex) Euclidean space, $n>1$, and indeed in any Hilbert space, every point x, $\|x\|\leqq 1$ may be written $x=\sum_{i=1}^{m}f_i(x)$, where $f_1(x), \cdots, f_m(x)$ is a fixed finite set of functions which are defined and uniformly continuous in $\|x\|\leqq 1$ and satisfy $\|f_i(x)\|=1$ for $i=1, \cdots, m$ and for all $\|x\|\leqq 1$.

From this result it follows at once that if $\mathfrak{C}$ is the Banach space of all bounded continuous functions $\phi(t)$ of a real variable t with values in any Hilbert space of dimension >1 (with the norm of the supremum), the set of $\phi(t)\in\mathfrak{C}$ with $\|\phi(t)\|=1$ identically contains a linear basis of $\mathfrak{C}$.

4826. *Proposed by M. S. Klamkin and L. A. Shepp, AVCO Research and Development, Wilmington, Mass.*

If $\phi(x)=x/1^2-x^3/3^2+x^5/5^2-x^7/7^2+\cdots$, express $\phi(1)$ in terms of $\phi(2-\sqrt{3})$, thus obtaining a more rapidly converging expansion.

4827. *Proposed by J. Gallego-Diaz, Vanderbilt University*

If the development of the function $y=f(x)$ is given by

$$y = a_1x + a_3x^3 + a_5x^5 + \cdots,$$

find a function y knowing that if we invert the series we get

$$x = a_1y - a_3y^3 + a_5y^5 - a_7y^7 + \cdots.$$

SOLUTIONS

Maximal Ideals in a Ring

4761 [1957, 676]. *Proposed by Alfredo Jones, Institute of Mathematics and Statistics, Montevideo, Uruguay*

In a ring with identity and with proper ideals, there always exist maximal ideals. Is the statement true for rings with a nontrivial multiplication and with no identity?

Solution by E. A. Walker, New Mexico A & M State College. Let R be the rational numbers. Define multiplication in the additive group $R\oplus R$ by $(r, s)\cdot(r', s')=(0, rr')$. With this multiplication, $R\oplus R$ becomes a ring with proper ideals, nontrivial multiplication, and no identity. Let I be a proper ideal in this ring. The set R_1 of first coordinates of the elements of I is not R, because then I would be $R\oplus R$. Let R_2 be a proper subgroup of R properly containing R_1. Then $R_2\oplus R$ is a proper ideal properly containing I. Thus the anwer is no.

Also solved by E. R. Gentile, K. G. Wolfson, and the proposer.

Editorial Note. The solver felt that the reader will be aware that the additive group R has no maximal subgroups (whence there must exist an R_2 as stated above.) However, at the Editor's request, he supplies the following proof.

Let R_1 be a proper subgroup of R. Let $a\in R$, $a\notin R_1$. Some non-zero multiple na is in R_1. Let R_2 be the subgroup of R generated by R_1 and a. If $a/n\in R_2$ then $a/n=r_1+ka$, $r_1\in R_1$, k an integer. Therefore $a=nr_1+kna$. But $nr_1\in R_1$, $kna\in R_1$. Hence $a\in R_1$, and this contradiction proves $a/n\notin R_2$. Hence $R_1<R_2<R$.

An Algebraic Identity

4778 [1958, 211]. *Proposed by R. C. Lyness, Preston, England*

Given $f(r)=\alpha^r(\beta-\gamma)+\beta^r(\gamma-\alpha)+\gamma^r(\alpha-\beta)$ in which α, β, γ are nonzero and distinct. If n is a positive integer and $f(n+1)=0$, prove that

$$f(n+2)f(n) = \alpha^n\beta^n\gamma^n(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)f(-n).$$

Solution by D. C. B. Marsh, Colorado School of Mines. From the product rule for determinants, the following identity is evident:

$$\begin{vmatrix} f(n+2) & f(n+1) & \gamma^{n+1} \\ f(n+1) & f(n) & \gamma^n \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} \alpha^{n+1} & \beta^{n+1} & \gamma^{n+1} \\ \alpha^n & \beta^n & \gamma^n \\ 1 & 1 & 1 \end{vmatrix} \cdot \begin{vmatrix} \alpha(\beta-\gamma) & (\beta-\gamma) & 0 \\ \beta(\gamma-\alpha) & (\gamma-\alpha) & 0 \\ \gamma(\alpha-\beta) & (\alpha-\beta) & 1 \end{vmatrix}.$$

Replacing each by its expansion, one obtains

$$\begin{aligned} f(n+2)f(n) - f^2(n+1) &= \{\beta^n\gamma^n(\beta-\gamma) + \gamma^n\alpha^n(\gamma-\alpha) + \alpha^n\beta^n(\alpha-\beta)\}\{(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)\} \\ &= \alpha^n\beta^n\gamma^n(\beta-\gamma)(\gamma-\alpha)(\alpha-\beta)\cdot f(-n). \end{aligned}$$

Setting $f(n+1)=0$ gives the announced result. There need be no restriction on n.

Also solved by A. H. Aheart, A. P. Boblétt, L. Carlitz, A. E. Danese, E. S. Eby, E. L. Ellis and D. L. Muench, A. B. Farnell, J. W. Haake, J. H. Hodges, R. D. James, Norbert Kaufman, Irving Kay, J. D. E. Konhauser, A. E. Landry, J. A. Larrivee, Gerald Leibowitz, O. E. Lewis, Joe Lipman, Yoshio Matsuoka, M. F. Neuts, Kyu Sam Park, F. D. Parker, Paul Payette, F. W. Ponting, Siya Ram, Benjamin Sapolsky, Marlow Sholander, Arnold Singer, T. H. Slook, Robert Spira, Chih-yi Wang, Harry Weingarten, David Zeitlin, and the proposer.

Total Variation

4779 [1958, 211]. *Proposed by Solomon Leader, Rutgers University*

Let $f(x)=|x-k|$ for $k-\frac{1}{2}\leqq x\leqq k+\frac{1}{2}$, where k runs through the integers, and

$$g_n(x) = \sum_{\nu=0}^{n-1} 10^{-\nu}f(10^\nu x).$$

($\text{Lim}_{n\to\infty} g_n(x)$ is Van der Waarden's example of a continuous, nowhere differentiable function.) Let $\|g_n\|$ be the total variation of g_n on the interval (0, 1). Find $\lim_{n\to\infty} n^{-1/2}\|g_n\|$.

Solution by the proposer. From the definition of $f(x)$ we have for every integer k:

$$f'(x) = \begin{cases} +1 & \text{for } k < x < k+\frac{1}{2} \\ -1 & \text{for } k+\frac{1}{2} < x < k+1. \end{cases}$$

Thus

$$f'(10^\nu x) = \begin{cases} +1 & \text{for } k/10^\nu < x < k/10^\nu + 5/10^{\nu+1} \\ -1 & \text{for } k/10^\nu + 5/10^{\nu+1} < x < (k+1)/10^\nu. \end{cases}$$

For x in (0, 1) and not a decimal fraction of order $\nu+1$, consider the decimal expansion of x, $x=.x_1x_2\cdots x_\nu x_{\nu+1}\cdots$. Then

$$f'(10^\nu x) = \begin{cases} +1 & \text{for } x_{\nu+1} = 0, 1, 2, 3, 4 \\ -1 & \text{for } x_{\nu+1} = 5, 6, 7, 8, 9. \end{cases}$$

For x not a decimal fraction of order n we have

$$g_n'(x) = \sum_{\nu=0}^{n-1} f'(10^\nu x).$$

Let m be the number of digits less than 5 occurring in the first n places of the decimal expansion of x. Then $g_n'(x)=m-(n-m)=2m-n$. Now there are

$\binom{n}{m}5^n$ decimal fractions of the form $.x_1x_2 \cdots x_n$ having m digits less than 5 and $n-m$ digits equal to 5 or more. For each such decimal fraction we have for $.x_1x_2 \cdots x_n < x < .x_1x_2 \cdots x_n999 \cdots$, $g_n'(x) = 2m-n$. Thus $g_n'(x)=2m-n$ over $\binom{n}{m}5^n$ intervals of length $1/10^n$, giving a total length $\binom{n}{m}/2^n$.

Now $\|g_n\| = \int_0^1 |g_n'(x)|\,dx$. So

$$\|g_n\| = \sum_{m=0}^{n} |2m-n| \binom{n}{m} \Big/ 2^n$$

which is just twice the mean deviation of the number m of heads in n tosses of a coin. The expectation of m is $n/2$ and the standard deviation of m is $\sqrt{n}/2$. Since

$$\frac{1}{\sqrt{n}}\|g_n\| = \sum_{m=0}^{n} \left|\frac{m-n/2}{\sqrt{n}/2}\right| \binom{n}{m}\frac{1}{2^n}$$

and the binomial distribution converges to the normal distribution, we have

$$\begin{aligned}\lim_{n\to\infty}\frac{1}{\sqrt{n}}\|g_n\| &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}|x|\,e^{-x^2/2}dx \\ &= \frac{2}{\sqrt{2\pi}}\int_0^{\infty} xe^{-x^2/2}dx = \sqrt{\frac{2}{\pi}}\,.\end{aligned}$$

Also solved by Robert Breusch.

Equiproduct Points

4780 [1958, 212]. *Proposed by M. S. Klamkin and D. J. Newman, AVCO Research and Development, Wilmington, Mass.*

An equiproduct point of a curve is defined to be a point such that the product of the two segments of any chord through the point is constant. (1) Show that if every point inside a curve is equiproduct, the curve must be a circle. (2) What is the maximum number of equiproduct points a noncircular oval can have?

Editorial Note. The problem is not new. For solution and discussion see problem E 705 [1946, 395] and [1947, 164]. In the latter reference will be found a discussion of a paper by K. Yanagihara in the Tohoku Mathematical Journal (1917) touching on the same problem, together with analogous theorems for three dimensional space. The principal result is the theorem: *A convex closed curve having two distinct interior equiproduct points is a circle.*

Solved by Robert Breusch, Michael Goldberg, Marlow Sholander, and (partially) by Sidney Glusman, Ronald Graf, and the proposers.

A Polynomial Assuming Positive Values Only

4781 [1958, 212]. *Proposed by J. L. Massera, Mathematics Institute, Montevideo, Uruguay*

Let $f(x, y, z, \cdots) = a_0(y, z, \cdots)x^n + a_1(y, z, \cdots)x^{n-1} + \cdots + a_n(y, z, \cdots)$, where the a_i are any real functions defined in any region G of

the $(y, z, \cdots)$-space. Let $g(y, z, \cdots)$ be any real function defined in G and construct $f^*(x, y, z, \cdots) = f + gf_x + g^2 f_{xx} + \cdots$. Then, if $f \geqq 0$ in the cylinder $K = G \times \{x: -\infty < x < +\infty\}$, we have $f^* \geqq 0$ in K. More precisely, if the a_i do not vanish simultaneously in G, $f^* > 0$ in K except at points where $f = g = 0$.

Solution by Robert Breusch, Amherst College. There seems to be no significant difference in content between the given theorem and the following:

If $f(x) \equiv a_0x^n + a_1x^{n-1} + \cdots + a_n$ (a_i constant, $a_0 \neq 0$), *if* $f(x) \geqq 0$ *for every* x (thus n even, $a_0 > 0$), *if g is a constant distinct from zero, then* $f^*(x) \equiv f(x) + g \cdot f'(x) + g^2 \cdot f''(x) + \cdots + g^n \cdot f^{(n)}(x) > 0$ for every x.

Proof. We have

$$f^*(x) = f(x) + g \cdot f^{*\prime}(x) \tag{1}$$

where $f^{*\prime}(x)$ is the derivative of $f^*(x)$. $f^*(x)$ is again a polynomial of even degree, with a positive highest coefficient. Thus $f^*(x)$ is positive for large x. If $f^*(x)$ has a minimum at $x = c$, then $f^{*\prime}(c) = 0$. Thus, from (1), $f^*(c) = f(c) \geqq 0$. Therefore $f^*(x)$ must be nonnegative for every x. In order to show that $f^*(x)$ must be positive (that is, $f(c) \neq 0$), we solve the differential equation (1), obtaining

$$f^*(x) = -\frac{1}{g} e^{x/g} \int_c^x e^{-x/g} f(x)dx + f^*(c) \cdot e^{(x-c)/g}.$$

Since the integrand is nonnegative, the integral will be positive or negative, depending on the sign of $x - c$. Thus, if $f^*(c)$ were zero, $f^*(x)$ would be negative for some x, which contradicts the previously-established fact that $f^*(x)$ is nonnegative for all x.

Also solved by the proposer.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

Note. Readers may be interested to know that section 135.215a of the new (August 1958) U. S. Postal Rates provides a special rate of 4 cents for the first pound and 1 cent for each additional pound on books and other library materials, whether printed, photographed, duplicated or typed (including unpublished manuscripts), which are being loaned or exchanged (not sold) *between educational institutions and/or nonprofit organizations or associations.* The wrapper should be marked *LIBRARY MATERIALS* to qualify for this special rate.

Problems in Euclidean Space. Application of Convexity. By H. G. Eggleston. Pergamon Press, New York, 1957. viii+165 pp. $6.50.

This book contains the Adams Prize Essay of the University of Cambridge 1955–6. Professor Eggleston presents the material contained in a series of his own research papers as discussions of ten problems. Problems range from ones concerned with meromorphic functions and planar homeomorphisms to closest packing problems. Mathematicians who proposed some of the problems include: W. Gross, S. M. Ulam, A. S. Besicovitch, C. H. Dowker, Ellen F. and R. C. Buck, and the reviewer. The book is of interest to research workers in geometry, topology, and analysis for its results and for the methods used.

PRESTON C. HAMMER
University of Wisconsin

Introduction to Riemann Surfaces. By George Springer. Addison-Wesley, Reading, Mass., 1957. viii+307 pp. $9.50.

The book deals with abstract Riemann surfaces and their connection with algebraic functions and their integrals. It is not a survey of the subject, but is "a modern presentation of the classical theory which will prepare the reader for further study in this and related fields." It is the first book of its nature written in English, and is certainly a long-awaited text due to the increase of interest in the subject.

The book begins with an Introduction, which bridges the gap between the classical and abstract notions and outlines the goals of the study together with their means. The second chapter, on General Topology, discusses the basic notions to be used for defining an abstract Riemann surface or analytic manifold. It is shown in Chapter 3 that the Riemann surface of an analytic function is also an abstract Riemann surface. Chapter 4 on Covering Manifolds and Chapter 5 on Combinatorical Topology include such topics as covering surfaces, fundamental groups, triangulation, orientability, normal forms of surfaces and homology groups. In Chapter 6, differentials and their integrals on a surface are considered. These differentials form a Hilbert space, which is studied in Chapter 7. The existence of harmonic and analytic differentials and functions on the abstract Riemann surfaces is given in Chapter 8; the problem of finding a uniformizing parameter over the whole surface and automorphic functions are taken up in Chapter 9; and, finally, the meromorphic functions and multiple-valued functions on the Riemann surface are studied in Chapter 10, which connects the study of closed Riemann surfaces and the study of algebraic functions as planned.

The book is written with unusual clearness. As in the Introduction, which outlines the whole book, similar lines appear in each Chapter. There are a list of references, an index, and a collection of exercises at the end of each chapter. The author spends almost one-third of the book presenting a modern treatment in a self-contained manner with a minimum assumption of knowledge—a knowl-

edge of elementary complex function theory and some real variables and algebra. He is most successful in this magnificent project. The book offers not only an excellent treatment of Riemann surfaces, but also an excellent introduction to Topology, Hilbert-space theory and other important mathematical notions. It is highly recommended as a text in its field.

T. K. PAN
University of Oklahoma and
National Taiwan University

Circles. By D. Pedoe. Pergamon Press, New York, 1957. x+78 pp. $3.75.

The prestige of the circle is great. It was deemed to be the only curve fit to be used in geometrical constructions, with the assistance of the straight line (Plato), and even that help may be dispensed with (Mascheroni). Celestial bodies could follow no other path but the perfect one—a circle. Biologists ungraciously point out that nature, in fashioning the organisms of the animal kingdom, has no use for the circle. But Pedoe is still sure that the circle can be used to make friends for—and influence people in favor of—mathematics.

He begins by treating his readers to a few choice bits (or bites?) of modern geometry of the triangle and the circle—and real titbits they are—served in a most attractive way. The elements of the theory of inversion, thrown in for good measure, enable the author, among other things, to exhibit later Poincare's model of Lobacevskian geometry. He shows the advantages that may be derived from the study of the circle by the fact that both the circle in the plane and a point in three-space are determined by three parameters. He even braves the difficulty of trying to convince the reader of the isoperimetric property of the circle.

All this is accomplished with elementary means, in an elegant manner, in the space of a small and slender volume of a few dozen pages. It is to be regretted that, presumably, this economy of space made the author resist the temptation of enhancing the reader's pleasure by including some historical data, or egging on the curiosity of the reader by a few bibliographical references.

NATHAN ALTSHILLER COURT
University of Oklahoma

Elements of Modern Abstract Algebra. By Kenneth S. Miller. Harper, New York, 1958. vii+188 pp. $5.00.

Professor Miller has produced in this text a display of mathematical austerity for "upperclassmen mathematics majors or beginning graduate students." The book is presented in four parts: groups, rings and ideals, fields, sets. It is a compact affair which hurries the reader along from the definition of a group (page 1), through homomorphism theory and the Jordan-Hölder Theorem, to the basis theorem for finite abelian groups (page 49). This pace keeps up throughout the book, and by the time page 184 is reached the reader has been

exposed to ideals, the Hilbert Basis Theorem, finite fields, Zorn's Lemma, and much more. To pursue the subject so relentlessly is not hard if one is willing to keep his eyes on the center of the road and dares only to glance at the countryside. From remarks in the preface (which contains a reference to "ideals in the field of real numbers"), this relentless pursuit seems to be the author's goal; and the reviewer must congratulate the author for standing by his promise. To do so much in so short a space, of course, entails the omission of something—and there isn't *much* choice. In presenting a theorem, one can hardly omit the statement of the theorem; once a theorem is stated, the reader expects shortly to see a proof; and examples are nice to have. That leaves motivation, history, application, *etc.*, as the only possible exclusions and, except for a slight bow now and then to these considerations, they are excluded. The chapters remain unspoiled by problems until the end.

Is this bad? It depends on how the text is used. With so little history and application, there is plenty of room for the *teacher* to ply his trade, and so some might like it. It's quite like "instant algebra" in that something needs to be added. On the other hand, the undergraduate who can read this book and enjoy it without instruction doesn't need to read it—he can go on to Jacobson for bigger and better thrills.

The author has some disconcerting habits. We mention a few.

(1) Every display is an "equation," a typical example being "Equation 1.16": G, G_{15}, G_5, G_1.

(2) Some theorems do not "fit in" as they should. For example, Theorem 12 of Chapter I would be better presented if it preceded Theorem 10. Theorem 4 of Chapter II implies Theorem 3.

(3) Examples are not always instructive: one example of a ring without identity is lost to the reader who is not familiar with the Dirac δ-function.

(4) Some results are never subsequently used, so that the reader can only wonder about their inclusion.

The book contains relatively few errors. Helpful "identifications" are made in cases where some authors persist in pedantry, and there are but a few theorems or proofs which are unnecessarily complicated. The section on sets (an appendix) is nice to have, and applications are given.

ROBERT J. WISNER
Haverford College

Commutative Algebra, Vol. I. By Oscar Zariski and Pierre Samuel. University Series in Higher Mathematics, Van Nostrand, Princeton, 1958. xi+330 pp. $6.95.

The authors approach modern algebra in this book from a sophisticated viewpoint. The theory of the structure of rings is developed in as general a way as feasible except for the assumption of commutativity. Since this book is intended as a companion to the second volume, to be published on algebraic geometry, the restriction is natural. Despite its dual purpose, the book is an excellent

reference book on commutative algebra.

As a reference book, it contains no problems; the function of illustrating the theory is adequately served by copious "remarks" and "examples." An occasional minute detailing of a proof is an interesting exercise but has little place in a reference book. Luckily, such tendencies are restricted to the first chapter.

Careful indexing of the extensive collection of definitions makes them readily available for reference. The definitions are introduced in the text as they are used; some are even dragged in where they might be omitted in order to make the list more complete.

The first three chapters contain "basic definitions and properties of algebraic structures." Chapter I introduces most of the basic systems: groups, rings, fields, polynomial rings, and vector spaces. Chapter II develops the theory of extension fields. Chapter III is concerned with ideals and modules and the structure of rings developable from them.

The last two chapters restrict the basic ring and develop further the structure theory of the specialized rings. Chapter IV is primarily restricted to Noetherian rings while Chapter V develops the theory of integral dependence and of Dedekind domains.

The writing is uneven in quality and at times suffers from the dry, frequently dull style so common in mathematical papers; but it has the overwhelming compensation of being almost always clear. In part the authors and in part the publisher are responsible for many small errors ranging from broken letters in the type to questionable or even incorrect statements. A random count on twenty pages yielded at least eight such flaws, none of which seriously affected the readability of the pages. Also, the format of the book is variable. In particular, Chapter V is quite different in form from the previous ones.

The book is clearly written, presenting in a sophisticated comprehensive way a connected body of material, much of which was not previously available in book form. The index of definitions alone would be sufficient to persuade this reviewer to recommend to every mathematician that he include *Commutative Algebra* in his reference library.

DONALD A. NORTON
University of California, Davis

BRIEF MENTION

Advances in Applied Mechanics, Vol. 5. Edited by H. L. Dryden, Th. von Karman, and G. Kuerti. Academic Press, New York, 1958. x+459 pp. $12.00.

The fifth volume of *Advances in Applied Mechanics* contains surveys and reviews of research in applied mechanics. The volume is divided into seven portions. These are: Supersonic Air Ejectors, Unsteady Airfoil Theory, The Theory of Distributions, Stress Wave Propagation in Rods and Beams, Problems in Hydromagnetics, Mechanics of Granular Matter, and Condensation in Super-

sonic and Hypersonic Wind Tunnels. The individual papers will undoubtedly be reviewed in appropriate journals. We call this volume to our readers' attention.

Algebraic Geometry and Topology. A symposium in honor of S. Lefschetz. Ed. by R. H. Fox, D. C. Spencer, and A. W. Tucker. Princeton University Press, New Jersey, 1957. viii+399 pp. $7.50.

What more appropriate birthday present could there be for a great mathematician than a collection of contemporary research papers which have developed from his own basic work? The individual papers in this volume will undoubtedly be reviewed in appropriate journals; indeed, several of them have already been so reviewed. We commend this collection to readers interested in mathematics in general as well as those interested in algebraic geometry and topology.

Algebra. By W. L. Ferrar. Oxford University Press, New York, 1958. vii+220 pp. $2.80.

Professor Ferrar has added a chapter on latent vectors to his 1941 textbook on determinants, matrices, and algebraic forms.

Analytic Geometry of Three Dimensions. By George Salmon. Seventh Edition. Chelsea, New York, 1958. xxiv+470 pp. $4.95.

The printing of this volume, which was already in its fourth edition in 1882, later revised by Reginald A. P. Rogers in 1911, and the revision re-edited into the (current) seventh edition by C. H. Rowe in 1927 probably sets some sort of record for longevity in 19th and 20th century books.

Calculus of Finite Differences. By George Boole. Fourth Edition. Chelsea, New York, 1958. xii+336 pp. $4.95.

Another old-timer, Boole's 1860 treatise, revised somewhat by John F. Moulton in 1872.

College Mathematics. By Kaj L. Nielsen. Barnes and Noble College Outline Series, New York, 1958. xviii+302 pp. $1.95.

This outline, keyed to the general course intended for students who do not plan to continue the study of mathematics, contains fifteen pages of sample examination questions along with answers thereto.

Modern Computing Methods. N. P. L. Staff. Philosophical Library, New York, 1958. vi+129 pp. $8.75.

These notes are based on lectures from a course on "Computers for Electrical Engineering Problems" presented to representatives of industrial firms. Perhaps the best feature of the volume is the twelve-page annotated bibliography (before 1956) which is appended.

Notes on Analog-Digital Conversion Techniques. Edited by Alfred K. Susskind. Wiley, New York, 1957. x+410 pp. $10.00.

These notes, the results of intensive one-week courses presented in the summers of 1956 and 1957 at Massachusetts Institute of Technology, present the subject matter for a practicing engineer having only moderate sophistication. In addition to an introductory chapter, which discusses systems in general, there are chapters on Sampling and Quantizing, Codes, Digital Circuits, Coding and Decoding Techniques for Electrical Signals, Coding and Decoding Techniques for Translational and Angular Motion, Design of a Digital Instrumentation System, and Optical Coders for DFTI.

Basic Geometry. By George D. Birkhoff and Ralph Beatley. Chelsea, New York, 1958. 294 pp. $3.95.

It seems most appropriate that this book, first published by the authors in an experimental edition in 1933 and later published by Scott, Foresman and Company in 1940–41, be reprinted at this time. Certainly anyone interested in teaching geometry or in writing a text on geometry could well spend a few hours studying this book.

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

MAGAZINES FOR FRIENDSHIP, INC.

Please do not throw away this issue. If you do not regularly file or pass your copies on after reading them, why not participate in the Magazines for Friendship Program? This nonprofit organization will provide you with selected names of foreign scholars, teachers, universities and libraries eager to receive learned U. S. publications, even old ones. For complete details, please send a stamped, self-addressed envelope to Magazines for Friendship, Occidental College, Los Angeles 41, California.

VISITING ASSOCIATESHIPS IN TEST DEVELOPMENT

Two Visiting Associateships in Test Development are being offered to secondary school or college teachers by the Educational Testing Service, one in Science and one in Mathematics. The appointments will be for July and August, 1959. The Associates will work primarily on tests at the college-entrance and higher levels. They will analyze existing tests and work on planning new ones. The stipend is $700 plus transportation to and from Princeton. Application forms must be submitted by February 27, 1959. All inquiries should be addressed to Mrs. W. Stanley Brown, Test Development Division, Educational Testing Service, 20 Nassau Street, Princeton, New Jersey.

NSF SUMMER INSTITUTES FOR MATHEMATICS AND STATISTICS

The National Science Foundation has announced the following institutes in the summer of 1959 for college and high school teachers of mathematics. Inquiries about a particular institute should be sent to the Director named for that institute. Unless otherwise indicated, the Director is located at the same institution as the institute. The group of teachers for whom an institute is intended is given following the dates according to the following code: C—College, JC—Junior College, HS—High School, JHS—Junior High School, SHS—Senior High School.

Mathematics

Arizona State University, Tempe, June 22–July 31: HS. Lloyd L. Lowenstein.

University of Arizona, Tucson, June 8–July 18: JHS. Millard G. Seeley.

Loyola University, Los Angeles, Calif., June 22–July 31: J&SHS. B. R. Wicker.

San Jose State College, San Jose, Calif., June 22–July 31: HS. Max Kramer.

University of Santa Clara, Santa Clara, Calif., June 22–August 1: HS. Irving Sussman.

University of California, Berkeley, June 22–July 31: J&SHS&C. Frantisek Wolf.

University of California, Los Angeles, July 6–August 28: C. C. B. Tompkins.

University of Southern California, Los Angeles, June 22–July 31: C. D. Victor Steed.

Catholic University of America, Washington, D. C., June 29–August 7: SHS. Raymond W. Moller.

Georgetown University, Washington, D. C., July 6–August 14: SHS. Malcolm W. Oliphant.

Eastern Illinois University, Charleston, Ill., June 15–August 7: JHS-HS. Lawrence A. Ringenberg.

Knox College, Galesburg, Ill., July 6–August 14: HS. Rothwell Stephens.

Northwestern University, Evanston, Ill., June 22–August 15: J&SHS. E. H. C. Hildebrandt.

Ball State Teachers College, Muncie, Ind., July 20–August 21: HS-C. Charles F. Brumfiel.

Indiana University, Bloomington, June 29–August 7: SHS. Mrs. Marie S. Wilcox, Thomas Carr Howe High School, Indianapolis, Ind.

University of Notre Dame, Notre Dame, Ind., June 19–August 4: J&SHS. Arnold E. Ross.

Purdue University, Lafayette, Ind., June 8–July 31: SHS. M. Wiles Keller.

Purdue University, Lafayette, Ind., June 8–July 31: J&SHS. G. N. Wollan.

Drake University, Des Moines, Iowa, June 8–July 17: J&SHS. Basil E. Gillam.

Iowa State College, Ames, June 8–July 18: C. J. A. Greenlee.

State University of Iowa, Iowa City, June 22–August 2: SHS. Lloyd A. Knowler.

Fort Hays Kansas State College, Hays, Kan., June 3–July 30: J&SHS. W. Toalson.

Kansas State College, Manhattan, June 15–August 7: HS. Leonard E. Fuller.

University of Kansas, Lawrence, June 8–August 1: SHS-C. G. Baley Price.

Washburn University of Topeka, Topeka, Kan., June 10–August 5: JHS. Laura Z. Greene.

Bowdoin College, Brunswick, Me., June 29–August 8: SHS. Dan E. Christie.

University of Maine, Orono, July 6–August 14: SHS. S. H. Kimball.

University of Maryland, College Park, June 22–July 31: JHS. John Brace.

Boston College, Chestnut Hill, Mass., July 6–August 14: J&SHS. Rev. Stanley J. Bezuszka, S.J.

Clark University, Worcester, Mass., June 28–August 14: SHS&JC. Charles T. Bumer.

College of the Holy Cross, Worcester, Mass., June 29–August 7: J&SHS. Rev. Raymond J. Swords, S.J.

University of Massachusetts, Amherst, June 29–August 14: J&SHS. Robert W. Wagner.

Central Michigan College, Mt. Pleasant, Mich., June 22–July 31: JHS-HS. Lester H. Serier.

Eastern Michigan College, Ypsilanti, Mich., June 22–July 31: SHS. Robert S. Pate.

University of Michigan, Ann Arbor, June 29–August 8: SHS. Bernard A. Galler.

Wayne State University, Detroit, Mich., June 22–August 14: J&SHS. Karl W. Folley.

Western Michigan University, Kalamazoo, Mich., June 22–July 31: SHS. Charles H. Butler.

St. Louis University, St. Louis, Mo., June 15–July 24: JHS-HS. Francis Regan.

Southwest Missouri State College, Springfield, Mo., June 15–July 24: JHS. Carl V. Fronabarger.

Montana State College, Bozeman, July 20–August 21: J&SHS. Adrien L. Hess.

Montana State University, Missoula, June 29–August 7: C. Frederick H. Young.

Princeton University, Princeton, N. J., June 29–August 7: HS&C. J. A. Farrington, Jr.

Rutgers, the State University, New Brunswick, N. J., June 29–August 7: J&SHS. Emory P. Starke.

Montclair State College, Upper Montclair, N. J., June 29–August 7: J&SHS. Max A. Sobel.

University of New Mexico, Albuquerque, June 20–August 14: J&SHS. Frank C. Gentry.

Brooklyn College, Brooklyn, N. Y., July 6–August 7: J&SHS. Carroll W. Grant.

University of Buffalo, Buffalo, N. Y., July 6–31: HS. Harriet F. Montague.

Teachers College, Columbia University, New York, July 6–August 14: SHS. Howard F. Fehr.

Hamilton College, Clinton, N. Y., June 29–August 22: J&SHS. Brewster H. Gere.

Hunter College, New York, July 1–August 8: J&SHS. Jewell Hughes Bushey.

New York State College for Teachers, Albany, N. Y., June 29–August 8: J&SHS. Edgar W. Flinton.

University of Rochester, Rochester, N. Y., June 29–August 8: SHS. William A. Fullagar.

Syracuse University, Syracuse, N. Y., June 29–August 8: J&SHS. Robert B. Davis.

Duke University, Durham, N. C., June 15–July 24: C. J. J. Gergen.

Baldwin-Wallace College, Berea, Ohio, June 22–July 31: SHS. Dean L. Robb.

Case Institute of Technology, Cleveland, Ohio, June 21–July 31: J&SHS. Paul E. Guenther.

University of Cincinnati, Cincinnati, Ohio, June 26–August 7: SHS. H. David Lipsich.

Kent State University, Kent, Ohio, June 22–August 14: SHS. Kenneth B. Cummins.

Oberlin College, Oberlin, Ohio, June 15–August 7: SHS. Wade Ellis.

Oklahoma State University, Stillwater, June 15–July 25: J&SHS-C. James H. Zant.

Oregon State College, Corvallis, June 22–August 14: J&SHS. Albert R. Poole.

University of Oregon, Eugene, June 22–August 15: J&SHS. A. F. Moursund.

Portland State College, Portland, Ore., July 13–August 28: HS. Robert W. Rempfer.

Reed College, Portland, Ore., June 20–August 14: J&SHS-JC. Burrowes Hunt.

Lehigh University, Bethlehem, Pa., June 22–August 1: SHS. Clarence A. Shook.

University of Pittsburgh, Pittsburgh, Pa., June 22–August 14: HS. John C. Knipp.

Seton Hill College, Greensburg, Pa., June 29–August 7: J&SHS. Sister Mary Thaddeus.

Catholic University of Puerto Rico, June 22–July 31: J&SHS. K. C. Schraut, University of Dayton, Dayton 9, Ohio.

University of Puerto Rico, Mayagüez, Puerto Rico, June 8–July 17: J&SHS. Mariano García.

University of South Carolina, Columbia, June 15–August 10: HS-C. W. L. Williams.

Memphis State University, Memphis, Tenn., June 8–July 10: SHS. H. S. Kaltenborn.

Vanderbilt University, Nashville, Tenn., June 8–July 31: J&SHS. E. Baylis Shanks.

East Texas State College, Commerce, Tex., June 3–July 14: SHS. Roy N. Jervis.

Our Lady of the Lake College, San Antonio, Tex., June 3–July 16: H.S. Sister M. Laetitia Hill.

Southern Methodist University, Dallas, Tex., July 15–August 25: J&SHS. Joe P. Harris, Jr.

University of Vermont, Burlington, June 29–August 14: SHS. N. James Shoonmaker.

State College of Washington, Pullman, June 15–August 7: SHS. Sidney G. Hacker.

Western Washington College of Education, Bellingham, Wash., June 22–August 21: H.S. Harvey M. Gelder.

University of Wyoming, Laramie, June 15–August 7: C. W. Norman Smith.

Statistics

North Carolina State College, Raleigh, June 8–July 17: C. F. E. McVay.

Oklahoma State University, Stillwater, June 8–July 31: C. Carl E. Marshall.

University of Wyoming, Laramie, June 15–August 7: C. Edward C. Bryant.

NATIONAL REGISTER OF SCIENTIFIC AND TECHNICAL PERSONNEL

On behalf of the Mathematical Association of America and various other mathematical organizations, the American Mathematical Society is assembling and maintaining a register of mathematicians and mathematical scientists. The mathematics register is a section of the National Register of Scientific and Technical Personnel, which is an official responsibility of the National Science Foundation. The purpose of the Register is to provide up-to-date information on the scientific manpower resources of the United States.

As a result of the splendid cooperation accorded to the project by most of the mathematicians and mathematical scientists who have received questionnaires to fill in, the mathematical section of the Register is now remarkably complete. However, there are still a few gaps to be filled in.

If you have received a National Register questionnaire from the American Mathematical Society, won't you please fill it in now and send it to the Headquarters Offices of the Society at 190 Hope Street, Providence 6, Rhode Island?

If you have never received a questionnaire and feel that you are qualified for inclusion in the Register, please drop us a note to that effect at the above address.

PERSONAL ITEMS

Professor L. W. Cohen of the University of Maryland was the representative of the Association at the annual meeting of the American Council on Education held in Chicago on October 9 and 10, 1958.

Dean Mina Rees of Hunter College represented the Association at the Inauguration of President H. W. Stoke of Queens College, New York, on Wednesday, October 22, 1958.

Brigham Young University: Associate Professor H. J. Fletcher has been appointed Chairman of the Mathematics Department; Assistant Professor Donald Robinson has been promoted to Associate Professor.

The Florida State University: Associate Professor E. P. Miles, Jr., Alabama Polytechnic Institute, has been appointed Associate Professor; Dr. John Greever, University of Virginia, Dr. Gabriel Margulies, Indiana University, Dr. L. L. Lasman, North Carolina State College, and Dr. M. F. Tinsley, Ohio State University, have been appointed Assistant Professors; Mr. C. A. Brown and Mr. B. L. Sanders have been appointed Instructors.

Georgetown University: Dr. F. G. Asenjo, University De La Plata, Argentina, Dr. J. E. LeBel, University of Toronto, and Mr. J. I. Hincke, United States Army, have been appointed to the Mathematics Faculty.

University of Georgia: Professor M. K. Fort, Jr. has been awarded an Alfred P. Sloan Fellowship for 1958–59 and 1959–60; Associate Professor M. L. Curtis has been promoted to Professor; Dr. R. P. Hunter and Dr. J. J. Andrews have been appointed Assistant Professors.

Rutgers, The State University: Assistant Professors Solomon Leader and K. C. Wolfson have been promoted to Associate Professors; Mr. W. R. Jones has been promoted to Instructor; Dr. J. M. Danskin, Jr., Institute for Advanced Study, has been appointed Assistant Professor; Mr. Terence Butler, Indiana University, Mr. M. J. Greenberg, Princeton University, Dr. P. E. Martin, Harvard University, and Mr. Stephen Weingram, Princeton University, have been appointed Instructors.

State College of Washington: Dr. Helmut Schaefer of Mainz, Germany, has been appointed Associate Professor; Dr. T. R. Jenkins, Lockheed Aircraft Corporation, and Assistant Professor T. A. Newton, Colorado State University, have been appointed Assistant Professors; Dr. Tanjiro Okubo of the National Defense Academy, Yokosuda, Japan, has been appointed Visiting Assistant Professor for the present academic year.

University of Texas: Associate Professor W. T. Guy, Jr. has been appointed Acting Chairman of the Department of Mathematics; Assistant Professor F. N. Edmonds, has been promoted to Associate Professor and is on a research leave for the academic year 1958–59; Assistant Professor E. J. Prouse has been promoted to Associate Professor; Dr. Ben Fitzpatrick, Jr. has been promoted to Assistant Professor; Dr. D. M. Young, Ramo-Woolridge, Los Angeles, has been appointed Professor and Director of the Computing Center.

Mathematics Research Center, United States Army: Dr. R. N. Buchal, New York University, Dr. H. F. Bueckner, General Electric Company, Schenectady, New York, and Dr. A. Ghaffari, National Bureau of Standards, Washington, D. C., have joined the staff; Dr. Miklos Hetenyi, on leave from Northwestern University, began an appointment at the Center and also holds a Visiting Professorship in Mechanics in the College of Engineering at the University of Wisconsin; Dr. J. A. Nohel, on leave from Georgia Institute of Technology, and Dr. A. L. Rabenstein, Massachusetts Institute of Technology, will spend a year at the Center; Dr. C. H. Wilcox, on leave from California Institute of Technology, and Dr. R. G. D. Steel, on leave from Cornell University, hold visiting appointments.

Mr. John Abramowich, University of California, Berkeley, has been appointed Instructor at the University of British Columbia, Vancouver.

Mr. R. J. Andree, Oklahoma State University, has accepted a position as Mathematical Analyst with Lockheed Missile Systems Division, Palo Alto, California.

Associate Professor I. L. Battin, Drew University, has been appointed Professor at New Jersey State Teachers College, Trenton.

Mr. Ralph Beals, University of Kentucky, is now a Teaching Assistant at Northwestern University.

Professor R. F. Bell, on leave from Eastern Washington College of Education, is a Lecturer at the University of Michigan.

Associate Professor Barney Bissinger, Lebanon Valley College, is spending the year on leave with the Statistical Research Group at Princeton University under a National Science Foundation Science Faculty Fellowship.

Mr. T. H. Blackburn, Case Institute of Technology, has been appointed Professor and Chairman of the Department of Mathematics at Lenoir Rhyne College.

Mr. K. Z. Bradford, University of Oklahoma, has accepted a position as Physicist with Hughes Aircraft Company, Culver City, California.

Professor J. A. Brown, State University of New York, Teachers College at Oneonta, has been appointed Associate Professor at the University of Delaware.

Mr. J. R. Brown, Oregon State College, is a University Fellow at Yale University Graduate School.

Mr. C. M. Bruen, I.B.M. Corporation, has been promoted to Associate Mathematician with I.B.M. Scientific Computation Laboratory, Endicott, New York.

Professor Herbert Busemann, on leave from the University of Southern California, is a Visiting Professor at Harvard University.

Mr. B. R. Buzby, Indiana University, has accepted a position as Research Mathematician with Metals Research, Electro Metallurgical Company, Niagara Falls, New York.

Mr. A. J. Carlan, Hoffman Semiconductor Division, Evanston, Illinois, is now a Fellow at the Mellon Institute.

Mr. R. K. Clark, Texas Christian University, has been appointed Teaching Assistant at the University of California, Berkeley.

Mr. W. L. Congleton, Bell Telephone Laboratories, Murray Hill, New Jersey, has accepted the position of Electrical Engineer with Sylvania Electrical Products, Needham, Massachusetts.

Dr. S. D. Conte has been promoted to Manager of the Mathematical Analysis Department of the Space Technology Laboratories, Los Angeles, California.

Associate Professor Byron Cosby, Jr., State University of Iowa, has been appointed Professor of Actuarial Science at the University of Texas.

Associate Professor C. H. Cunkle, Dickinson College, has accepted a position as Research Mathematician with Cornell Aeronautical Laboratory, Inc., Buffalo, New York.

Mr. Charles Drescher, Western Electric Company, New York City, is now a Member of the Technical Staff of the Ramo-Wooldridge Corporation, Los Angeles, California.

Miss Ann Farek, Laredo Junior College, has been appointed Instructor at the Texas College of Arts and Industries.

Mr. W. E. Felling, Parks College of Aeronautical Technology, has accepted a position as Research Scientist with the McDonnell Aircraft Corporation, St. Louis, Missouri.

Mr. W. E. Ferguson, on leave from Newton High School, Newtonville, Massachusetts, is a Visiting Lecturer at the University of Illinois, 1958–59.

Mr. H. E. Fleming, Harpur College, is now a Teaching Assistant at the University of Maryland.

Dr. Abraham Franck, Senior Mathematician with Engineering Research Associates of Remington Rand, St. Paul, Minnesota, has been promoted to Manager, Systems and Mathematics Research.

Mr. D. A. Freedman, McGill University, Montreal, is now a Rand Fellow in Mathematical Statistics, Princeton University.

Mr. J. D. Gilbert, Alabama Polytechnic Institute, has been appointed Assistant Professor at Louisiana Polytechnic Institute.

Associate Professor W. M. Gilbert, on leave from Iowa State College, is a Visiting Fellow at Princeton University.

Associate Professor E. H. Gilmore, Oklahoma State University, has been appointed Assistant Professor at Texas Technological College.

Mr. M. L. Glasser, University of Miami, has been appointed Research Assistant at Carnegie Institute of Technology.

Mr. J. B. Goebel, University of Oregon, has been appointed Teaching Fellow at Oregon State College.

Professor M. O. Gonzalez, University of Havana, has been appointed Professor at the University of Alabama.

Mr. William Granet, Office of Statistical and Research Services, Boston University, has been appointed Director of the Computing Center, Oklahoma State University.

Mr. J. R. Hanne, Dartmouth College, is now a Teaching Fellow at the University of Michigan.

Dr. S. M. Harmon, University of California, Los Angeles, has been appointed Assistant Professor at Fresno State College.

Mr. F. H. Hildebrand, Kent State University, is an NSF Fellow at the University of Illinois for the academic year 1958–59.

Mr. Raymond Hirschkop, Pratt Institute, is now on the Research Staff at Lincoln Laboratories, Massachusetts Institute of Technology.

Assistant Professor T. C. Holyoke, Miami University, Ohio, has been appointed Associate Professor at Antioch College.

Mr. Norman Johnson, Geneva College, is now a Teaching Fellow at the University of Toronto.

Assistant Professor J. B. Johnston, University of Kansas City, has been appointed Assistant Professor at the University of Kansas.

Mr. A. P. Jones, U. S. Army Computing Laboratory, is now a Mathematician with the National Heart Institute, Bethesda, Maryland.

Assistant Professor R. P. Kelisky, University of Texas, has accepted the position of Associate Mathematician with I.B.M. Research Center, Yorktown Heights, New York.

Dr. Naoki Kimura, Tokyo Institute of Technology, Japan, has been appointed Acting Assistant Professor at the University of Washington.

Mr. G. D. King, Brevard College, has been appointed Assistant Professor at Clemson College.

Associate Professor O. M. Klose, Seattle University, has been appointed Associate Professor at Humboldt State College.

Dr. G. R. Lehner, University of Wisconsin, has been appointed Instructor at the University of Maryland.

Mr. R. J. Libera, University of Massachusetts, has been appointed Teaching Assistant at Rutgers, The State University.

Associate Professor A. C. Lindberg, Dana College, has been appointed Assistant Professor at Mankato State College.

Mr. W. M. Lowney, University of Notre Dame, has accepted the position of Scientist with Lockheed Missiles System Division, Palo Alto, Califfornia.

Mr. J. C. Mairhuber, University of Rochester, has been appointed Assistant Professor at the University of New Hampshire.

Mr. Dale Maness, Chance Vought Aircraft Inc., Dallas, Texas, has been appointed Professor and Chairman of the Department of Mathematics at Howard Payne College.

Mr. H. T. Mathews, Georgia Institute of Technology, has been appointed Associate in Mathematics at Louisiana State University in New Orleans.

Dr. Elliott Mendelson, Harvard University, has been appointed Instructor at Columbia University.

Dr. Mabel D. Montgomery, University of Buffalo, has been appointed Associate Professor at State University of New York, College for Teachers at Buffalo.

Dr. H. S. Moredock, Jr. has been appointed Professor at Sacramento State College.

Mr. D. L. Muench, Weapon Systems Laboratory, Aberdeen Proving Ground, Maryland, has been appointed Graduate Assistant at St. John's University, New York.

Mr. M. J. Pascual has been appointed Assistant Professor at Siena College.

Dr. M. J. Poliferno, Williams College, has been appointed Instructor at Trinity College, Connecticut.

Dr. G. Y. Rainich, on leave from the University of Michigan, is a Visiting Professor at the University of Notre Dame.

Mr. C. R. Riehm, University of Toronto, has been appointed Assistant in Research at Princeton University.

Mr. E. L. Roetman, Indiana University, has been appointed Teaching Assistant at the University of Minnesota.

Mr. H. D. Ruderman, Bronx High School of Science, New York, has been appointed Chairman of the Mathematics Department of Hunter College High School.

Mr. P. T. Rygg has been appointed Instructor at Iowa State College.

Mr. R. T. Sandberg, University of Buffalo, has been appointed Instructor at the University of Arizona.

Dr. B. D. Seckler, Brooklyn College, has been appointed Assistant Professor at Pratt Institute of Technology.

Mr. Anthony Sepan, Temple University, has been appointed Teaching Assistant at the University of Wisconsin.

Mr. W. T. Sharp, on leave from Atomic Energy of Canada Ltd., is an Instructor at Princeton University for the academic year 1958–59.

Dr. M. W. Shelly II, Laboratory of Aviation Psychology, Columbus, Ohio, has been appointed Psychologist at the Office of Naval Research, Deerfield, Illinois.

Mr. M. G. Shults, Northern Oklahoma Junior College, has been appointed Assistant Professor at Panhandle Agricultural and Mechanical College.

Dr. H. F. Simmons, Iowa State College, has been appointed Assistant Professor at California State Polytechnic College.

Mr. R. H. Sprague, University of Kentucky, has been appointed Assistant Professor at New Mexico Agricultural and Mechanical College.

Mr. H. R. Stevens, University of Buffalo, has been appointed Part-Time Instructor at Duke University.

Dr. J. R. Stock, Union Carbide and Carbon Corporation, New York, has been appointed Engineer at the Stock Equipment Company, Cleveland, Ohio.

Mrs. Doris S. Stockton, Brown University, has been appointed Assistant Professor at the University of Massachusetts.

Dr. Karl Stromberg, University of Washington, is now Postdoctoral Research Associate at Yale University.

Mr. L. R. Tappan, University of Michigan, has been appointed Assistant Professor at Nicholls State College.

Mr. J. D. Thomas, University of Oklahoma, has accepted a position as Staff Member at Los Alamos Scientific Laboratory.

Mr. C. C. Thompson, Oklahoma State University, has been appointed Assistant Instructor at Yale University.

Dr. R. N. Townsend, University of Illinois, has been appointed Assistant Professor at San Jose State College.

Dr. R. N. Walter, Manhattan High School of Aviation Trades, New York, has been appointed Professor at Paterson State College.

Professor S. E. Warschawski, on leave from the University of Minnesota, has been appointed Visiting Professor at the University of California at Los Angeles for the academic year 1958–59.

Mr. D. B. Wells, University of Kentucky, has been appointed Instructor at Western Carolina College.

Dr. G. M. Wing, Los Alamos Scientific Laboratory, has been appointed Associate Professor at the University of New Mexico.

Assistant Professor Mildred M. Sullivan, Queens College, New York, died on August 30, 1958. She was a member of the Association for eleven years.

Professor G. C. Vedova, Newark College of Engineering, died on September 5, 1958. He was a member of the Association for twenty years.

Professor Emeritus A. H. Wilson, Haverford College, died on September 22, 1958 at the age of 86 years. He was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 204 persons have been elected to membership by the Board of Governors on applications duly certified.

JEAN E. ABELL, A.A. (California, Davis) Student, University of California, Berkeley.

JACK A. ADJAMI, Student, Brooklyn College.

ARNOLD C. AHLIN, B.A. (Eastern Washington) Mathematician, Lincoln Lab.

ETHAN O. ALLEN, M.A. (Cornell) Chief, Math. Analysis Section, Bell Aircraft Corp.

JOE H. ALLEN, B.S. (Middle Tennessee S.C.) Teacher, Donelson High School, Tennessee.

BROTHER ANDREW, B.S. (Fordham) Head of Dept., St. Anthony's High School, Smithtown, New York.

JERRY G. BAILS, M.A. (Kansas City) Instr., University of Kansas.
CAMERON C. BARR, JR., B.S. (Roanoke) Instr., Roanoke College.
LT. COL. LASZLO BERES, Asst. Military and Air Attache, Hungarian Legation, Washington, D. C.
BERNARD BERGER, Student, University of Pittsburgh.
MARTIN F. BERMAN, B.A. (U.C.L.A.) Asst. Mathematician, System Development Corp.
JOSEPH R. BIENAS, Student, Indiana Technical College.
BERNARD E. BJORK, B.S. (St. Cloud S.C.) Instr., Falls High School, International Falls, Minnesota.
MARTIN BLUMBERG, M.S. in E.E. (Stanford) Eng. Specialist, Electronic Defense Lab., Mountain View, California.
DAVID B. BOOTHBY, M.A. (Massachusetts) Instr., University of Vermont.
MARK BRIDGER, Student, Bronx High School of Science, New York.
JAMES D. BRISTOL, M.A. (Western Reserve) Teacher, Shaker Heights Senior High School, Ohio.
CARL N. BROOKS, Student, University of Maine.
RICHARD A. BROWN, M.S. (S.U. of Iowa) Asso. Professor, Bluefield State College.
ROBERT F. BROWN, A.B. (Harvard) Mathematician, Bureau of Supplies and Accounts, U. S. Navy.
O. LEXTON BUCHANAN, JR., M.Ed. (Georgia) Asst., University of Kansas.
J. BERNARD BUCKY, B.S. (Queens, New York) Grad. Student, New York University.
MRS. JOYCE M. BURCHENAL, B.S. (Michigan Coll. of Mining & Tech.) Head of Dept., L'Anse Creuse High School, Mt. Clemens, Michigan.
NEWBERN W. BUSH, M.S. (Auburn) Head of Dept., Jacksonville State College.
MAGDELHAYNE F. BUTEAU, M.S. (Montreal) Professor, St. Joseph Teachers College.
RICHARD E. CADY, M.S. (West Virginia) Instr., West Virginia University.
HUGH G. CAMPBELL, M.S. (Florida S.U.) Asso. Professor, Virginia Polytechnic Institute.
JAY O. CASEY, Student, Oklahoma State University.
BERNARD C. CHALOUPKA, B.S. (Creighton) Engr., Martin Co.
GEORGE L. CHANEY, M.S. (Kansas S.T.C.) Instr., Coffeyville College.
RICHARD M. CHESSER, Student, Central State College, Oklahoma.
JOSEPH B. CHICCARELLI, M.A. (Boston Coll.) Asst. Professor, Fordham University.
KEEWHAN CHOI, B.S. (Case I.T.) Grad. Student, Massachusetts Institute of Technology.
ALICE M. CHRISTIANSEN, M.S. (Northwestern) Instr., University of Illinois.
MARSHALL B. COHEN, B.S. (Wisconsin) Mathematician, Cornell Aeronautical Lab.
MARSHALL M. COHEN, Student, University of Chicago.
HARVEY COHN, Ph.D. (Harvard) Head of Dept., University of Arizona.
ALEX C. COMPTON, Student, Columbia University.
LEWIS H. COON, M.S. (Indiana) Asst. Professor, Southwestern State College.
DENSEL G. CORBIN, B.S. (Oklahoma) Res. Engr., Jersey Production Research Center.
PATRICIA A. COUGER, M.A. (Wichita) Instr., Wichita University.
WILLIAM D. CRAVEN, B.S. (Pennsylvania S.U.) Grad. Student, Pennsylvania State University
ARTHUR CRAWSHAW, Engr., Washington State Highway Dept., Seattle, Washington.
STERLING C. CRIM, M.A. (Peabody) Asst. Professor, West Georgia College.
JOHN P. CUELLAR, JR., B.S. (St. Mary's, Texas) Junior Res. Engr., Southwest Research Institute.
ETHA A. DAHLGREN, B.A. (Coll. of Mt. St. Vincent) Asst. Supervisor, New York City Dept. of Welfare.
BENJAMIN DAMSKY, B.S. (Glassboro S.T.C.) Teacher, Roosevelt Junior High School, New Brunswick, New Jersey.
STEPHEN P. DILIBERTO, Ph.D. (Princeton) Asso. Professor, University of California, Berkeley.
RUSSELL D. F. DINEEN, M.A. (Delaware) Teacher, Board of Public Schools, Wilmington, Delaware
LESLIE A. DWIGHT, Ph.D. (Geo. Peabody) Head of Dept., Southeastern State College.
BENJAMIN F. EDWARDS, JR., B.S. (S.F. Austin S.C.) Senior Engr., Chance Vought Aircraft Inc.
VALMA Y. EDWARDS, B.S. (Florida S.U.) Grad. Asst., Florida State University.
MRS. FLORENCE L. ELDER, M.A. (Columbia) Chairman of Dept., W. Hempstead Junior-Senior High School, New York.
EDWIN L. ELLIS, Ballistic Research Lab., Aberdeen Proving Ground, Maryland.
JOHN C. ESTY, JR., M.A. (Yale) Asso. Dean and Instr., Amherst College.
CLEMENT E. FALBO, M.A. (Texas) Instr., San Antonio College.
LT. NEAL A. FARMER, B.S. (Tusculum Coll.) United States Marine Corps.
JOHN D. FERRUCCI, M.S. (Kansas S.C.) Instr., Teachers College of Connecticut.
WILLIAM B. FLOYD, M.S. (Emory) Senior Res. Engr., Melpar Inc..
JOHN A. FLYNN, Engr., Martin-Denver.
JOHN C. FOGARTY, Student, Harvard University.
DONALD FORBES, Asso. Editor and Senior Economist, Econometric Institute.
STANLEY P. FRANKLIN, Student, Memphis State University.
RICHARD B. FREY, M.S. (Illinois) High School Instr., Des Plaines, Illinois.
HERMAN P. FRIEDMAN, M.A. (Brooklyn) Senior Mathematician, Bulova Research and Development Labs.
KURT O. FRIEDRICHS, Ph.D. (Göttingen) Professor, New York University; Asso. Director, Institute of Mathematical Sciences.
PHILLIP H. C. FUNG, Student, Idaho State College.
JOHN W. GAMMILL, B.S. (Delta S.C.) Instr., Mississippi State College.
CHARLES E. GARDNER, JR., Student, Oklahoma State University.
JOHN B. GARNER, Res. Asst., Carleton College.
JOHN M. GARY, Ph.D. (Michigan) Instr., California Institute of Technology.
ROBERT M. GASPER, Student, Butler University.
JOHN H. GAY, Ph.D. (Columbia) Dean of Instruction, Cuttington College, Monrovia, Liberia.
JOSEPH B. GEISER, M.S. (Chicago) Teacher, University of Chicago Lab. School.
CALVIN W. GILLARD, B.S. (U.C.L.A.) Engr., Sylvania Electric Products Inc.; Grad. Student, Stanford University.
ABRAHAM GOLDRICH, B.A. (Montreal) Lecturer and Grad. Student, McGill University.
PAUL J. GOWEN, Student, Georgetown University.
REV. PHILIP M. GRIMES, M.S. (Loyola) Instr., Servite Seminary
NORMAN GROSSMAN, Ph.D. (New York) Chief Equipment Engr., Republic Aviation Corp.
TAN GRZESIK, Student, University of California, Los Angeles.
EDWARD P. GUETTLER, Student, Georgetown University.
A. GLEN HADDOCK, M.S. (Oklahoma S.U.) Asst. Professor, Arkansas College.
PAUL A. HAEDER, M.A. (South Dakota) Asst. Professor, University of South Dakota.
HARRY C. HARRISON, B.S. (Texas Western) Grad, Student, University of Kansas.
WILLIAM G. HAZLETT, M.Ed. (Southwest Texas S.T.C.) Computational Engr., Chance Vought.
MRS. RUTH E. HEINTZ, B.A. (Buffalo) Instr., University of Buffalo.
MERWYN H. HEMP, Student, University of Wisconsin.
MRS. SAMMIE R. HENDRICKS, B.S. (North Texas S.C.) Aerophysics Engr., Convair.
PAUL F. HENNING, JR., M.A. (Penna. State) Mathematician, Gannett Fleming Corddry Carpenter Inc.
THOMAS L. HICKS, M.A. (Alabama) Asso. Professor, Jacksonville State College.
DONALD W. HINKKANEN, M.A. (Wisconsin) Asso. Research Engr., Boeing Airplane Co.
HENRY HIZ, Ph.D. (Harvard) Asst. Professor, Pennsylvania State University.
SHAREEN R. HODGE, Student, St. Anthony's Girls' High School, Lakewood, California.
DOROTHY M. HORN, M.A. (Drake) Instr., Grand View College.
EDWIN A. HORN, B.A. (Western S.C. of Colorado) Teacher, Reliance High School, Wyoming.
HENRY HOSEK, Jr., B.S. in Ed. (Ball S.T.C.) Teaching Asst., Ball State Teachers Coll.
CHARLES M. HUGHEY, Student, The Citadel.

JOHN D. HWANG, M.A. (California, Berkeley) Asst. Professor, Sacramento State College.
HORACE B. IRVING, Asst. Treasurer-Controller, Van Camp Hardware & Iron Co., Indianapolis, Indiana.
WILLIAM H. JOBE, B.A. (Tulsa) Grad. Student, Tulsa University.
ELGY S. JOHNSON, Ph.D. (Catholic) Instr., Public School System, Washington D. C.; Lecturer, The American University.
MRS. ROXEE W. JOLY, M.A. (Columbia) Chairman of Dept., Walton High School, New York City.
ARNOLD P. JONES, A.B. (Oberlin) Mathematician, National Heart Institute.
BILLY P. JONES, B.S. (Alabama Polytech.) Aeronautical Res. Engr., Army Ballistic Missile Agency.
KEITH KENDIG, Student, University of California, Los Angeles.
MRS. CAROL H. KIPPS, M.A. (Mills, California) Instr., Pasadena City College.
HAROLD T. KNIGHT, M.A. (Geo. Peabody) Instr., Waverly Central High School, Tennessee.
MILDRED E. KOCORNIK, B.A. (Bethany) Teacher, Nutley High School, New Jersey.
CARL KONOVE, M.A. (Montclair S.T.C.) Asso. Professor, Newark College of Engineering
ANDREW J. KORSAK, Student, University of Toronto.
HENRY E. KYBURG, JR., Ph.D. (Columbia) Asst. Professor, Wesleyan University.
JAMES H. LAMB, B.S. (Fairleigh Dickinson) Tech. Asst., Western Electric Co.
JOHN E. LEBEL, Ph.D. (Toronto) Asst. Professor, Georgetown University.
ROMUALD G. LESAGE, M.S. (Vermont) Asso. Professor, State University Teachers College, Plattsburgh, New York.
JOEL LEVY, M.A. (Johns Hopkins) Mathematician, U. S. Navy.
DONALD R. LITTLE, B.S. (Pittsburgh) Senior Mathematician, Curtiss-Wright Corp.
MICHAEL W. LODATO, A.B. (Colgate) Grad. Student, University of Rochester.
ROSS LOMANITZ, Ph.D. (Cornell) Norman, Oklahoma.
JESSE L. LONG, M.A. (Southern Methodist) Asso. Professor, Midwestern University.
FRED J. LOTZ, B.S. (St. Josephs) Mathematician, Holloman Air Force Base.
MRS. JOSEPHINE H. MAGNIFICO, M.Ed. (Virginia) Grad. Student, University of Virginia.
WILLIAM A. MARKLEY, JR., M.Litt. (Pittsburgh) Instr., Mount Union College.
JOSEPH MAYER, Ph.D. (Columbia) Professor and Head of Dept., Western College.
DENNIS M, MCCASKILL, M.S. (North Carolina Coll., Durham) Principal, Washington High School, Shelby, North Carolina.
MICHAEL M. MENKE, Student, Webster Groves High School, Missouri.
G. H. MILLER, Ph.D. (Southern California) Assoc. Professor, Western Illinois University.
ROSE M, MILLER, M.Ed. (Vermont) Teacher, Monmouth College.
GEORGE L. MILLICAN, M.S. (Southern Methodist) Engr., Texas Instruments Inc.
BARBARA A. MOORE, M.A. (New York S.T.C.) Res. Asst., Los Alamos Scientific Lab.
RICHARD C. MORGAN, Student, Stevens Institute of Technology.
MARY E. MORRIS, M.A. (Geo. Peabody) Teacher, Webster Groves Public Schools, Missouri.
STEWART NAGLER, Student, Madison High School, Brooklyn, New York.
WILLIAM NEWMAN, Student, Brooklyn College.
JOEL NIEDELMAN, B.A. (C.C.N.Y.) Engr., Burroughs Corp.
MRS. LOIS J. NIEMANN, M.S. (Purdue) Shrewsbury, Massachusetts.
LT. ROBERT J. O'BRIEN, A.B. (Sacramento S.C.) U. S. Air Force.
REV. FRANCIS C. O'CONNOR, S.J., B.S. (Springhill) Clergy, N. Y. Province of Society of Jesus.
MRS. BARBARA L. OSOFSKY, Student, Cornell University.
CRISTINA P. PAREL, Ph.D. (Michigan) Asst. Professor, University of the Philippines.
BERNARD PELLETIER, Draftsman, Canadair Ltd.
LOUIS J. RATLIFF, JR., M.S. (S.U. of Iowa) Teacher, University High School, Iowa City, Iowa.
JOHN L. RAVESLOOT, M.S. (Northwestern) I.M.B. Research Lab.
MIRIAM M. REIK, Student, Sarah Lawrence College; Member Res. Staff. Tex McCrary Inc.
MELVIN D, REIN, M.A. (Columbia) Teacher, Pekin Community High School, Illinois.
KENNETH A. RETZER, M.Ed. (Illinois) Teacher, Saunemin High School, Illinois.
N. WAYNE RHODUS, B.S. (Ohio State) Grad. Student, Ohio State University.
ARTHUR W. ROBERTS, M.S. (Wisconsin) Instr., Morton Junior College.
GERALD S. ROGERS, Ph.D. (S.U. of Iowa) Asst. Professor, University of Arizona.
JOHN L. ROPER, Lab. Tech., Chrysler Engineering.
JACOB J. ROSENBERG, New York City.
DAVID ROSENTHAL, Student, University of Chicago.
ADOLPH ROSTENBERG, JR., M.D. (McGill) Professor of Dermatology, University of Illinois.
EUGENE E. RYGWALSKI, B.S. (Alliance) Statistician, General Electric.
ALFRED SCHILD, Ph.D. (Toronto) Professor, University of Texas.
NORMA P. SCHMID, B.A. (Hunter) Teacher, Valley Stream North High School, New York.
EDWARD SCHNEIDER, M. Fiscal Ad. (Columbus) Analytical Statistician, St. Elizabeths Hospital, Washington, D. C.
HANS W. E. SCHWERDTFEGER, Ph.D. (Bonn) Assoc. Professor, McGill University.
PHILIP D. SCOTT, Student, South Oak Cliff High School, Dallas, Texas.
ROBERT L. SCOTT, M.S. (Atlanta) Instr., Tuskegee Institute.
ROBERT D. SHORTNACY, Eng. Checker, Southern Asso. Engrs. Inc.
GERALD H. SILBERBERG, B.A. (Buffalo) Eng. Computer, Bell Aircraft Corp.
DAVID L. SILVERMAN, M.A. (U.C.L.A. Teaching Asst., University of California, Los Angeles.
SISTER M. LOUISE SCHMIDT, M.S. (Kansas S.T.C., Pittsburg) Instr., St. Mary of the Plains College.
SISTER MARIA WILLIAM WHITE, M.S. (St. John's, N.Y.) Instr., Ladycliff College.
SISTER MARIE BLANCHE, B.A. (St. Mary-of-the-Woods Coll.) Teacher, Providence High School, Chicago, Illinois.
SISTER MARY OLIVIA, B.A. (Sacred Heart Coll.) Teacher, Academia Perpetuo Socorro, Puerto Rico.
SISTER MARY SERAPHINE BENNETT, M.S. (Catholic) Asst. Professor, Mt. St. Agnes College.
WALTER A. SKALSKI, B.S. (Columbia) Development Engr., ITT Labs.
JOEL A. SMOLLER, M.S. (Ohio) Grad Asst., Purdue University.
JOHN R. STAGNER, A.A. (Pasadena C.C.) Student, University of Redlands.
WAYNE J. STANLEY, B.E. (Yale) Development Engr., Western Electric Co.
MRS. EMMA G. STANTON, M.S. (Chicago) Instr., Portland State College.
SAMUEL T. STERN, B.A. (Buffalo) Instr., University of Buffalo.
JAMES J. STOKER, Ph.D. (Technische Hochschule) Director, Institute of Mathematical Sciences.
DONALD W. STOKES, Instr., Blue Island Community High School, Illinois.
FELICIANO A. SUBANG, B.S. (Mindanao) Student and Employee, San Miguel Brewery Inc., Philippines.
MELVIN TAINITER, B.S. (Brooklyn) Aide, Fairchild Guided Missile Div.
STANLEY L. TAYLOR, A.A. (Santa Ana Jr. Coll.) Lab. Tech., Eastman Kodak Co.
MRS. ARLEEN G. THOMPSON, Student, University of Detroit.
HOWARD E. THOMPSON, M.S. (Wisconsin) Mathematician, A. O. Smith Corp.
PHILIP J. THORSON, M.S. (Purdue) Asst. Professor, Michigan State University.
PETER G. THURNAUER, Res. Asst., Carleton College.
RAYMOND D. TRAVIS, Student, Wayne State University.
ROBERT L. TRUAX, M.A. (Mississippi) Chairman of Dept., Crossett Public Schools, Arkansas.
BRUCE M. TYNDALL, M.S. (S.U. of Iowa) Instr., Elizabethtown College.
ALBINO UGGE, Ph.D. (Pavia) Professor, University of Padova, Italy.
ERNEST E. UNDERWOOD, B.A. (Montana S.U.) Instr., Northern Montana College.
GEORGE VAN ZWALENBERG, M.A. (Florida) Res. Asst., University of California, Berkeley.

RAYMOND W. VENN, M.Ed. (Wisconsin S.C., Superior) Teacher, Evanston Township High School Illinois.
JAMES D. VINEYARD, A.A. (Blackburn) Student, Blackburn College.
WILLIAM B. WALLACE, B.A. (Minnesota) Mathematician, Remington Rand Univac.
CHARLES D. WATKINS, B.A. (S.U. of Iowa) Instr., Mt. Pleasant High School, Iowa.
LYNDON J. WATSON, Jr., M.A. in Ed. (Arizona S.C., Tempe) Instr., Phoenix Union High Schools and College District, Arizona.
DAVID B. WEBSTER, M.A. (Wisconsin) Instr., Oberlin College.
LEO R. WHICHER, Student, Roosevelt University.
JOHN S. WHITE, Ph.D. (Minnesota) Engr., Minneapolis-Honeywell Regulator Co.
BOOKER T. WHITTINGTON, B.S. (Tuskegee) Teacher, Chilton County Training School, Clanton, Alabama.
ARTHUR F. WICKERSHAM, Jr., Ph.D. (California) Eng. Specialist, Sylvania Electronic Defense Lab.
DON A. WITCRAFT, B.S. (Oregon S.C.) Math. Analyst, Lockheed Aircraft Corp.
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CALENDAR OF FUTURE MEETINGS

Forty-second Annual Meeting, University of Pennsylvania, Philadelphia, Pennsylvania, January 22–23, 1959.

Fortieth Summer Meeting, University of Utah, Salt Lake City, Utah, August 31–September 3, 1959.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, University of Pittsburgh, May 2, 1959.
ILLINOIS, Millikin University, Decatur, May 8–9, 1959.
INDIANA, Valparaiso University, May 2, 1959.
IOWA, Iowa Wesleyan University, Mount Pleasant, April 17, 1959.
KANSAS, Marymount College, Salina, April 11, 1959.
KENTUCKY, Centre College of Kentucky, Danville, April, 1959.
LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 13–14, 1959.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK, Polytechnic Institute of Brooklyn, April 18, 1959.
MICHIGAN, Michigan State University of Agriculture and Applied Science, East Lansing, March 28, 1959.
MINNESOTA, University of Minnesota, Minneapolis, April 25, 1959.
MISSOURI, Lindenwood College, St. Charles, April 25, 1959.
NEBRASKA, University of Nebraska, Lincoln, April 18, 1959.
NEW JERSEY
NORTHEASTERN
NORTHERN CALIFORNIA, Stanford University, January 17, 1959.
OHIO, Miami University, Oxford, May 9, 1959.
OKLAHOMA, Tulsa University, Tulsa, Oklahoma, Spring, 1959.
PACIFIC NORTHWEST, University of Oregon, Eugene, June 19, 1959.
PHILADELPHIA
ROCKY MOUNTAIN, Utah State University of Agriculture and Applied Science, Logan, May 8–9, 1959.
SOUTHEASTERN, East Tennessee State College, Johnson City, March 20–21, 1959.
SOUTHERN CALIFORNIA, University of Redlands, March 14, 1959.
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COMPLETENESS IN TOPOLOGICAL VECTOR LATTICES*

CASPER GOFFMAN, Purdue University and University of Oklahoma

One of the early successes of the Lebesgue integral was the observation that the space $L_2(a, b)$ of square-summable functions on the interval (a, b) is complete. The initial historical interest in this fact was the discovery that it constitutes the main point in the proof that if $\{\phi_n\}$ is an orthonormal basis for $L_2(a, b)$, then if $\{a_n\}$ is any sequence of real numbers for which $\sum_{n=1}^{\infty} a_n^2 < \infty$, it follows that $\sum_{n=1}^{\infty} a_n\phi_n$ is the Fourier expansion, with respect to this basis, of a function $f \in L_2(a, b)$, and the sequence $f_n = \sum_{k=1}^{n} a_k\phi_k$ converges to f in the space $L_2(a, b)$; *i.e.*,

$$\lim_{n\to\infty} \int_a^b [f(x) - f_n(x)]^2 dx = 0.$$

For, in order to have this result, it is only necessary to show that $\{f_n\}$ is a Cauchy sequence in $L_2(a, b)$. It then follows from the completeness of $L_2(a, b)$ that the limit of $\{f_n\}$ in $L_2(a, b)$ is the required f.

Although the proof of the completeness of $L_2(a, b)$ is not difficult, its interest is emphasized by the large number of eminent mathematicians (von Neumann, Weyl, *etc.*) who wrote proofs of their own. The proof most often quoted nowadays consists in showing that if $\{g_n\}$ is a Cauchy sequence in $L_2(a, b)$ then it has a subsequence which converges almost everywhere to a function f, thus locating the precious object. It is then shown that $f \in L_2(a, b)$, and finally that f is the limit of $\{g_n\}$ in $L_2(a, b)$.

This argument applies to all L_p spaces, $p \geqq 1$, and to Orlicz spaces, which we need not define here, as well.

Köthe has defined an important class of sequence spaces and has developed their properties in a series of interesting papers, [1], [2], [3]. Dieudonné [4] has extended the definition to spaces of locally summable functions on a locally compact space X (which is, however, restricted to be the union of a countable set of compact subspaces) with a specified nonnegative Radon measure [7] on X. These spaces admit various topologies in a natural way, and are complete for all these topologies. We shall be concerned in this paper only with spaces of real functions.

A very interesting theorem has been given by Nakano [5] who showed that, in a certain kind of topological vector lattice, a sort of completeness in terms of the order relation implies topological completeness.

Now, Köthe spaces à la Dieudonné, even without the countability restriction, fall within the Nakano framework and are easily seen to satisfy his type of order completeness. If this exposition can make any claim to making a new contribution, it is the observation of this connection, which establishes the completeness property for a wider class of Köthe spaces than previously considered.

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Nakano's theorem. We merely present the definitions needed to state the theorem. The proof will not be given here. Let X be a real vector space (we assume the definition is known). Let $P \subset X$ be a subset of X such that $x, y \in P$ and $a \in R$, $a \geqq 0$, (R is the space of real numbers) implies $x+y \in P$ and $ax \in P$. In particular, the zero vector θ of X belongs to P. Let $-P$ be the set of all $x \in X$ for which $-x \in P$; we suppose that θ is the only element belonging to both P and $-P$. Then P is called a positive cone in X. P defines an order relation in X by $x \geqq y$ if and only if $x-y \in P$. A vector space X together with a positive cone $P \subset X$ is called a *vector lattice* if every pair $x, y \in X$ has a least upper bound; it is designated as $x \cup y$. It then easily follows that every $x, y \in X$ has a greatest lower bound which is designated as $x \cap y$. A vector lattice is called *conditionally complete* if every set which has an upper bound has a least upper bound. An important fact about a vector lattice X is that every $x \in X$ has a canonical representation as the difference $x = x^+ - x^-$ of two elements in P; *i.e.*, $x^+, x^- \geqq \theta$. In this representation, $x^+ = x \cup \theta$ and $x^- = -(x \cap \theta)$. For every $x \in X$, we can define the absolute value of x as $|x| = x^+ + x^-$. (General references on vector lattices where these facts and many others are proved are [6], [7]).

A *convex topological vector space* X is a vector space with a topology such that the mappings

α) $(x, y) \to x+y$ *of* $X \times X$ *into* X,
β) $x \to -x$ *of* X *into* X,
γ) $(a, x) \to ax$ *of* $R \times X$ *into* X,

are continuous, and where for every $x, y \in X$, $y \neq x$, and open G, with $x \in G$, there is an open convex H, with $H \subset G$, $x \in H$, and $y \notin H$. In such a space, the topology may also be defined in terms of certain sets, containing the identity θ, which we call balls.

We now give the definition of a convex topological vector space in this form. In a vector space X, a *ball* U is a set which is convex, symmetric, and absorbing; *i.e.*,

a) *if* $x, y \in U$ *and* $0 \leqq a \leqq 1$, *then* $ax+(1-a)y \in U$,
b) *if* $x \in U$ *then* $-x \in U$,
c) *if* $x \in X$, *then there are* $y \in U$ *and* $a \in R$ *for which* $x = ay$.

Now, the topology in a convex topological vector space is determined by a collection $\mathfrak{U}$ of balls for which

i) *if* $x \neq \theta$, *there is a* $U \in \mathfrak{U}$ *with* $x \notin U$,
ii) *if* $U, V \in \mathfrak{U}$, *there is a* $W \in \mathfrak{U}$ *with* $W \subset U \cap V$,
iii) *if* $U \in \mathfrak{U}$, *there is a* $V \in \mathfrak{U}$ *with* $V+V \subset U$.

(The set $V+V$ consists of all $x+y$ where $x, y \in V$).

A set $G \subset X$ is open if for every $x \in G$ there is a $U \in \mathfrak{U}$ with $U_x \subset G$, where U_x is the set of all $y = z+x$, $z \in U$. In other words, G is open means that G is the union of a family of translations of sets in $\mathfrak{U}$. It is a routine task to show

that the topology defined in this way satisfies conditions α), β), γ). (As a general reference for topological vector spaces we can cite only [8]).

Having defined vector lattice and convex topological vector space, we can now define *convex topological vector lattice*. Let X be a vector lattice with a set $\mathfrak{U}$ of balls satisfying i), ii), iii), and which conform to the order relation in X. Specifically,

iv) *for every* $U \in \mathfrak{U}$, *if* $x \in U$ *and* $|y| \leqq |x|$ *then* $y \in U$.

If we define a set $A \subset X$ to be *order-closed* if $x \in A$ and $|y| \leqq |x|$ implies $y \in A$ the condition iv) simply asserts that every U is order-closed. A convex topological vector lattice is a vector lattice and a topological vector space with topology given by a set $\mathfrak{U} = [U]$ of order-closed balls.

A Nakano space is a special kind of convex topological vector lattice. This involves specialization of the kind of vector lattice allowed as well as of the admissible set $\mathfrak{U}$ of balls. The vector lattice is assumed to be conditionally complete. The restriction on the set of $U \in \mathfrak{U}$ requires some discussion. A *directed set* D is a partly ordered set with order relation "$\geqq$" such that for every $\lambda, \lambda' \in D$ there is $\lambda'' \in D$ with $\lambda'' \geqq \lambda$, $\lambda'' \geqq \lambda'$. A *net* in X is a function on a directed set D with values $x_\lambda \in X$. A net is *increasing* if $\lambda' \geqq \lambda$ implies $x_{\lambda'} \geqq x_\lambda$. A set $U \subset X$ is a *Nakano ball* if it is an order-closed ball, and if for every increasing net x_λ with values in $U \cap P$, which has an upper bound in X, the least upper bound, $x = \sup_\lambda x_\lambda$, is in U. A *Nakano space* is a convex topological vector lattice which is conditionally complete as a vector lattice and for which the sets in $\mathfrak{U}$ are Nakano balls.

The theorem of Nakano connects two kinds of completeness, topological completeness and monotone completeness, notions which we now define.

In a topological vector space X, a set $S \subset X$ is *bounded* if it is absorbed by every $U \in \mathfrak{U}$. That is, if for every $U \in \mathfrak{U}$ there is an $a \in R$ such that $S \subset aU$. (For any $A \subset X$, aA means, as always, the set of all ax, $x \in A$). A Nakano space X is called *monotone complete* if every increasing net in P which is topologically bounded is order bounded. In other words, if x_λ is an increasing net in P whose value set is topologically bounded, then x_λ has an upper bound so that, since X is conditionally complete, $\sup_\lambda x_\lambda$ exists.

A net x_λ in a topological vector space is called a *Cauchy net* if for every $U \in \mathfrak{U}$ there is a ξ such that $\lambda \geqq \xi$, $\lambda' \geqq \xi$ implies $x_\lambda - x_{\lambda'} \in U$. x is the limit of a net x_λ if for every $U \in \mathfrak{U}$ there is ξ such that $\lambda \geqq \xi$ implies $x - x_\lambda \in U$. The space X is *topologically complete* if every Cauchy net in X has a limit.

We may now state

THEOREM 1. (Nakano). *If a Nakano space is monotone complete then it is topologically complete.*

We shall not prove this theorem but feel it is fair to say that the reader who has come this far should be able to read the original proof in [5].

Köthe spaces. Let S be a locally compact Hausdorff space, and let μ be a nonnegative Radon measure on S. Then $\mu(C) < \infty$ for every compact $C \subset S$. Let Ω be the vector lattice of equivalence classes of *locally summable* functions on S. The function f is locally summable if, for every compact $C \subset S$, $\int_C |f| \, d\mu < \infty$, and f is equivalent to g if, for every compact $C \subset S$, $\int_C |f-g| \, d\mu = 0$. For convenience, we speak of functions themselves as elements of Ω, when there is no danger of confusion, but mean the equivalence classes to which the functions belong.

Köthe spaces come as pairs K, K^* of subspaces of Ω, which we shall call Köthe duals of each other and which we now define. Let A be any set of functions in Ω. We associate with A a Köthe space K as follows:

$f \in K$ if and only if $f \in \Omega$ and $\int_S |fg| \, d\mu < \infty$ for every $g \in A$. If f, $g \in K$, an immediate calculation shows that $f+g \in K$ and $af \in K$ for every $a \in R$. Hence, K is a vector space which is a subspace of Ω. Now, if $f \geqq \theta$ means $f(x) \geqq 0$ on a set whose complement meets every compact subset of S in a set of measure 0, then Ω is a vector lattice. It is clear from the definition that if $f \in K$ and $|f| \geqq |g|$ then $g \in K$. This, together with the fact that K is a vector space implies, since $|f| + |g| \geqq |f| \cup |g|$, that K is a vector lattice. Corresponding to K is the set K^* of all $f \in \Omega$ such that $\int_S |fg| \, d\mu < \infty$ for all $g \in K$. Then K^* is also a vector lattice. Since $\int_S |fg| \, d\mu < \infty$ for all $f \in A$ and $g \in K$ it follows that $A \subset K^*$. The vector lattice K^* is the Köthe dual of K. We show that K is the Köthe dual of K^*; *i.e.*, $(K^*)^* = K$. First, if $f \in K$ then, for every $g \in K^*$, $\int_S |fg| \, d\mu < \infty$, so that $f \in (K^*)^*$. Next, if $f \in (K^*)^*$, then $\int_S |fg| \, d\mu < \infty$ for every $g \in K^*$. In particular, $\int_S |fg| \, d\mu < \infty$ for every $g \in A$, so that $f \in K$. This proves $(K^*)^* = K$.

The Köthe pair K, K^* is thus a pair of vector lattices, contained in Ω, each of which is the Köthe dual of the other. As an example, let $S = R$, the space of real numbers, and let Ω be the vector lattice of locally summable real functions on R. Then the vector lattice of summable functions on R and the vector lattice of bounded measurable functions on R are a Köthe pair K, K^*. (Thus, the space of summable functions is "reflexive" as a Köthe space, since every Köthe space has this property; while, as a Banach space with norm $\|f\| = \int_R |f| \, dt$, it is not reflexive). If K is the set of pth power integrable functions on R, then K^* is the set of qth power integrable functions, where $p^{-1} + q^{-1} = 1$. In addition, various other spaces of functional analysis are included in the Köthe system.

Thus far, K is merely a vector lattice. We now show how K may be given many topologies, for each of which it becomes a convex topological vector lattice. These topologies are defined via the Köthe duality between K and K^*. A set $A \subset K^*$ is called *admissible* if for every $f \in K$,

$$\sup_{g \in A} \int_S |fg| \, d\mu < \infty.$$

In particular, the definition of K^* assures that every set in K^* consisting of a single element is admissible. Now, every admissible $A \subset K^*$ and every $k > 0$

determine a ball in K. This is the set U_k of all $f\in K$ for which $\sup_{g\in A}\int_S|fg|\,d\mu \leqq k$. That U_k is a ball is easily established. For, if $\sup_{g\in A}\int_S|fg|\,d\mu\leqq k$, $\sup_{g\in A}\int_S|f'g|\,d\mu\leqq k$, and $0\leqq a\leqq 1$, then $\sup_{g\in A}\int_S|hg|\,d\mu\leqq k$, where $h=af+(1-a)f'$, so that U_k is convex. If $\sup_{g\in A}\int_S|fg|\,d\mu\leqq k$ then $\sup_{g\in A}\int_S|-fg|\,d\mu\leqq k$, so that U_k is symmetric. For every $f\in K$, $\sup_{g\in A}\int_S|fg|\,d\mu=\alpha<\infty$. Now, $\alpha>\hat{0}$, if there is a $g\neq\theta$ in A and if $f\neq\theta$. Hence, $k\alpha^{-1}f\in U_k$. If A consists only of θ, then $U_k=K$. If $f=\theta$, then $f\in U_k$ so that U_k is absorbing, in any case.

We next show that a collection $\mathfrak{A}$ of admissible sets in K^* defines a convex topology in K if:

I) *every* $g\in K^*$ *belongs to at least one* $A\in\mathfrak{A}$,
II) *if* A, $A'\in\mathfrak{A}$, *then* $A\cup A'\in\mathfrak{A}$.

Proof. (i) Let $f\in K$, $f\neq\theta$. There is a $g\in K^*$ with $\int_S|fg|\,d\mu>1$. But there is an $A\in\mathfrak{A}$ with $g\in A$. Thus, $\sup_{g\in A}\int_S|fg|\,d\mu>1$ so that $f\notin U_1(A)$.

(ii) Let r, $s>0$, and let $t=\min(r, s)$ and A, $A'\in\mathfrak{A}$. Then $U_t(A\cup A')\subset U_r(A)\cap U_s(A')$. For, if

$$\sup_{g\in A\cup A'}\int_S|fg|\,d\mu\leqq t,$$

then $\sup_{g\in A}\int_S|fg|\,d\mu\leqq r$ and $\sup_{g\in A'}\int_S|fg|\,d\mu\leqq s$.

(iii) Let $A\in\mathfrak{A}$ and $k>0$. Then $U_{k/2}(A)+U_{k/2}(A)\subset U_k(A)$. We leave this last simple verification to the reader.

A topology in K given in this way by a collection $\mathfrak{A}$ of admissible sets satisfying (I) and (II) is called a *Köthe topology* and the space K with any such topology is called a *topological Köthe space*. Furthermore, the balls U_k are order-closed in the vector lattice K. For, if $\sup_{g\in A}\int_S|fg|\,d\mu\leqq k$ and $|f'|\leqq|f|$ then $\sup_{g\in A}\int_S|f'g|\,d\mu\leqq k$. Hence, every topological Köthe space is a convex topological vector lattice.

Examples of collections $\mathfrak{A}$ of admissible sets satisfying (I) and (II) are the collection of all finite subsets of K^* and the collection of all admissible subsets of K^*. These are the extreme cases, giving the coarsest and finest Köthe topologies for K, respectively.

We now prove:

THEOREM 2. *Every topological Köthe space is a Nakano space.*

Proof. We first show that K is conditionally complete. We need only consider positive elements for this. Let $B\subset K$ be a bounded set of positive elements, and let $u\in K$ be an upper bound. Since Ω is conditionally complete, B has a least upper bound $f\in\Omega$, $f\leqq u$. But K is order-closed, since $\int_S|hg|\,d\mu<\infty$ for every $g\in K^*$ and $|h'|\leqq|h|$ implies $\int_S|h'g|\,d\mu<\infty$ for every $g\in K^*$. Hence $f\in K$, and so K is conditionally complete.

It remains only for us to show that every U_k is a Nakano ball. For this, let f_λ be an increasing net of positive elements in U_k, with an upper bound. Let f

be the least upper bound of f_λ. Then, for every $g \in K^*$, $\int_S |fg| \, d\mu = \sup_\lambda \int_S |f_\lambda g| \, d\mu$ so that $\sup_{g \in A} \int_S |f_\lambda g| \, d\mu \leqq k$, for every λ, implies $\sup_{g \in A} \int_S |fg| \, d\mu \leqq k$. This completes the proof.

Completeness of Köthe spaces. We are now ready to show that every topological Köthe space is topologically complete. Since every topological Köthe space is a Nakano space, it suffices, by Nakano's theorem, to show that K is monotone complete. We proceed to do this. (No countability assumptions are made for the locally compact space S on which the functions in Ω are defined.) Let $\mathfrak{A} = [A]$ be a collection of admissible sets in K^* satisfying (I) and (II). Let f_λ be an increasing net of positive elements in K which is bounded in the associated topology. This means that for every $A \in \mathfrak{A}$ there is a constant $M(A)$ such that $\int_S |f_\lambda g| \, d\mu < M(A)$ for every λ and every $g \in A$. In particular, for every $g \in K^*$, there is a constant $M(g)$ such that $\sup_\lambda \int_S |f_\lambda g| \, d\mu < M(g)$. Let $C \subset S$ be compact, and χ_C the characteristic function of C. We then have $\sup_\lambda \int_S |f_\lambda \chi_C| \, d\mu < M(\chi_C)$, or $\sup_\lambda \int_C f_\lambda d\mu < M(\chi_C)$. The restrictions of the functions f_λ to C, call them $f_\lambda | C$, then converge almost everywhere to a function $f | C$ whose domain is C. If C_1 and C_2 are compact, then $f | C_1$ and $f | C_2$ agree almost everywhere on $C_1 \cap C_2$. The collection $\{f | C\}$ of functions thus defines a function f on S (of course, modulo a set which meets every compact set in a set of measure 0). The function f belongs to Ω and is the least upper bound of f.

It remains only to show that $f \in K$. For every compact $C \subset S$ and $g \in K^*$, $\int_C |fg| \, d\mu = \sup_\lambda \int_C |f_\lambda g| \, d\mu \leqq \sup_\lambda \int_S |f_\lambda g| \, d\mu \leqq M(g)$. But, $\int_S |fg| \, d\mu = \sup_C \int_C |fg| \, d\mu \leqq M(g)$ so that $f \in K$. This completes the proof that K is monotone complete, and we have:

THEOREM 3. *Every topological Köthe space is topologically complete.*

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POSTULATES FOR COMMUTATIVE GROUPS*

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In a set of postulates one does not expect to find the laws of classical logic, the rules of grammar, or definitions of various words. Often omitted are the statements that equality is an equivalence relation and that, in a binary operation, one is free to replace elements by equal elements. For the purposes of this paper it seems natural to go further and ignore, for operations introduced elsewhere, the postulated closure of the system under these operations. We count as postulates only those which are identities or those which have one conditional equation as a consequence of another.

This paper gives two one-postulate systems which characterize commutative groups. The author's interest in one-postulate systems began with alternation groupoids ([2], Postulate II), a generalization of Abelian quasigroup which independently caught the interest of O. Frink [1]. Since then, parthenogenetic postulates for Boolean groups and Boolean algebras have been found [3]. Single postulates for ordinary groups have not yet been found.

1. Ordinary operations. Consider a set S closed under an operation denoted by multiplication. To each element x in S corresponds an element x' in S. The following postulate holds:

(G) $$\text{If } (aa')b' = (rs')t', \text{ then } b = (tr')s.$$

From the case $r=s=a$ and $t=b$ we have

(1.1) $$b = (ba')a.$$

We define a'' as $(a')'$, *etc.* Applying identity (1.1) in two ways to $((ba'')a')a$,

(1.2) $$ba'' = ba.$$

This implies $(aa')b' = (aa')b'''$. From (G), $b = (b''a')a$, and from (1.1),

(1.3) $$b = b''.$$

Hence, using (1.1), $(bb')b'' = b = (ba')a''$. From (G), $b' = (a'b')a$. This is equivalent to

(1.4) $$b = (ab)a'.$$

Proceeding as before, $(bb')b'' = b = (ab'')a'$ and from (G), $b' = (aa')b'$ or

(1.5) $$b = (aa')b.$$

Multiplying (1.4) on the right by a and using (1.1), we obtain

(1.6) $$ba = ab.$$

* This paper was written while the author was under contract to the United States Air Force Office of Scientific Research.

Multiplying (1.5) on the right by b' and using (1.1), we obtain $bb' = aa'$. We denote this common value by e. As a consequence of (1.6) and (1.5), we have two theorems.

$$e = xx' = x'x. \tag{1.7}$$

$$x = ex = xe. \tag{1.8}$$

We may now restate (G) in the form: If $b' = (rs')t'$, then $b = (tr')s$. This implies $[(rs')t']' = (tr')s$. Taking primes again, $(rs')t' = [(tr')s]' = (s't')r$. This may be restated as $(ab)c = (ca)b$ and, similarly, $(ca)b = (bc)a$. These identities and (1.6) imply

$$(ab)c = a(bc). \tag{1.9}$$

Thus S is a commutative group and since (G) holds for such groups, (G) characterizes such groups.

2. The inverse operation. Consider a set S closed under an operation, denoted by subtraction, for which we have the identity:

$$y = x - [(x - z) - (y - z)]. \tag{H}$$

We define x^* as $x - x$. Placing $y = x$ in (H), we have

$$x = x - (x - z)^*. \tag{2.1}$$

If $z = (x - y)^*$, it follows from (2.1) that $x - z = x$. And replacing z in (2.1) by this expression gives

$$x = x - x^*. \tag{2.2}$$

Hence, from (H) with $z = x^*$,

$$y = x - [x - (y - x^*)]. \tag{2.3}$$

From this, if $z = x - (y - x^*)$, we have $x - z = y$. Choosing this expression for z in (2.1), we obtain

$$x = x - y^*. \tag{2.4}$$

From this and (2.3) it follows that

$$y = x - (x - y). \tag{2.5}$$

The following theorem is proved by subtracting both differences from x and applying (2.5).

(2.6) If $x - y = x - z$, then $y = z$.

From this and (2.4), we have

$$y^* = z^*. \tag{2.7}$$

We use 0 to denote this common value. We have shown that $z-0=z$. Since $x-y=0$ implies $x=x-(x-y)=y$, we see that

$$x - y = 0 \text{ if and only if } x = y. \tag{2.8}$$

Now consider the expression $x-(x-[(x-z)-(y-z)])$. From (H), it equals $x-y$. From (2.5), it equals $(x-z)-(y-z)$. We have proved

$$x - y = (x - z) - (y - z). \tag{2.9}$$

But (2.5), (2.8), and (2.9) characterize a commutative group in terms of its inverse operation [4]. Thus we have shown that the single identity (H) characterizes commutative groups in terms of the subtraction operation.

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THE PROBABILITY DISTRIBUTION OF THE PRODUCT OF n RANDOM VARIABLES

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In many cases one must be satisfied with approximate results when treating the distributions of n random variables by n-dimensional mapping. However some problems, in particular this one, yield an exact solution by standard techniques employing characteristic functions and contour integration as indicated in this note.

Assume n random variables $x_1, \cdots, x_n$, each distributed with uniform density over an interval of unit length. That is, denoting the density-function of x_k by f_k:

$$f_k(x) = \begin{cases} 1 \text{ for } |x - a_k| \leqq \frac{1}{2}, \\ 0 \text{ elsewhere}, \end{cases} \qquad k = 1, \cdots, n. \tag{1}$$

Putting $y=x_1x_2 \cdots x_n$, we ask for the probability distribution of y under the assumption that the $x_1, \cdots, x_n$ are stochastically independent. Let x_k be positive and therefore $a_k>\frac{1}{2}$, $k=1, \cdots, n$ in (1), so that we can form

$$u = \log y = \log x_1 + \cdots + \log x_n = u_1, \cdots + u_n, \qquad u_k = \log x_k; \tag{2}$$

then we will have the joint distribution of the $u_1, \cdots, u_n$

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So we have two possible orderings of the eight products. Let us choose one of them by assuming the last order relation also to be valid. Then we get from (14) and (15) by means of the functional equation of the logarithmic function:

$$f(y) = \begin{cases} 0 & (y \leqq B_1B_2B_3), \\ \dfrac{1}{2}\log^2\dfrac{y}{B_1B_2B_3} & (B_1B_2B_3 \leqq y \leqq B_1B_2A_3), \\ \dfrac{1}{2}\log^2\dfrac{y}{B_1B_2B_3} - \dfrac{1}{2}\log^2\dfrac{y}{B_1B_2A_3} & (B_1B_2A_3 \leqq y \leqq B_1A_2B_3), \\ \log\dfrac{A_2}{B_2}\log\dfrac{A_3}{B_3} - \dfrac{1}{2}\log^2\dfrac{y}{B_1A_2A_3} & (B_1A_2B_3 \leqq y \leqq B_1A_2A_3), \\ \log\dfrac{A_2}{B_2}\log\dfrac{A_3}{B_3} & (B_1A_2A_3 \leqq y \leqq A_1B_2B_3), \\ \log\dfrac{A_2}{B_2}\log\dfrac{A_3}{B_3} - \dfrac{1}{2}\log^2\dfrac{y}{A_1B_2B_3} & (A_1B_2B_3 \leqq y \leqq A_1B_2A_3), \\ \dfrac{1}{2}\log^2\dfrac{y}{A_1A_2A_3} - \dfrac{1}{2}\log^2\dfrac{y}{A_1A_2B_3} & (A_1B_2A_3 \leqq y \leqq A_1A_2B_3), \\ \dfrac{1}{2}\log^2\dfrac{y}{A_1A_2A_3} & (A_1A_2B_3 \leqq y \leqq A_1A_2A_3), \\ 0 & (A_1A_2A_3 \leqq y). \end{cases}$$

FINITE MARKOV CHAINS AND THEIR APPLICATIONS*

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Recent applications of mathematics to the behavioral and biological sciences have emphasized the importance of Markov chains as a stochastic model. Because of this interest, it seemed important to develop a unified treatment of the descriptive quantities of a Markov chain. I shall describe briefly such a treatment developed with John G. Kemeny.†

A finite Markov chain is a special type of stochastic process which may be described briefly as follows. There are a finite number of *states* which we label $1, \cdots, r$. The process moves through these states in a sequence of *steps*. If

* Invited address at the Thirty-ninth Summer Meeting of the Mathematical Association of America, August 25, 1958.

† The details will appear in a forthcoming book, Finite Markov Chains by John G. Kemeny and J. Laurie Snell.

at any time it is in state i, it moves, on the next step, to state j, with probability p_{ij}. The matrix $P = \{p_{ij}\}$ is called the *matrix of transition probabilities.* The process is completely determined by specifying the transition matrix P and a starting state.

The importance of matrix theory to Markov chains comes from the fact that the ijth entry of the nth power P^n of P represents the probability that the process will be in state j after n steps if it is started in state i.

The study of the general Markov chain can be reduced to the study of two special types of chains. These are *absorbing chains* and *ergodic chains.* I shall first discuss absorbing chains.

An absorbing chain is defined as follows. A state is *absorbing* if, once entered, it cannot be left. A chain is an *absorbing chain* if it has at least one absorbing state and if, from every state, it is possible (not necessarily in one step) to reach an absorbing state.

An absorbing chain will, with probability one, eventually reach an absorbing state, no matter how it is started. Once in an absorbing state, it stays there. Hence we are interested in the behavior of the process up to the time it reaches an absorbing state. A quite complete description can be given in terms of certain random variables determined by the process. These are:

u_j: The number of times the process is in the nonabsorbing state j before being absorbed.

v: The number of steps taken before absorption.

w: The number of different nonabsorbing states entered before absorption.

x: The state in which the process is absorbed.

For these random variables, only x has a simple distribution. For the others we content ourselves with finding the first two moments (and hence also the variance). These moments will depend upon the starting state. Let me introduce the following notation:

$Pr_i[p]$: The probability that p occurs when the process is started in state i.

$M_i[f]$: The mean value of the random variable f when the process is started in state i.

$C_i[f, g]$: The covariance of f and g when the process is started in state i.

The mean values of the random variables u_j, v, w, are to be represented by vectors and matrices since they depend on the starting state. For example, the random variables u_j determine a matrix $N = \{M_i[u_j]\}$ while the random variable v determines a vector $\tau = \{M_i[v]\}$.

We begin by putting the transition matrix in a canonical form: we put the absorbing states first and then partition the matrix as follows.

$$\begin{array}{cc} & \begin{array}{cc}\text{absorbing} & \text{nonabsorbing}\end{array} \\ P = & \left(\begin{array}{c|c} I & 0 \\ \hline R & Q \end{array}\right). \end{array}$$

Here I is an indentity matrix and 0 a matrix with all entries zero.

We can then represent all of our interesting quantities in terms of a single fundamental matrix. This is the matrix $N=(I-Q)^{-1}$. In terms of this matrix we can show that,

$$\begin{aligned} N &= \{M_i[u_j]\} = (I-Q)^{-1}, & N_2 &= \{M_i[u_j^2]\} = N(2N_{dg}-I), \\ \tau &= \{M_i[v]\} = Ne, & \tau_2 &= \{M_i[v^2]\} = (2N-I)Ne, \\ H &= \{M_i[w]\} = NN_{dg}^{-1}e, & B &= \{Pr_i[x=j]\} = NR. \end{aligned}$$

Here e is a vector with each component 1, and N_{dg} is a matrix with diagonal entries the same as N, but zeros for all other entries. The important fact to notice is that all of this information about the process can be obtained by very simple matrix operations from the single fundamental matrix $N=(I-Q)^{-1}$. Hence only one matrix inversion is necessary. This makes it easy to write a general-purpose machine program to compute all of these quantities for a given absorbing chain. We have prepared such a program and have found it invaluable in studying specific applications.

I shall not verify all of these formulas, but let me indicate the proof of the first assertion. The ijth entry of Q^n represents the probability of being in the nonabsorbing state j after n steps, if the chain is started in the nonabsorbing state i. Let $e_j^{(n)}$ be a random variable which has the value 1, if the chain is in state j after n steps, and 0, if it is not. Then $q_{ij}^{(n)}=M_i[e_j^{(n)}]$ and $u_j=\sum_{j=0}^{\infty} e_j^{(n)}$. From this it follows that $M_i[u_j]=\sum_{n=0}^{\infty} M_i[e_j^{(n)}]=\sum_{n=0}^{\infty} q_{ij}^{(n)}$. But $N=(I-Q)^{-1}=I+Q+Q^2+\cdots$. Hence $N=\{M_i[u_j]\}$. That is, the ijth entry of N represents the mean number of times in state j when the process is started in stage i. The fact that the fundamental matrix itself has such a simple probabilistic interpretation is helpful in finding its relation to other basic quantities. The use of eigenvalues which do not have simple interpretations often leads to unnecessarily complicated formulas.

I shall next discuss ergodic chains, and we shall see that their behavior is quite different from that of absorbing chains. An *ergodic chain* is one such that it is possible to go (not necessarily in one step) from any state to any other state. An important special case of an ergodic chain, called a *regular chain*, is one such that, for some N, P^N has no zero entries.

For an ergodic chain

$$\lim_{n\to\infty} \frac{P+P^2+\cdots+P^n}{n} = A,$$

where A is a matrix with each row the same vector $\alpha=(a_1,\cdots,a_r)$. The vector α has all components positive and is the unique vector with components adding to 1 and such that $\alpha P=\alpha$. For regular chains this limit may be strengthened to $\lim_{n\to\infty} P^n=A$. In this case we see that the probability $p_{ij}^{(n)}$ of going from i to j in a large number of states is essentially independent of the starting state. In either case, a_j gives the fraction of the times that the process can be expected

to be in each of the states in the long run. Since we expect, in an ergodic chain, that the process will continually wander around through all of the states, we are interested in quite different random variables than we considered for absorbing chains.

Interesting descriptive random variables for an ergodic chain deal, for example, with the number of times a process is in a particular state in the first n steps, and in the behavior of the process between occurrences of two states, or before returning to a starting state for the first time. I shall discuss two such random variables. These are:

v_j: The time required to reach j for the first time.
$s_j(n)$: The number of times in state j in the first n steps.

As in the case of an absorbing chain we are interested in means and variances of these random variables. The quantity $M_i[v_j]$, for $i \neq j$, represents the average length of time, starting at i, to reach j for the first time. It is called the *mean first passage time.* $M_i[v_i]$ represents the mean time to return to state i and is called the *mean recurrence time.* We represent these quantities by the matrix $M = \{M_i[v_j]\}$.

We define first a fundamental matrix Z for regular chains to play the role of N for absorbing chains. This matrix is given by $Z = (I - P + A)^{-1}$.

For the regular case,

$$Z = I + (P - A) + (P^2 - A) + \cdots .$$

In this case the entry z_{ij} of Z may then be interpreted as a measure of the deviations of $p_{ij}^{(n)}$ from their limiting probabilities a_j. From Z we can obtain the following information about $s_j^{(n)}$ and v_j.

$$M = \{M_i[v_j]\} = (I - Z + EZ_{dg})D,$$

$$W = \{M_i[v_j^2]\} = M(2Z_{dg}D - I) + 2(ZM - E(ZM)_{dg}),$$

$$C = \left\{\lim_{n\to\infty} \frac{C_k[s_i^{(n)}, s_j^{(n)}]}{n}\right\} = \{a_i z_{ij} + a_j z_{ji} - a_i\delta_{ij} - a_i a_j\}.$$

(Here D is a diagonal matrix with entry $1/a_j$ as the jth diagonal entry, E is a matrix with all 1's, and $\delta_{ij} = 1$ if $i = j$ and 0 otherwise.)

The first two formulas provide us with the mean and variances for the first passage times and the recurrence times. One of the advantages of simple matrix expressions is that they suggest simply relations between basic quantities. For example, it can be shown that the mean first passage times completely determine the transition matrix, and a formula for P in terms of these means can be given.

An important use of the covariance matrix C is the following. The central limit theorem for Markov chains states that $(s_j^{(n)} - na_j)/\sqrt{nc_{jj}}$ is asymptotically normally-distributed with mean 0 and standard deviation 1. Thus the diagonal

entries of C provide us with the proper norming constants to apply the central limit theory.

Finally I should like to mention a few of the areas of recent applications of finite Markov chain theory.

In physics, the Ehrenfest urn model has been used as a simple model to describe diffusion. In this model there are two urns which contain altogether N balls. Each second, one of the N balls is chosen at random and moved from the urn which contains it to the other urn. A Markov chain may be formed in two different ways. The first I shall call the *microscopic chain.* For this chain we assume that the balls are distinguishable and label them $1, \cdots, N$. A state is then an N-component vector which has ith component 1 if the ith ball is in the first urn and 0 otherwise. The second chain, the *macroscopic chain*, is formed by taking as state the number of balls in the first urn. The macroscopic chain is obtained from the microscopic chain by combining states. This is not always a legitimate procedure, but in this case it is. Most interesting questions for applications relate to the macroscopic chain. For example, an interesting quantity is the mean time it takes to go from all balls in one urn to as near an equal number in each urn as possible. It is interesting to compare these with the time required to go from an equal number to one of the extreme cases where all are in one urn. These mean first passage times are most easily obtained from the microscopic chain. This chain is represented as a random walk on an N-dimensional cube and the mean first passage time from state i to state j in the macroscopic chain becomes the mean length of time to go a distance $|i-j|$ in the microscopic chain.

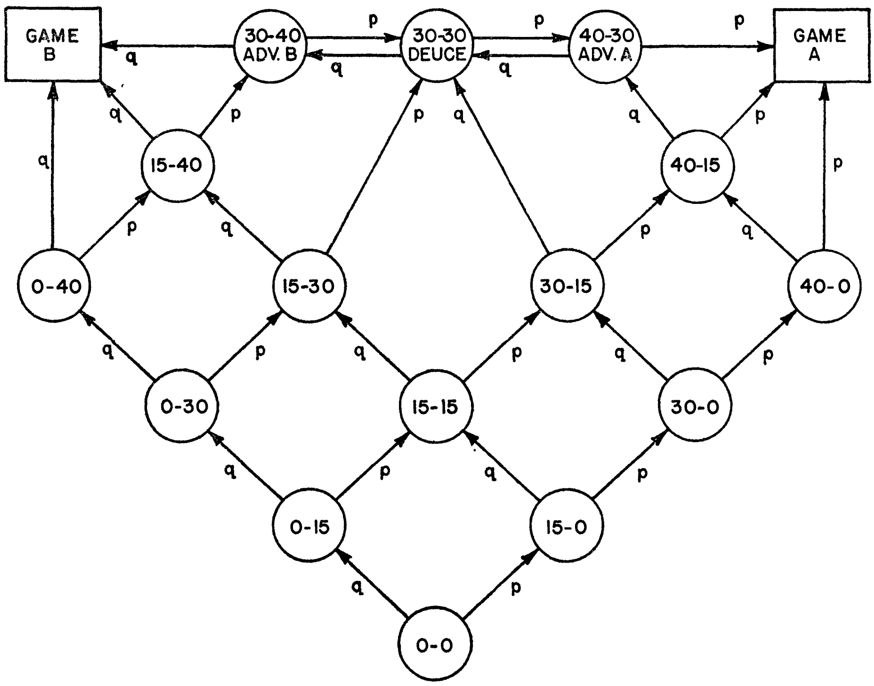
In psychology, the Estes learning model leads to a finite Markov chain. The model has been used to predict the learning habits of animals as well as humans. For example, consider the behavior of rats in a T maze. Different types of feeding schedules lead to different kinds of Markov chains. When the animal is fed always on one side, the model leads to an absorbing chain, and the mean length of time to absorption determines the mean time to complete learning. Other feeding schedules lead to ergodic chains where the number of times in a state related to the number of correct choices the rat makes.

In economics, Markov chain theory has been applied to study the Leontief model for the economy. In this case the transition probability represents the way that a one-dollar order of goods from a single industry spreads its effect through the other industries. Markov chain theory determines, for example, conditions under which economy can meet a given demand.

In genetics, Markov chain models have been used to describe the effect of inbreeding. A pair of offspring are chosen and mated; then a pair of their offspring chosen, *etc.* The state is the genetic type of the parents. An absorbing chain is obtained, and the absorbing states are states where both parents are dominant or both are recessive. The absorption probabilities give the probability that the process will lead to each of these two pure types. The mean time to

absorption gives the mean time to reach a pure type. The number of different types that occur before absorption is also of interest.

Finally, let me mention an application on the lighter side. This is the application of Markov chain theory to the game of tennis. The states and transition probabilities are shown below. (We assume that 30-30 is deuce). The transition probabilities have been determined by assuming that a player has a fixed probability p for a win on each point. (Obvious modifications of this assumption suggest themselves and lead to new Markov chains.)



We have here an absorbing chain. The absorption probabilities give the probabilities of a win, and the mean time to absorption gives the mean length of a game. One would hope that this crude model is not realistic, for it leads to the following predictions.

	$p=.51$	$p=.60$
Probability of winning a point	.510	.600
Probability of winning a game	.525	.736
Probability of winning a set	.573	.966
Probability of winning a match	.635	.9996

One feels that there would be less tennis played between unequal players if this model were correct.

TRIGONOMETRIC SUMS IN ELEMENTARY NUMBER THEORY

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1. Introduction. Let n and r be integers with $r>0$. We say that a complex-valued arithmetical function $f(n)=f(n, r)$ is a *periodic* function of n (mod r) if $f(n+r, r)=f(n, r)$ for all n. The class of periodic functions (mod r) is characterized by the fact that such functions possess a trigonometric (finite Fourier) expansion. For a discussion of the periodic functions (mod r) and their relation to elementary geometry, we mention I. J. Schoenberg [16].

An arithmetical function $f(n, r)$ is said to be an *even* function of n (mod r) if $f((n, r), r)=f(n, r)$ for all n, where (n, r) denotes the greatest common divisor of n and r. Clearly the even functions (mod r) form a subclass of the periodic functions (mod r). This class of functions was studied in a previous paper [2]. The purpose of the present paper is to illustrate the theory of even functions (mod r) with some instructive examples at an elementary level.

First we recall some results proved in [2] which will be needed in our discussion. Ramanujan's trigonometric sum $c(n, r)$ is defined by

$$c(n, r) = \sum_{(x,r)=1} e(nx, r) \qquad (e(n, r) = e^{2\pi i n/r}), \tag{1}$$

where the summation is over a reduced residue system (mod r). The function $c(n, r)$ is even (mod r); moreover ([2], Th. 1), a function $f(n, r)$ is even (mod r) if and only if it possesses a trigonometric representation of the form

$$f(n, r) = \sum_{d|r} \alpha(d, r)c(n, d). \tag{2}$$

The Fourier coefficients $\alpha(d, r)$ are determined uniquely by the formula ([2], (8)),

$$\alpha(d, r) = \frac{1}{r\phi(d)} \sum_{a \,(\mathrm{mod}\, r)} f(a, r)c(a, d), \tag{3}$$

where the summation is over a complete residue system (mod r) and $\phi(r)$ denotes the Euler ϕ-function.

It was also proved in [2] that $f(n, r)$ is even (mod r) if and only if it possesses an arithmetical representation of the form ([2], Th. 3)

$$f(n, r) = \sum_{d|(n,r)} g(d, r/d), \tag{4}$$

where $g(a, b)$ is an arithmetical function of two variables, and the summation is over the common divisors of n and r. It was also shown ([2], (10)) that the Fourier coefficients $\alpha(d, r)$ of $f(n, r)$, as defined in (4), are given by the formula

* The author wishes to express his appreciation to the Institute for Advanced Study, Princeton, N. J., for the use of its facilities in completing this paper, 1957.

$$\alpha(d, r) = \frac{1}{r} \sum_{e \mid (r/d)} g(r/e, e)e. \tag{5}$$

We now give a brief description of the content of this paper. Suppose that k is a positive integer. In Sec. 2 we list a number of results concerned with the Jordan totient function $J_k(r)$ ([11], pp. 95–97]; [7], pp. 147–155), which will be required in the later discussion. In Sec. 3 we discuss a function $c^{(k)}(n, r)$ generalizing $c(n, r)$, and prove two results concerning this function (Th. 1, 2), which yield, in case $k=1$, two fundamental evaluations of $c(n, r)$. It is noted that Theorem 1 was proved in an equivalent form by Carmichael [1] (see the remark in Sec. 2) and later by Daniloff ([6], (67)). The proof given in this paper is along the lines of Ramanujan's proof for the special case $k=1$.

Suppose that s is an arbitrary nonnegative integer. In Sec. 4 we obtain the trigonometric expansion (Th. 3) of the function $\sigma_s^*(n, r)$, defined to be the number of solutions $x_0, y_0, \cdots, x_s, y_s \pmod r$ of the congruence,

$$n \equiv x_0 y_0 + \cdots + x_s y_s \pmod r. \tag{6}$$

An arithmetical representation of $\sigma_s^*(n, r)$ is deduced in Theorem 4. For a discussion of an extension of this problem using other methods we refer to [3], Sec. 3 and [4], Sec 3. Denote now the greatest common divisor of integers $a_0, \cdots, a_s$ by $(a_0, \cdots, a_s)$ and let $\phi_s^*(n, r)$ represent the number of solutions $x_0, \cdots, x_s \pmod r$ of the congruence,

$$n \equiv x_0 + \cdots + x_s \pmod r, \tag{7}$$

under the restriction $((x_0, \cdots, x_s), r)=1$. In Theorems 5 and 6 of Sec. 4 we obtain results for $\phi_s^*(n, r)$ analogous to those proved for $\sigma_s^*(n, r)$.

It is of interest to note that the finite expansions (2) of even functions (mod r) correspond to Ramanujan's infinite series expansions [15] for certain types of arithmetical functions of a single variable. In fact, Theorems 3 and 6 provide finite analogues of two of Ramanujan's most familiar expansions. More precisely, let $\mu(r)$ denote the Möbius μ-function and define

$$\text{(8)} \qquad \text{(a) } \sigma_s(r) = \sum_{d \mid r} d^s, \qquad \text{(b) } \phi_s(r) = \sum_{d \mid r} d^s \mu(r/d),$$

or equivalently,

$$\text{(9)} \qquad \text{(a) } \frac{\sigma_s(r)}{r^s} = \sum_{d \mid r} \frac{1}{d^s}, \qquad \text{(b) } \frac{\phi_s(r)}{r^s} = \sum_{d \mid r} \frac{\mu(d)}{d^s}.$$

In Sec. 6, by a special limiting process, we deduce Ramanujan's expansion of $\sigma_s(r)/r^s$ from the results of this paper for $\sigma_s^*(n, r)/r^s$ and his expansion of $\phi_s(n)/n^s$ from the corresponding results for $\phi_s^*(n, r)/r^s$ (Th. 7, 8, respectively). Thus, in a certain sense, the congruence problems (6) and (7) may be conceived of as the ultimate arithmetical sources of Ramanujan's expansions.

Further investigations in the theory of even functions (mod r) will form the substance of a paper to be published later.

2. The Jordan function $J_k(r)$. Let k denote an arbitrary positive integer. As in [5] we define a k-vector, $\{a_i\} = \{a_1, \cdots, a_k\}$, to be an ordered set of k integers, and call two k-vectors $\{a_i\}$, $\{b_i\}$ congruent (mod k, r) provided $a_i \equiv b_i$ (mod r), $i = 1, \cdots, k$. A *complete* set of residues (mod k, r) is defined to be a set of k-vectors $\{a_i\}$ where the a_i range independently over a complete residue system (mod r). The Jordan function $J_k(r)$ may be defined as the number of elements in a *reduced* residue system (mod k, r); that is, the number of $\{a_i\}$ in a complete residue system (mod k, r) such that $((a_i), r) = 1$, where $(a_i) = (a_1, \cdots, a_k)$, $(0) = 0$. It is also convenient to define $J_0(r) = 1$ or 0 according as $r = 1$ or $r > 1$. Clearly $J_1(r) = \phi(r)$.

In addition to the above notation we define the sum of two k-vectors by $\{a_i + b_i\} = \{a_i\} + \{b_i\}$ and scalar multiples by $c\{a_i\} = \{ca_i\}$ for integers c. We now collect a number of known results relating to $J_k(r)$.

LEMMA 1. *The k-vectors $\{a_i\} = \{rx_i/d\}$ where d ranges over the divisors of r and, for each d, x_i ranges over a reduced residue system* (mod k, d), *form a complete residue system* (mod k, r).

This lemma ([16], Sec. 2) can be proved by the method used in treating the familiar special case $k = 1$ ([9], Sec. 16.2). An immediate consequence of this result is

LEMMA 2.

$$\sum_{d|r} J_k(d) = r^k. \tag{10}$$

By the Möbius inversion formula ([9], Sec. 16.4) there follows the evaluation,

LEMMA 3.

$$J_k(r) = \phi_k(r) \equiv \sum_{d|r} d^k \mu(r/d). \tag{11}$$

For other derivations of Lemma 3 we mention ([5], Th. 2) and the references listed in that paper.

A function $f(r)$ is said to be factorable if $f(1) = 1$ and $f(r_1 r_2) = f(r_1) f(r_2)$ for all positive integers r_1, r_2, such that $(r_1, r_2) = 1$. In view of the factorability of $\mu(r)$, one deduces ([9], Sec. 16.3, Th. 265) from Lemma 3 the factorability of $J_k(r)$.

LEMMA 4.

$$J_k(r_1 r_2) = J_k(r_1) J_k(r_2), \qquad (r_1, r_2) = 1. \tag{12}$$

Lemmas 2, 3, and 4 generalize familiar properties of $\phi(r)$ ([9], Ch. 16). We note that $J_k(r)$ satisfies the more general factorability law ([8], (30)) contained in the following

LEMMA 5. *For all positive* r_1, r_2,

$$J_k(r_1r_2) = J_k(r_1)J_k(r_2)\frac{\delta^k}{J_k(\delta)}, \qquad \delta = (r_1, r_2). \tag{13}$$

We next prove two useful lemmas concerning residue systems (mod k, r). The term *minimal* (complete) *residue system* (mod k, r) is used to denote the complete residue system $\{a_i\}$ such that $0 \leqq a_i < r$. Similarly, a *minimal reduced residue system* is defined to be the reduced system $\{a_i\}$ satisfying $0 \leqq a_i < r$.

LEMMA 6. *If* $(a, b) = 1$, *then a reduced residue system* (mod k, ab) *is generated by the set* $\{ah_i + bh_i'\}$ *where* $\{h_i\}$ *and* $\{h_i'\}$ *range over reduced residue systems* (mod k, b) *and* (mod k, a) *respectively.*

Proof. The elements of the set $\{ah_i + bh_i'\}$ are distinct (mod k, ab); moreover, $((ah_i + bh_i'), ab) = 1$ for each element of the set. Since, by Lemma 4, there are $J_k(ab)$ elements in the set, it must form a reduced residue system (mod ab). This proves Lemma 6.

The preceding lemma generalizes a well-known result in the case $k = 1$ ([**9**] Sec. 5.5, Th. 61).

LEMMA 7. *A reduced residue system* (mod k, ab) *can be decomposed into* $J_k(ab)/J_k(b)$ *reduced residue systems* (mod k, b).

Proof. Let $\gamma(r)$ denote the core of r; that is, $\gamma(r) = 1$ if $r = 1$ while $\gamma(r)$ is the product of the distinct prime factors of r if $r > 1$. We separate the proof into three cases.

(i) If $(a, b) = 1$, the lemma results from Lemma 6.

(ii) Next, suppose $\gamma(a) \mid b$. Let S denote the set $\{x_i + by_i\}$ where $\{y_i\}$ ranges over a minimal residue system (mod k, a) while x_i ranges over a minimal reduced residue system (mod k, b). It follows that the elements of S are distinct (mod k, ab) and are contained in a minimal reduced residue system T (mod k, ab). Conversely, by the division algorithm, any element of T can be expressed in the form $\{X_i\} = \{r_i + q_ib\}$ where $0 \leqq q_i < a$, $0 \leqq r_i < b$, $((r_i), b) = 1$. Hence $S = T$ and the lemma follows in this case.

(iii) In the general case, place $a = \alpha\beta$ where $\gamma(\beta) \mid b$ and $(\alpha, b) = 1$. Then by (i) there are $J_k(ab)/J_k(\beta b)$ reduced residue systems (mod k, βb) contained in such a system (mod k, ab). By (ii) there are $J_k(\beta b)/J_k(b)$ reduced residue systems (mod k, b) contained in such a system (mod k, βb). Hence, multiplication gives $J_k(ab)/J_k(b)$ systems (mod k, b) contained in a single reduced residue system (mod k, ab) and the proof is complete.

For a proof of the preceding result in the case $k = 1$, we refer to Nagell ([**14**], p. 24). The following analogue of Lemma 7 follows from definition.

LEMMA 8. *A complete residue system* (mod k, ab) *contains* a^k *complete residue systems* (mod k, b).

3. A generalization of $c(n, r)$. Ramanujan's trigonometric sum $c(n, r)$ can be generalized by placing

$$c^{(k)}(n, r) = \sum_{((x_i),r)=1} e(n(x_1 + \cdots + x_k), r), \tag{14}$$

where $\{x_i\}$ ranges over a reduced residue system (mod k, r). It is observed that $c^{(1)}(n, r) = c(n, r)$. We shall obtain two evaluations of $c^{(k)}(n, r)$; first we state a simple lemma. Let $\xi(n, r)$ be defined by $\xi(n, r) = r$ or 0 according as $r \mid n$ or $r \nmid n$.

LEMMA 9.

$$\eta(n, r) \equiv \sum_{a \,(\mathrm{mod}\, r)} e(na, r) = \xi(n, r). \tag{15}$$

This result is well-known ([**9**], Sec. 16.6).

THEOREM 1.

$$c^{(k)}(n, r) = \sum_{d \mid (n,r)} d^k \mu(r/d). \tag{16}$$

Proof. Place

$$\eta^{(k)}(n, r) = \sum_{a_i \,(\mathrm{mod}\, k,r)} e(n(a_1 + \cdots + a_k), r), \tag{17}$$

where the summation is over a complete residue system (mod k, r). Note that $\eta^{(1)}(n, r) = \eta(n, r)$. By (17) and the additivity property $e(a+b, r) = e(a, r)e(b, r)$,

$$\eta^{(k)}(n, r) = \eta^k(n, r) = \xi^k(n, r). \tag{18}$$

Moreover, by Lemma 1 and (17), we have, since $e(na, ab) = e(n, b)$,

$$\eta^{(k)}(n, r) = \sum_{d \mid r} c^{(k)}(n, d). \tag{19}$$

Comparing (18) and (19) we have

$$\sum_{d \mid r} c^{(k)}(n, d) = \xi^k(n, r). \tag{20}$$

Application of the Möbius inversion formula to (20) gives

$$c^{(k)}(n, r) = \sum_{d \mid r} \xi^k(n, d)\mu(r/d) = \sum_{d \mid r, d \mid n} d^k \mu(r/d),$$

and the theorem is proved.

The case $k=1$ yields Ramanujan's evaluation ([**15**], (2.7); [**9**], Sec. 16.6, Th. 271) of $c(n, r)$.

COROLLARY 1.

$$c(n, r) = \sum_{d \mid (n,r)} d\mu(r/d). \tag{21}$$

If n and r are relatively prime, one obtains

COROLLARY 2. *For all k,*

$$c^{(k)}(n, r) = c^{(k)}(1, r) = \mu(r) \qquad (n, r) = 1. \tag{22}$$

Remark. Carmichael ([1], Sec. 4) introduced a function $c_r^{(k)}(n)$, defined by means of Dirichlet series, which was shown to be equivalent to the function appearing on the right of (16). It therefore follows by Theorem 1 that $c_r^{(k)}(n) = c^{(k)}(n, r)$.

We now deduce a second evaluation of $c^{(k)}(n, r)$.

THEOREM 2.

$$c^{(k)}(n, r) = \frac{J_k(r)\mu(m)}{J_k(m)} \qquad \left(m = \frac{r}{(n, r)}\right). \tag{23}$$

Proof. By (14) we may write

$$c^{(k)}(n, r) = \sum_{((x_i), r)=1} e\left(\frac{n}{(n, r)}(x_1 + \cdots + x_k), m\right).$$

But by Lemma 7 and (22) it follows that

$$c^{(k)}(n, r) = \frac{J_k(r)}{J_k(m)} c^{(k)}\left(\frac{n}{(n, r)}, m\right) = \frac{J_k(r)\mu(m)}{J_k(m)}.$$

This completes the proof.

We observe that in case $k=1$, Theorem 2 reduces to the Hölder evaluation ([10], (13)) of $c(n, r)$ contained in

COROLLARY 3.

$$c(n, r) = \frac{\phi(r)\mu(m)}{\phi(m)} \qquad \left(m = \frac{r}{(n, r)}\right). \tag{24}$$

In case $n=0$, one obtains

COROLLARY 4.

$$c^{(k)}(0, r) = J_k(r). \tag{25}$$

This result also follows from Theorem 1 or by the definition of $c^{(k)}(n, r)$. We remark finally that the Hölder relation (24) was actually proved as early as 1907 by Kluyver ([12], p. 410); however, it was Hölder who first recognized the intrinsic interest of this formula.

4. Representations of $\sigma_s^*(n, r)$ and $\phi_s^*(n, r)$. In this section we obtain expansions of the functions $\sigma_s^*(n, r)$ and $\phi_s^*(n, r)$ in the trigonometric form (2) and in the arithmetical form (4). We first consider $\sigma_s^*(n, r)$.

THEOREM 3. *The number of solutions in* x_i, y_i (mod r) *of the congruence* (6) *is given by*

$$\sigma_s{}^*(n, r) = r^{2s+1} \sum_{d|r} (c(n, d)/d^{s+1}). \tag{26}$$

Proof. It is evident that $\sigma_s^*(n, r)$ is even (mod r). Therefore, by (2) and (3), $\sigma_s^*(n, r)$ has a trigonometric representation of the form

$$\sigma_s{}^*(n, r) = \sum_{d|r} \alpha(d, r)c(n, d), \tag{27}$$

$$\alpha(d, r) = \frac{1}{r\phi(d)} \sum_{a\,(\mathrm{mod}\, r)} \sigma_s^*(a, r)c(a, d). \tag{28}$$

By the definitions of $\sigma_s^*(n, r)$ and $c(n, r)$ we obtain

$$\begin{aligned} \alpha(d, r) &= \frac{1}{r\phi(d)} \sum_{x_i, y_i (\mathrm{mod}\; s+1, r)} c(x_0y_0 + \cdots + x_sy_s, d) \\ &= \frac{1}{r\phi(d)} \sum_{(z,d)=1} \sum_{x_i, y_i (\mathrm{mod}\; s+1, r)} e(z(x_0y_0 + \cdots + x_sy_s), d), \end{aligned}$$

where, as in (17), $\{x_i\}$, $\{y_i\}$ range over complete residue systems (mod $s+1$, r). By the additivity of $e(a, r)$ and Lemmas 8 and 9, it follows that

$$\begin{aligned} \alpha(d, r) &= \frac{r^{2s+1}}{\phi(d)d^{2s+2}} \sum_{(z,d)=1} \prod_{i=0}^{s} \Bigl(\sum_{y_i (\mathrm{mod}\; d)} \eta(zy_i, d) \Bigr) \\ &= \frac{r^{2s+1}}{\phi(d)d^{2s+2}} \sum_{(z,d)=1} \Bigl(\sum_{y (\mathrm{mod}\; d)} \xi(zy, d) \Bigr)^{s+1}. \end{aligned}$$

Since $(z, d) = 1$, by definition of $\xi(n, r)$ and $\phi(r)$ one obtains

$$\alpha(d, r) = r^{2s+1}/d^{s+1}. \tag{29}$$

The theorem follows from (27) and (29).

THEOREM 4.

$$\sigma_s^*(n, r) = r^s \sum_{d|(n,r)} dJ_{s+1}(r/d). \tag{30}$$

Proof. Applying (21) to (26) and recalling the definition (8b) of $\phi_s(r)$, it follows that

$$\begin{aligned} \sigma_s^*(n, r) &= r^{2s+1} \sum_{d|r} (1/d^{s+1}) \sum_{e|(n,d)} e\mu(d/e) \\ &= r^{2s+1} \sum_{e|(n,r)} e \sum_{d|r,\; d=\delta e} (\mu(d/e)/d^{s+1}) \\ &= r^s \sum_{e|(n,r)} e \sum_{\delta|(r/e)} (r/(e\delta))^{s+1}\mu(\delta) = r^s \sum_{e|(n,r)} e\phi_{s+1}(r/e). \end{aligned}$$

The theorem follows on applying Lemma 3.

The following corollaries are special cases of Theorem 4 (and also Theorem 3).

COROLLARY 5. *If* $(n, r) = 1$, *then the number of solutions of* (6) *is given by*

$$\sigma_s^*(n, r) = r^s J_{s+1}(r). \tag{31}$$

COROLLARY 6. *If* $n \equiv 0 \pmod r$, *then the number of solutions of* (6) *is given by*

$$\sigma_s^*(n, r) = r^s \sum_{d|r} dJ_{s+1}(r/d). \tag{32}$$

Before considering the function $\phi_s^*(n, r)$, we sketch an alternative method for proving the two preceding theorems. Theorem 4 can be proved first by a direct arithmetical approach, on the basis of Lemma 8 and the following lemma:

If $((a_i), r) = 1$, *then the number of solutions* $B_s(r)$ *of the congruence* $n \equiv a_0y_0 + \cdots + a_sy_s \pmod r$ *in* $y_i \pmod r$ *is* $B_s(r) = r^s$.

It is sufficient to observe that this result is valid for r a prime power, because $B_s(r)$ is a factorable function of r, by the Chinese Remainder Theorem ([**9**], Sec. 8.1, Th. 121). Theorem 3 can now be proved on applying (4) and (5) to the arithmetical representation (32) and using Lemma 2. The details are left to the reader.

We illustrate this alternative method by applying it to the function $\phi_s^*(n, r)$. First we introduce the notation,

$$J_s(n, r) = J_s((n, r)), \qquad \phi_s(n, r) = \phi_s((n, r)). \tag{33}$$

THEOREM 5. *The number of solutions of the congruence* (7) *in* $x_i \pmod r$ *such that* $((x_0, \cdots, x_s), r) = 1$ *is given by*

$$\phi_s^*(n, r) = \left(\frac{r}{(n, r)}\right)^s J_s(n, r). \tag{34}$$

Proof. If $s = 0$, then it is obvious that $\phi_0^*(n, r) = J_0(n, r) = 1$ or 0 according as $(n, r) = 1$ or $\neq 1$. In the remainder of the proof we may therefore suppose that $s > 0$. Under this assumption, the function $\phi_s^*(n, r)$ may be redefined as the number of s-vectors $\{x_i\}$, distinct (mod s, r), such that $((n - x_1 - \cdots - x_s, x_1, \cdots, x_s), r) = 1$. Now a divisor d of r is a divisior of $((n - x_1 - \cdots - x_s, x_1, \cdots, x_s), r)$ if and only if d is a divisor of $((n, x_1, \cdots, x_s), r)$. Therefore, since $((n, x_1, \cdots, x_s), r) = (x_1, \cdots, x_s, (n, r)) \equiv ((x_i), (n, r))$, it follows that $\phi_s^*(n, r)$ represents the number of s-vectors $\{x_i\}$, (mod s, r), such that $((x_i), (n, r)) = 1$. Applying Lemma 8, we obtain then, by the definition of $J_s(n, r)$ and the later reformulation of $\phi_s^*(n, r)$,

$$\phi_s^*(n, r) = \left(\frac{r}{(n, r)}\right)^s \phi_s^*(n, (n, r)) = \left(\frac{r}{(n, r)}\right)^s J_s(n, r).$$

THEOREM 6.

$$\phi_s^*(n, r) = \frac{J_{s+1}(r)}{r} \sum_{d|r} \left(\frac{\mu(d)}{J_{s+1}(d)} \right) c(n, d). \tag{35}$$

Proof. By (9b) we may rewrite (34) in the form

$$\phi_s^*(n, r) = r^s \sum_{d|(n,r)} \mu(d)/d^s. \tag{36}$$

Applying (4) and (5) to (36) with $g(a, b) = \mu(a)a^{-s}$ we obtain on the basis of (16),

$$\phi_s^*(n, r) = \sum_{d|r} \alpha(d, r)c(n, d), \tag{37}$$

where

$$\begin{aligned} \alpha(d, r) &= (1/r) \sum_{e|(r/d)} \mu(r/e)e^{s+1} \\ &= (1/r) \sum_{e|(r/d,r)} \mu(r/e)e^{s+1} = (1/r)c^{(s+1)}(r/d, r). \end{aligned} \tag{38}$$

Hence by Theorem 2 it follows that

$$\alpha(d, r) = \{J_{s+1}(r)\mu(d)\} / \{rJ_{s+1}(d)\}. \tag{39}$$

The theorem follows by (37) and (39).

We mention the following special cases.

COROLLARY 7. *If* $(n, r) = 1$, *then the number of solutions of* (7) *is* $\phi_s^*(n, r) = r^s$.

COROLLARY 8. *If* $n \equiv 0 \pmod r$, *then the number of solutions of* (7) *is* $\phi_s^*(n, r) = J_s(r)$.

It will be observed that the method used in proving Theorems 3 and 4 can also be applied to prove Theorems 5 and 6. More precisely, Theorem 6 can be proved by applying (2) and (3) to $\phi_s^*(n, r)$ and making use of Lemma 7 and (22). Then Theorem 5 can be proved by applying (21) to (35) with the aid of [**4**], Th. 9 and (13). The reader may supply the details.

5. Series expansions of $\sigma_s(n)/n^s$ and $\phi_s(n)/n^s$. In this section n will denote a positive integer. First we restate the results of the preceding section in a form convenient for use in this section. Combining Theorems 3 and 4 on the basis of Lemma 3, one obtains

$$\frac{\sigma_s^*(n, r)}{r^s} = \sum_{d|(n,r)} d\phi_{s+1}(r/d) = r^{s+1} \sum_{d|r} \frac{c(n, d)}{d^{s+1}}. \tag{40}$$

Similarly, by Theorems 5 and 6 and Lemma 3, we have

$$\frac{\phi_s^*(n, r)}{r^s} = \frac{\phi_s(n, r)}{(n, r)^s} = \frac{\phi_{s+1}(r)}{r^{s+1}} \sum_{d|r} \left(\frac{\mu(d)}{\phi_{s+1}(d)} \right) c(n, d). \tag{41}$$

The principal tool in passing from (40) and (41) to Ramanujan's expansions of $\sigma_s(n)/n^s$ and $\phi_s(n)/n^s$ will be the following lemma.

LEMMA 10. *Let $\lambda(k)$ be a sequence of positive integers, $k=1, 2, \cdots$, such that for all sufficiently large k, $\lambda(k)$ is divisible by $d=1, \cdots, k$. If, in addition, the series $\sum_{m=1}^{\infty} f(m)$ converges absolutely, then*

$$\lim_{k\to\infty} T_k = \sum_{m=1}^{\infty} f(m) \equiv S, \qquad T_k = \sum_{d\mid\lambda(k)} f(d). \tag{42}$$

Proof. Let S_k denote the kth partial sum of the series in (42) and let R_k denote the remainder after k terms of the series $\sum_{m=1}^{\infty} |f(m)|$. For k sufficiently large, $\lambda(k)>k$ and

$$\begin{aligned} |T_k - S_k| &= \Big| \sum_{d\mid\lambda(k),d>k} f(d) \Big| \leqq \sum_{d\mid\lambda(k),d>k} |f(d)| \\ &\leqq \sum_{m=k+1}^{\lambda(k)} |f(m)| < \sum_{m=k+1}^{\infty} |f(m)| \equiv R_k. \end{aligned}$$

But by the hypothesis of absolute convergence, $\lim R_k=0$ as $k\to\infty$; hence $\lim (T_k-S_k)=0$ or, equivalently, $\lim T_k=S$ ([**13**], Ch. 3, 4). This completes the proof.

We shall also need some additional facts.

LEMMA 11. *For fixed $n>0$, $c(n, r)$ is bounded as a function of r.*

Proof. By (21)

$$|c(n, r)| \leqq \sum_{d\mid(n,r)} d \leqq \sum_{d\mid n} d \equiv \sigma(n).$$

LEMMA 12 ([**18**], Th. 10). *If $s>0$, then for every $\epsilon>0$,*

$$\lim_{r\to\infty} \frac{\phi_s(r)}{r^{s-\epsilon}} = \infty. \tag{43}$$

A proof of Lemma 12 in the case $s=1$ appears in ([**9**], Sec. 18.4, Th. 327). The general case can be proved in an analogous manner.

Let $\zeta(t)$ denote the Riemann ζ-function, $\zeta(t)=\sum_{m=1}^{\infty} n^{-t}$, $t>1$. We shall need the well-known fact ([**9**], Sec. 17.5, Th. 287; [**13**], p. 446)

$$\zeta^{-1}(t) = \sum_{m=1}^{\infty} \frac{\mu(m)}{m^t} \qquad (t > 1). \tag{44}$$

The series in (44) is absolutely convergent.

We are now in a position to derive Ramanujan's expansions ([**15**], (6.1), (9.6)) of $\sigma_s(n)/n^s$ and $\phi_s(n)/n^s$. (Also see [**9**], Sec. 17.5, Th. 292 in the case of $\sigma_s(n)/n^s$.)

THEOREM 7. *If* $s>0$, $n>0$, *then*

$$\frac{\sigma_s(n)}{n^s} = \zeta(s+1)\sum_{m=1}^{\infty}\frac{c(n, m)}{m^{s+1}}. \tag{45}$$

Proof. By (40) we have, on transforming slightly and applying (9b),

$$\sum_{d|(n,r)}\frac{1}{d^s}\left(\sum_{e|(r/d)}\frac{\mu(e)}{e^{s+1}}\right) = \sum_{d|r}\frac{c(n, d)}{d^{s+1}}. \tag{46}$$

We shall take limits on both sides of (46) as r ranges over the sequence $r_k = k!$, using $L = L(n)$ to denote the resulting limit. Passing to the limit first on the left of (46), one obtains

$$L = \lim_{k\to\infty}\sum_{d|(n,r_k)}\frac{1}{d^s}\left(\sum_{e|(r_k/d)}\frac{\mu(e)}{e^{s+1}}\right) = \sum_{d|n}\frac{1}{d^s}\lim_{k\to\infty}\left(\sum_{e|(r_k/d)}\frac{\mu(e)}{e^{s+1}}\right).$$

Applying Lemma 10, (44), and (9a), it thus follows, for $s>0$, that

$$L = \left(\sum_{d|n}\frac{1}{d^s}\right)\zeta^{-1}(s+1) = \frac{\sigma_s(n)}{n^s}\zeta^{-1}(s+1) \qquad (s>0). \tag{47}$$

Passing now to the limit on the right of (46), it follows by Lemmas 10 and 11 that

$$L = \lim_{k\to\infty}\sum_{d|r_k}\frac{c(n, d)}{d^{s+1}} = \sum_{m=1}^{\infty}\frac{c(n, m)}{m^{s+1}} \qquad (s>0). \tag{48}$$

Theorem 7 results on comparing (47) and (48).

THEOREM 8. *If* $s>0$, $n>0$, *then*

$$\frac{\phi_s(n)}{n^s} = \zeta^{-1}(s+1)\sum_{m=1}^{\infty}\left(\frac{\mu(m)}{\phi_{s+1}(m)}\right)c(n, m). \tag{49}$$

Proof. By (41) and (9b) we have

$$\frac{\phi_s(n, r)}{(n, r)^s} = \left(\sum_{d|r}\frac{\mu(d)}{d^{s+1}}\right)\sum_{d|r}\left(\frac{\mu(d)}{\phi_{s+1}(d)}\right)c(n, d). \tag{50}$$

We again take limits in (50) as r ranges over the sequence $r_k = k!$ Denoting the limit by $L = L(n)$, we obtain on the left,

$$L = \lim_{k\to\infty}\frac{\phi_s(n, r_k)}{(n, r_k)^s} = \frac{\phi_s(n)}{n^s}. \tag{51}$$

By Lemma 10 and (44), we have for $s>0$,

(52) $$\lim_{k\to\infty} \sum_{d|r_k} \frac{\mu(d)}{d^{s+1}} = \zeta^{-1}(s+1).$$

Also, by Lemma 12, the series $\sum_{m=1}^{\infty} \phi_{s+1}^{-1}(m)$ is absolutely convergent for $s>0$. Hence, application of Lemmas 10 and 11 gives, for $s>0$,

(53) $$\lim_{k\to\infty} \sum_{d|r_k} \left(\frac{\mu(d)}{\phi_{s+1}(d)}\right) c(n, d) = \sum_{m=1}^{\infty} \left(\frac{\mu(m)}{\phi_{s+1}(m)}\right) c(n, m).$$

Passing to the limit on the right of (50), we obtain by (52) and (53),

(54) $$L = \zeta^{-1}(s+1) \sum_{m=1}^{\infty} \left(\frac{\mu(m)}{\phi_{s+1}(m)}\right) c(b, m) \qquad (s > 0).$$

Comparison of (51) and (54) yields the theorem.

Remarks. We mention that, although s has been assumed integral in this paper, Theorems 7 and 8 are actually valid for all real $s>0$. The preceding proofs remain valid in the more general case, because it can be shown that (46) is true for all s, while (50) is true for all $s\neq -1$. However, it is the case of integral s that is of chief interest, because it is in this case that the identities (46) and (50) have an arithmetical interpretation in terms of the congruences (6) and (7).

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AN ELEMENTARY TREATMENT OF THE IMBEDDING OF A GRAPH IN A SURFACE

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1. Introduction. The object of this paper is to give a simple proof of the following theorem of König ([1], p. 20; [2], p. 198):

Any connected graph may be imbedded in an orientable surface so as to form the vertices and edges of a map.

2. Definitions. By a *graph* we understand a collection of $N_0(\geqq 1)$ vertices, joined in pairs by N_1 $(\geqq 0)$ edges.

It is *connected* if every two vertices, say A and Z, are joined by a chain of consecutively adjacent edges

$$AB,\ BC,\ \cdots,\ XY,\ YZ.$$

An edge is called an *isthmus* if its removal leaves the graph disconnected. If every edge of a connected graph is an isthmus, the graph is a *tree* (and $N_0 - N_1 = 1$).

A *map* is the decomposition of an unbounded surface into N_2 simply-connected regions by the vertices and edges of a graph. In other words, the complement of the graph on the surface consists of N_2 simply-connected open regions, namely, polygons whose sides are whole edges of the graph. The *characteristic* of the surface is

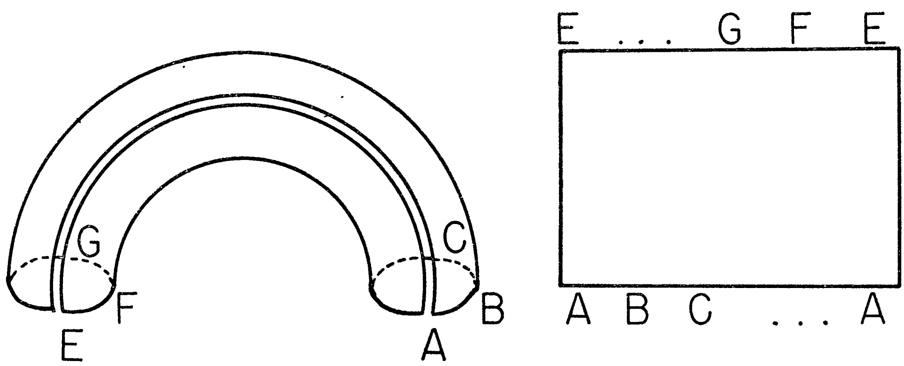
$$\chi = N_0 - N_1 + N_2.$$

If the surface is orientable, its *genus* is

$$p = 1 - \tfrac{1}{2}\chi.$$

(Such a surface may be regarded as a sphere with p handles.) We allow an edge of the map to occur twice among the sides of a region. For instance, the graph that has one vertex and one edge may be imbedded in a sphere to form the map $\{1, 2\}$ whose two regions are monogons. Similarly ([1], p. 101) the graph that has two vertices and one edge may be imbedded in a sphere to form the dual map $\{2, 1\}$ which has only one region, a digon, the sphere being slit by the edge. More generally, any tree can be imbedded in a sphere to form a map whose single region is a $2N_1$-gon.

3. Proof of the theorem. Observing that any graph of one edge may be imbedded in a sphere, we use induction over the number of edges. We assume that every connected graph of N_1-1 edges can be imbedded in some surface. Consider a given connected graph of N_1 edges. If it is a tree, the result is obvious. If not, there must be at least one edge AE that is not an isthmus. By our inductive assumption, we can imbed the rest of the graph in a certain surface to form a map M. If the two vertices A and E both belong to the same region of M, we simply join them on the same surface, thus dissecting that region into two smaller regions. If not, we complete our proof thus. There are two regions $ABC \cdots A$ and $EFG \cdots E$ with one end of AE on the boundary of each. Punch out these two regions and replace them by a tubular handle so as to increase the genus of the surface by 1. This can be done by expanding AE into a thin tube and flaring it at both ends to meet the edges of $ABC \cdots A$ and $EFG \cdots E$.



The modified surface contains a map having the same vertices as M, the same edges plus the extra one AE, and the same regions except that $ABC \cdots A$ and $EFG \cdots E$ (either of which may meet itself along an edge) are replaced by the single region $ABC \cdots AEFG \cdots EA$, which meets itself along the edge AE. Thus the whole graph has been imbedded as desired.

4. Concluding remarks. It may happen that a given graph can be imbedded several distinct ways. For instance, a "figure of eight," having one vertex and two edges, can be imbedded either in a sphere (so that the regions consist of two monogons and a digon) or in a torus (so that there is just one region, a quadrangle). For some graphs, such as the complete 6-point, the simplest imbedding is in a nonorientable surface ([1], p. 116).

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MATHEMATICAL NOTES

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ANOTHER PROOF OF CAUCHY'S GROUP THEOREM

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Since $ab=1$ implies $ba=b(ab)b^{-1}=1$, the identities are symmetrically placed in the group table of a finite group. Each row of a group table contains exactly one identity and thus if the group has even order, there are an even number of identities on the main diagonal. Therefore, $x^2=1$ has an even number of solutions.

Generalizing this observation, we obtain a simple proof of Cauchy's theorem. For another proof see [1].

CAUCHY'S THEOREM. *If the prime p divides the order of a finite group G, then G has kp solutions to the equation $x^p=1$.*

Let G have order n and denote the identity of G by 1. The set

$$S = \{(a_1, \cdots, a_p) \mid a_i \in G,\ a_1a_2 \cdots a_p = 1\}$$

has n^{p-1} members. Define an equivalence relation on S by saying two p-tuples are equivalent if one is a cyclic permutation of the other.

If all components of a p-tuple are equal then its equivalence class contains only one member. Otherwise, if two components of a p-tuple are distinct, there are p members in the equivalence class.

Let r denote the number of solutions to the equation $x^p=1$. Then r equals the number of equivalence classes with only one member. Let s denote the number of equivalence classes with p members. Then $r+sp=n^{p-1}$ and thus $p\mid r$.

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A REMARK ON BOUNDED FUNCTIONS

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Denote by E the class of functions regular and bounded by unity in $|z|<1$. Denote by E^* the subclass of functions of E which are in addition univalent in $|z|<1$. Analogies of various inequalities which are known to hold for functions in the class E have been obtained for functions of the class E^*. For example, it is known [3] that there exist functions in E for which the sequence $\{a_0+ \cdots +a_n\}$ $(f(z)=\sum a_nz^n)$ is unbounded. On the other hand, it is shown by Fejér in [1] that if $f\in E^*$ then $|a_0+ \cdots +a_n|<1+(1/\sqrt{2})$ for all n.

Hardy [2] has shown that if $f \in E$ then $\lim_{r\to 1}\sqrt{(1-r)}\,\overline{M}(r)=0$, where $\overline{M}(r) = \sum |a_n| r^n$. This raises the question of the behavior of $\overline{M}(r)$ (as $r \to 1$) for $f \in E^*$. To this end we prove the following theorem

THEOREM. *Let $f(z) = \sum a_n z^n$ map the unit disk upon a region of finite area. Then*

$$\lim_{r\to 1} \overline{M}(r)\left(\log \frac{1}{1-r^2}\right)^{-1/2} = 0. \tag{1}$$

Remark. We note that if $f \in E^*$ then f maps $|z| \leqq 1$ upon a finite area. Also if $f(z)$ is merely bounded it need not necessarily map the unit disk upon a finite area.

Proof. Since the unit disk is mapped upon a finite area we have

$$\infty > \iint_{|z|<1} |f'(z)|^2 dz = \sum_{n=1}^{\infty} n |a_n|^2.$$

Let $\epsilon > 0$ be given. Choose an integer k such that $\sum_{n=k+1}^{\infty} n|a_n|^2 < \epsilon$. Now

$$\begin{aligned}\overline{M}(r) &= \sum_{n=0}^{\infty} |a_n| r^n \\ &= \sum_{n=0}^{k} |a_n| r^n + \sum_{n=k+1}^{\infty} n^{1/2} |a_n| n^{-1/2} r^n.\end{aligned}$$

Upon applying the Schwarz inequality we have

$$\begin{aligned}\overline{M}(r) &\leqq \sum_{n=0}^{k} |a_n| r^n + \left(\sum_{n=k+1}^{\infty} n |a_n|^2 \sum_{n=k+1}^{\infty} n^{-1} r^{2n}\right)^{1/2} \\ &\leqq \sum_{n=0}^{k} |a_n| r^n + \left(\sum_{n=k+1}^{\infty} n |a_n|^2\right)^{1/2} \left(\log \frac{1}{1-r^2}\right)^{1/2}.\end{aligned}$$

Hence

$$\limsup_{r\to 1} \overline{M}(r)\left(\log \frac{1}{1-r^2}\right)^{-1/2} \leqq \left(\sum_{n=k+1}^{\infty} n |a_n|^2\right)^{1/2}.$$

But since the left member of the above inequality is independent of k, it follows that (1) holds. This completes the proof.

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GROUPS WHICH INDUCE A PARTITION OF A SET

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A group of transformations of a set can be used in a natural manner to define a partition of the set (*i.e.*, a division of the set into disjoint subsets), by specifying that two elements are in the same subset if there exists a transformation in the group which takes one element into the other.

An abstract group may be realized as a group of transformations in different ways so as to bring about several different partitions of a set. (By realization we mean that the abstract group and the group of transformations are isomorphic, so that distinct members of the abstract group give distinct transformations.) And, conversely, a given partition of a set may sometimes be accomplished by several different abstract groups. If a group G can be realized as a group of transformations of a set in a manner so that it causes a certain partition of the set, then we say that this partition is induced* by G. One method of approach to the problem of determining which groups induce a given partition is to investigate the possible orders for such a group and to limit these possible orders. Some results in this direction are given here.

We first establish an upper limit for these orders. Suppose that a set M is partitioned into n subsets M_i where M_i has m_i elements. If G is a group inducing the partition, then any transformation g in G can be factored into a product of cycles on disjoint letters, and the factorization is unique except for the order of the cycles. Any given cycle must contain only elements of one subset. There are $m_i!$ possible cycles on elements of M_i. Hence there can be at most $m_1! \cdots m_n!$ transformations g in G.

The possible orders for a group inducing a given partition are further limited by the theorem below. We consider the case where M is partitioned into two subsets.

THEOREM. *Let M be divided into subsets M_1 and M_2 having m_1 and m_2 elements, respectively, and let r be the least common multiple of m_1 and m_2. Then the order of any group which induces this partition is a multiple of r.*

Proof. Suppose the subsets are given by $M_1 = [a_1^1, a_2^1, \cdots, a_{m_1}^1]$ and $M_2 = [a_1^2, a_2^2, \cdots, a_{m_2}^2]$, and let G be any group inducing the given partition. Let H be the subgroup of G consisting of all ϕ_i such that $\phi_i(a_1^1) = a_1^1$.

We now examine the index of H in G. For each a_i^1, let π_i be a transformation in G such that $\pi_i(a_1^1) = a_i^1$. These π_i exist since G is transitive on M_1. Now $\pi_1 H = H$, and $\pi_i H \neq \pi_j H$ if $i \neq j$, since

$$\pi_i\phi_u(\overset{1}{a_1}) = \pi_i(\overset{1}{a_1}) = \overset{1}{a_i} \neq \overset{1}{a_j} = \pi_j(\overset{1}{a_1}) = \pi_j\phi_v(\overset{1}{a_1}),$$

where ϕ_u and ϕ_v are any transformations in H. Thus H has index at least m_1.

* Note that this usage of the word induce is different from that in which we say that a transformation of a set M induces a transformation in a subset of M.

Now let ρ be any transformation in G. Then $\rho(a_1^1) = a_i^1$ for some a_i^1 in M_1. But

$$\pi_i^{-1}\rho(a_1^1) = \pi_i^{-1}[\rho(a_1^1)] = \pi_i^{-1}(a_i^1) = a_1^1$$

since $\pi_i(a_1^1) = a_i^1$. Thus $\pi_i^{-1}\rho = \phi_j$ for some ϕ_j in H, and $\rho = \pi_i\phi_j$, which is in $\pi_i H$. This demonstrates that every transformation in G is in one of these cosets $\pi_i H$, $i = 1, \cdots, m_1$. Hence H is of index m_1 in G, so that the order of G is divisible by m_1.

Similarly the order of G is divisible by m_2, and therefore by $r = \text{l.c.m.}\ (m_1, m_2)$.

The previous theorem generalizes to any finite number of subsets, giving the corollary:

If M is partitioned into subsets $M_1, \cdots, M_n$ where M_i has m_i elements, and if r is the least common multiple of $m_1, \cdots, m_n$, then the order of any group inducing this partition is a multiple of r.

As might be expected, the restriction that a group G inducing a partition must be abelian results in a much smaller number of possible orders for G. The following theorem is concerned with this case.

THEOREM. *Let M be partitioned into n subsets M_i, where M_i has m_i elements, and let G be an abelian group inducing this partition. If k is the order of G, then $k \leqq m_1 \cdots m_n$.*

Proof. Let the subsets be given by $M_i = [a_1^i, a_2^i, \cdots, a_{m_i}^i]$, and let $g = k_1 \cdots k_n$ be a transformation in G, where k_i is a product of cycles on elements of M_i.

Suppose now that two transformations g and g' in G map a given element of M_i, say a_1^i, in the same way. For any element a_j^i of M_i, there exists h in G such that $h(a_1^i) = a_j^i$ since G is transitive on M_i. Thus

$$g(a_j^i) = g[h(a_1^i)] = h[g(a_1^i)] = h[g'(a_1^i)] = g'[h(a_1^i)] = g'(a_j^i),$$

so that g and g' map any element in M_i in the same way. This implies that g and g' have the same factor k_i, since k_i is the only factor which involves elements of M_i. That is, there are at most m_i factors k_i, and therefore no more than $m_1 \cdots m_n$ transformations $g = k_1 \cdots k_n$ in G.

The particular case where M is partitioned into one subset gives the corollary:

If G is an abelian group which is transitive on a set M of m elements, then G has order m.

Proof. From the theorem we have that the order k of G cannot be greater than m. But G is transitive on M so that $k \geqq m$. Hence, $k = m$.

It is known that any group of order m can be written transitively on m

elements. Thus the problem of finding all abelian groups which are transitive on a set of m elements is the same as that of finding all abelian groups of order m.

A special case of the partitioning in the theorem above yields the following result.

THEOREM. *Let M be partitioned into n subsets M_i of m_i elements where the m_i are relatively prime in pairs. If G is an abelian group inducing this partition, then G is a direct product of n subgroups, each of which induces a subset of the partition.*

The complete proof is omitted since it is rather long. Groups H_i of order m_i are constructed so that H_i is transitive on M_i and leaves all other subsets fixed. Then it is shown that these H_i are actually subgroups of G, and that the direct product of these subgroups is G.

In each of the theorems above, groups can be constructed which achieve the indicated bound.

The author wishes to express his appreciation to Professor R. W. Ball for suggesting the problem and for subsequent helpful remarks concerning it.

DESMIC SYSTEMS OF TETRAHEDRONS

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1. Eight lines on a quadric cone. (a) If two tetrahedrons $(T)=DABC$, $(M)=MM'M''M'''$ are harmonic, an edge, say, MM' of (M) meets two opposite edges, say, DA, BC of (T) in two points U', X' which separate the points M, M' harmonically ([1], p. 235). Hence the lines DM, DM' meet the plane ABC on the line AX' in two points S, S' harmonically separated by A, X'; that is, the point S' is an harmonic associate of S for the triangle ABC ([2], p. 246).

Similarly the projections of the vertices M'', M''' of (M) from D upon the plane ABC are the other two harmonic associates of S for the triangle ABC. Thus:

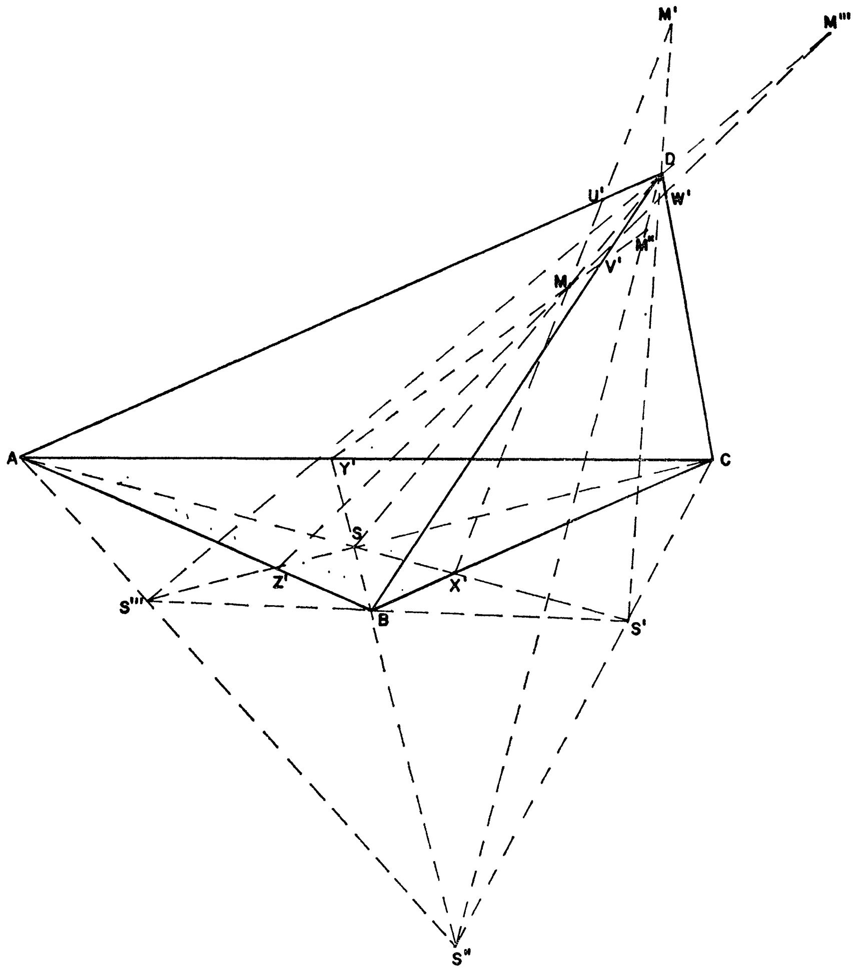
The four lines which project from a vertex of a tetrahedron (T) upon the opposite face the four vertices of a second tetrahedron harmonic to (T), meet that face of (T) in four points harmonically associated for the triangle of that face of (T).

(b) Four points harmonically associated with respect to a triangle are the vertices of a complete quadrangle whose diagonal triangle coincides with the given triangle. Thus if two tetrahedrons (M) and (N) are each harmonic to a given tetrahedron $(T)=DABC$ the projections of their vertices from the point D upon the plane ABC will form two complete quadrangles both having the triangle ABC for their diagonal triangle (art. 1a); hence those eight projections lie on a conic ([3], p. 203). Thus:

If two tetrahedrons are each harmonic to a third tetrahedron, the eight lines joining a vertex of the latter to the vertices of the first two tetrahedrons lie on a cone of the second degree.

2. A skew quartic curve of the first kind through sixteen points. (a) Let (M_i), (N_i) be the tetrahedrons which form a desmic system with the pairs of tetrahedrons (T), (M); (T), (N) (art. 1), respectively ([1], p. 237).

A line joining the vertex D of (T) to a vertex of the tetrahedron (M) passes through a vertex of (M_i) ([1], p. 238, art. 732); hence the vertices of (M_i) lie on the cone (D) having the vertex D of (T) for its vertex (art. 1). The vertices of the tetrahedron (N_i) lie on the cone (D) for analogous reasons. Thus the cone (D) contains all the vertices of the four tetrahedrons (M), (M_i), (N), (N_i).



Moreover, with the vertices A, B, C, of (T) we may associate cones (A), (B), (C), analogous to the cone (D) for the vertex D of (T). Thus the sixteen vertices of the four tetrahedrons lie on four distinct cones. Hence:

If a tetrahedron (T) belongs to two distinct desmic systems, the sixteen vertices of the remaining four tetrahedrons involved lie on a skew biquadratic curve of the first kind (C_4).

(b) *The four quadratic cones (D), (A), (B), (C) belong to the pencil of quadric surfaces determined by (C_4), and the tetrahedron (T) is conjugate to all the quadrics of the pencil* ([4], p. 699).

That (T) is conjugate to all the quadrics of the pencil may be shown directly. Any quadric (Q) passing through the skew quartic (C_4) is circumscribed to the tetrahedrons (M) and (M_i) of the desmic system (T), (M), (M_i). If we join a vertex, say, A of (T) to a vertex, say, M of (M), the line AM passes through a vertex, say, M_i of (M_i) and the trace A_0 of the line AMM_i in the face BCD is harmonically separated from A by the two points M, M_i ([1], p. 239, art. 734).

Since the points M, M_i lie on the quadric (Q), the point A_0 is conjugate to A for (Q). Now joining A to the other three vertices of (M) we obtain in the plane BCD three other points analogous to A_0 which proves, superabundantly, that the face BCD of (T) is the polar plane of A for the quadric (Q).

In a like manner it may be shown that the vertices B, C, D, of (T) have for their polar planes with respect to (Q) the faces CDA, DAB, ABC of (T); hence (T) is conjugate to (Q) (cf. [5], p. 307).

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CLASSROOM NOTES

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AN ADJUSTED TRAPEZOIDAL RULE USING FUNCTION VALUES WITHIN THE RANGE OF INTEGRATION*

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The presentation of the trapezoidal rule in textbooks on elementary calculus generally fails to include an estimate of error except in cases where the exact value of the definite integral can be determined by other methods. While such a

* See W. E. Milne, Numerical Calculus, Princeton University Press, 1949, pp. 116–120, for corrections to the trapezoidal rule requiring the knowledge of ordinates *outside* the range of integration.

procedure may tend to give the student an intuitive feeling about the usefulness of the trapezoidal rule, it is hardly a specific guide in situations where the student cannot obtain the exact value of the definite integral or cannot obtain it readily.

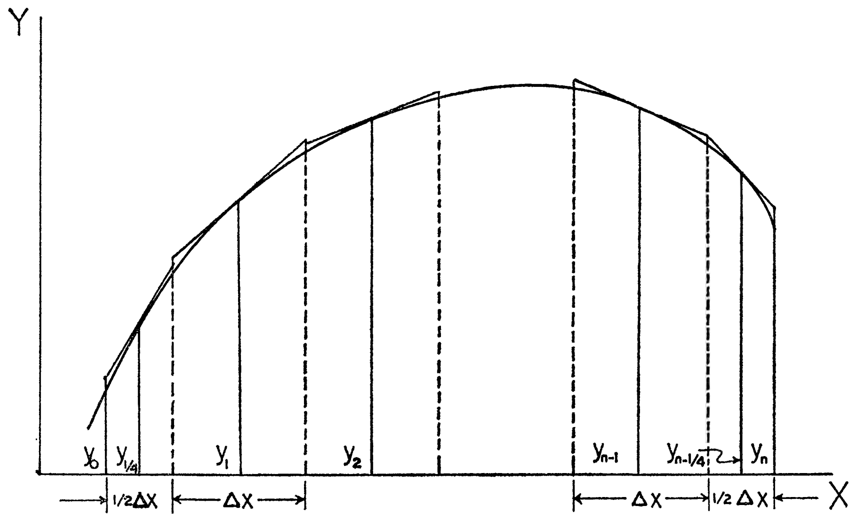
Let us consider the curve $y=f(x)$ in the interval $a\leqq x\leqq b$ throughout which the function is continuous and y'' does not change sign. The exact area under the curve will lie between the value given by the trapezoidal rule

$$K_1 = (\tfrac{1}{2}y_0 + y_1 + \cdots + y_{n-1} + \tfrac{1}{2}y_n)\Delta x \tag{1}$$

and the value given by the tangent trapezoids

$$K_2 = (\tfrac{1}{2}y_{1/4} + y_1 + \cdots + y_{n-1} + \tfrac{1}{2}y_{n-1/4})\Delta x, \tag{2}$$

where $y_{1/4}$ is the ordinate at $x=a+\frac{1}{4}\Delta x$, and $y_{n-1/4}$ is the ordinate at $x=b-\frac{1}{4}\Delta x$. All the ordinates in equation (2) are the medians of the respective tangent trapezoids formed by tangents at the extremities of these ordinates. In addition to y_0 and y_n, the bases of these trapezoids are the perpendiculars to the x-axis erected at the midpoint of each Δx interval.



Let us define $K=\frac{1}{2}(K_1+K_2)$ as the estimate of the area. Then

$$K = [\tfrac{1}{4}(y_0 + y_{1/4} + y_{n-1/4} + y_n) + y_1 + \cdots + y_{n-1}]\Delta x \tag{3}$$

may be called "an adjusted trapezoidal rule."

An upper bound of the error associated with the chosen value of Δx is

$$E = \left| K - K_1 \right| = \left| \tfrac{1}{4}([y_{1/4} + y_{n-1/4}) - (y_0 + y_n)]\Delta x \right| . \tag{4}$$

Thus, by determining only two additional values of the function, namely, at

$x=a+\frac{1}{4}\Delta x$ and at $x=b-\frac{1}{4}\Delta x$, not only is an adjusted approximation obtained for the area but an upper bound of the magnitude of error is provided as well.

Illustrative example. Evaluate $\int_1^5 \ln x dx$ by the trapezoidal rule and by the adjusted trapezoidal rule (3), using $\Delta x=.5$. (The value obtained by integration is 4.0472.)

x	y	Trapezoidal Rule	Adjusted Trapezoidal Rule (3)
1.0	$y_0=$ .00000	.00000	.00000
1.125	$y_{1/4}=$ .11778		.11778
4.875	$y_{n-1/4}=$1.58412		1.58412
5.0	$y_n=$1.60944	1.60944	1.60944
		2)1.60944	4)3.31134
		.80472	.82784
	$y_1+\cdots+y_{n-1}$	7.25665	7.25665
	Sum to be multiplied by Δx	8.06137	8.08449
Estimate of definite integral		4.0307	4.0422
Upper bound of error		No estimate	.0116 (by (4))
Actual error		.0165	.0050

DIFFERENTIATION OF THE REPEATED INTEGRAL

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Most texts on analysis include the rule for differentiating the single integral, namely if

$$\phi(\alpha) = \int_{a(\alpha)}^{b(\alpha)} f(x, \alpha)dx, \tag{1}$$

then under suitable conditions

$$\frac{d\phi}{d\alpha} = \int_a^b f_\alpha(x, \alpha)dx + f(b, \alpha)\frac{db}{d\alpha} - f(a, \alpha)\frac{da}{d\alpha}\cdot \tag{2}$$

Differentiation of the repeated integral is usually omitted. This author found it interesting and instructive to carry out for the case where the limits of integration are functions of a parameter.

If $f(x, y, \alpha)$ and its partial derivative $f_\alpha(x, y, \alpha)$ are continuous in the region $x_1(\alpha)\leqq x\leqq x_2(\alpha)$, $y_1(x, \alpha)\leqq y\leqq y_2(x, \alpha)$ and x_1, x_2, y_1, y_2 are differentiable functions of α within the region $c\leqq\alpha\leqq d$, then if

$$\phi(\alpha)=\int_{x_1(\alpha)}^{x_2(\alpha)}\int_{y_1(x,\alpha)}^{y_2(x,\alpha)} f(x,y,\alpha)dydx, \tag{3}$$

$$\begin{aligned}\frac{d\phi}{d\alpha}=&\int_{x_1(\alpha)}^{x_2(\alpha)}\int_{y_1(x,\alpha)}^{y_2(x,\alpha)}\frac{\partial f}{\partial\alpha}dydx+\int_{x_1(\alpha)}^{x_2(\alpha)} f(x,y_2,\alpha)\frac{\partial y_2}{\partial\alpha}dx\\ &-\int_{x_1(\alpha)}^{x_2(\alpha)} f(x,y_1,\alpha)\frac{\partial y_1}{\partial\alpha}dx+\int_{y_1(x_2,\alpha)}^{y_2(x_2,\alpha)} f(x_2,y,\alpha)\frac{\partial x_2}{\partial\alpha}dy\\ &-\int_{y_1(x_1,\alpha)}^{y_2(x_1,\alpha)} f(x_1,y,\alpha)\frac{\partial x_1}{\partial\alpha}dy.\end{aligned} \tag{4}$$

To show this let

$$\psi(x,\alpha)=\int_{y_1(x,\alpha)}^{y_2(x,\alpha)} f(x,y,\alpha)dy. \tag{5}$$

Now using (5), (3) may be written as

$$\phi(\alpha)=\int_{x_1(\alpha)}^{x_2(\alpha)}\psi(x,\alpha)dx,$$

$$\begin{aligned}\phi(\alpha+\Delta\alpha)&=\int_{x_1+\Delta x_1}^{x_2+\Delta x_2}\psi(x,\alpha+\Delta\alpha)dx=\int_{x_1+\Delta x_1}^{x_1}\psi(x,\alpha+\Delta\alpha)dx\\ &\quad+\int_{x_1}^{x_2}\psi(x,\alpha+\Delta\alpha)dx+\int_{x_2}^{x_2+\Delta x_2}\psi(x,\alpha+\Delta\alpha)dx.\end{aligned}$$

Hence

$$\begin{aligned}\frac{\Delta\phi}{\Delta\alpha}=&\int_{x_1}^{x_2}\frac{\psi(x,\alpha+\Delta\alpha)-\psi(x,\alpha)}{\Delta\alpha}dx+\frac{\Delta x_2}{\Delta\alpha}\psi(\epsilon_2,\alpha+\Delta\alpha)\\ &-\frac{\Delta x_1}{\Delta\alpha}\psi(\epsilon_1,\alpha+\Delta\alpha),\end{aligned} \tag{6}$$

where $|x_1-\epsilon_1|<|\Delta x_1|$ and $|x_2-\epsilon_2|<|\Delta x_2|$.

The last two terms of (6) follow from the fact that

$$\int_a^b f(x)dx=(b-a)f(\xi),\qquad a<\xi<b.$$

Now as $\Delta\alpha\to 0$, $\epsilon_1\to x_1$ and $\epsilon_2\to x_2$. Thus

$$\frac{d\phi}{d\alpha}=\int_{x_1}^{x_2}\frac{\partial\psi}{\partial\alpha}dx+\psi(x_2,\alpha)\frac{\partial x_2}{\partial\alpha}-\psi(x_1,\alpha)\frac{\partial x_1}{\partial\alpha}\cdot \tag{7}$$

If we use (5) in (7) we arrive at

$$
(8)\quad \begin{aligned}\frac{d\phi}{d\alpha} = \int_{x_1(\alpha)}^{x_2(\alpha)} \frac{\partial}{\partial\alpha}\left[\int_{y_1(x,\alpha)}^{y_2(x,\alpha)} f(x, y, \alpha)dy\right] dx + \int_{y_1(x_2,\alpha)}^{y_2(x_2,\alpha)} f(x_2, y, \alpha)\frac{\partial x_2}{\partial\alpha}\, dy \\ - \int_{y_1(x_1,\alpha)}^{y_2(x_1,\alpha)} f(x_1, y, \alpha)\frac{\partial x_1}{\partial\alpha}\, dy.\end{aligned}
$$

Using (2) in the first term of (8) we have the result (4).

It follows that under similar conditions this method of repeated application of (2) can be extended to include multiple integrals of any order.

NOTES ON THE FIBONACCI SEQUENCE

SAM E. GANIS, Ohio Wesleyan University

The Fibonacci sequence $\{f_n\}$ is defined by $f_1=f_2=1$, $f_n=f_{n-1}+f_{n-2}$, $n\geqq 3$. Simson noted that

$$
(1)\qquad f_{n-1}f_{n+1} - f_n^2 = (-1)^n.
$$

This can be proved by induction, for it is obviously true for $n=2$. If (1) is assumed true for $n=k$, then

$$
\begin{aligned}(-1)^k = f_{k-1}f_{k+1} - f_k^2 &= f_{k+1}(f_{k-1} + f_k) - f_k(f_k + f_{k+1}) \\ &= f_{k+1}f_{k+1} - f_kf_{k+2} = -(f_kf_{k+2} - f_{k+1}^2),\end{aligned}
$$

which proves (1) for $n=k+1$.

We can also prove by induction two well-known relations [(2) and (3) below].

$$
(2)\qquad f_{n+1} = {}_nC_0 + {}_{n-1}C_1 + {}_{n-2}C_2 + \cdots.
$$

Proof. The relation (2) is obviously true for $n=1$ and 2. If it is assumed true for $n=k-1$ and $n=k$, then

$$
f_{k+1} = {}_kC_0 + {}_{k-1}C_1 + {}_{k-2}C_2 + \cdots, \qquad f_k = {}_{k-1}C_0 + {}_{k-2}C_1 + \cdots.
$$

Adding the two equations we have

$$
f_{k+1} + f_k = {}_kC_0 + ({}_{k-1}C_0 + {}_{k-1}C_1) + ({}_{k-2}C_1 + {}_{k-2}C_2) + \cdots.
$$

Since ${}_kC_0={}_{k+1}C_0$, ${}_{k-1}C_0+{}_{k-1}C_1={}_kC_1$, *etc.*, then

$$
f_{k+2} = {}_{k+1}C_0 + {}_kC_1 + {}_{k-1}C_2 + \cdots,
$$

which proves (2) for $n=k+1$.

$$
(3)\qquad f_1 + \cdots + f_n = f_{n+2} - 1.
$$

Proof. The relation (3) is obviously true for $n=1$. If it is assumed true for $n=k$, then $f_{k+2}-1=f_1+\cdots+f_k$. Adding f_{k+1} to both sides of this equation, we have

$$f_{k+2} + f_{k+1} - 1 = f_1 + \cdots + f_k + f_{k+1}.$$

Hence $f_1 + \cdots + f_{k+1} = f_{k+3} - 1$, which proves (3) for $n = k+1$.

Two other relations [(4) and (5) below] can now be proved.

$$f_{n-2}f_{n+2} - f_n^2 = (-1)^{n+1}. \tag{4}$$

The proof of (4) can be made to depend on (1) as follows:

$$\begin{aligned} f_{n-2}f_{n+2} - f_n^2 &= f_{n-2}(f_{n+1} + f_n) - f_n(f_{n-2} + f_{n-1}) \\ &= f_{n-2}f_{n+1} - f_nf_{n-1} = f_{n-2}(f_{n-1} + f_n) - (f_{n-2} + f_{n-1})f_{n-1} \\ &= f_{n-2}f_n - f_{n-1}^2 = f_{n-2}f_n + f_{n-1}f_n - f_{n-1}f_n - f_{n-1}^2 \\ &= f_n(f_{n-1} + f_{n-2}) - f_{n-1}(f_n + f_{n-1}) = f_n^2 - f_{n-1}f_{n+1} = -(-1)^n, \end{aligned}$$

which proves (4). Adding (1) and (4), we have

$$f_{n-2}f_{n+2} + f_{n-1}f_{n+1} = 2f_n^2 \quad \text{or} \quad f_n = \{(f_{n-2}f_{n+2} + f_{n-1}f_{n+1})/2\}^{1/2}.$$

Also, subtracting (4) from (1), we have

$$f_{n-1}f_{n+1} - f_{n-2}f_{n+2} = 2(-1)^n. \tag{5}$$

ROLLING POLYGONS

ROBERT C. YATES, The College of William and Mary

Our purpose here is to obtain perimeters and areas of members of the family of cycloids by rolling polygons upon polygons. Ideas involved consist of elementary plane geometry, some trigonometry, and a speaking acquaintance with the limit concept. The ordinary cycloid, the nephroid, and the astroid serve as examples.

Each demonstration makes use of the trigonometric identities:

$$\sum_{k=1}^{n-1} \sin kx = \frac{\sin \frac{n}{2} x \sin \frac{n-1}{2} x}{\sin \frac{x}{2}}, \tag{I}$$

$$\sum_{k=1}^{n-1} \cos kx = \frac{\cos \frac{n}{2} x \sin \frac{n-1}{2} x}{\sin \frac{x}{2}}. \tag{II}$$

1. The cycloid.* A regular n-gon with circumradius a "rolls" without slipping upon a line (Fig. 1). A vertex P describes a set of $(n-1)$ circular arcs with

* See Problem E 1269, this MONTHLY, vol. 65, 1958, p. 45.

$$A_n = 4\left(\frac{a^2}{4}\right)\left(\frac{\pi}{4n}\right)\left[\sum_{k=1}^{4n-1} r_k^2 - \sum_{k=1}^{n-1} r_{4k}^2\right].$$

Setting

$$r_k^2 = \frac{a^2}{4}\sin^2 k\cdot\frac{\pi}{4n} = \frac{a^2}{8}\left(1 - \cos k\cdot\frac{\pi}{2n}\right)$$

and using (II), $A_n = 3\pi a^2/8$.

The sum of the areas of the triangular portions for each quarter arc is that of the rolling polygon $(na^2/8)\sin(\pi/2n)$, and for the entire trip $(na^2/2)\sin(\pi/2n)$. The fixed polygon has area $2na^2\sin(\pi/2n)$. The area interior to the path of P is then

$$\left(2n\cdot\sin\frac{\pi}{2n}\right)a^2 - \left(\frac{n}{2}\cdot\sin\frac{\pi}{2n}\right)a^2 - \frac{3\pi a^2}{8}\cdot$$

As $n\to\infty$, this has the limit $3\pi a^2/8$, the area of the astroid.

ON PLANE AREA IN POLAR COORDINATES

David Zeitlin, Remington Rand UNIVAC, St. Paul, Minnesota

Let $r = g(\theta)$ be the polar equation of a single valued, continuous curve, whose equation in rectangular coordinates is $y = f(x)$. As usual, with the pole at the origin of rectangular coordinates, we have $x = r\cos\theta$, $y = r\sin\theta$. Let $P_1(x_1, y_1)$ be a fixed point and $P(x, y)$ a variable point on $y = f(x)$; let $[r_1, \theta_1]$ and $[r, \theta]$ be their polar coordinates, respectively.

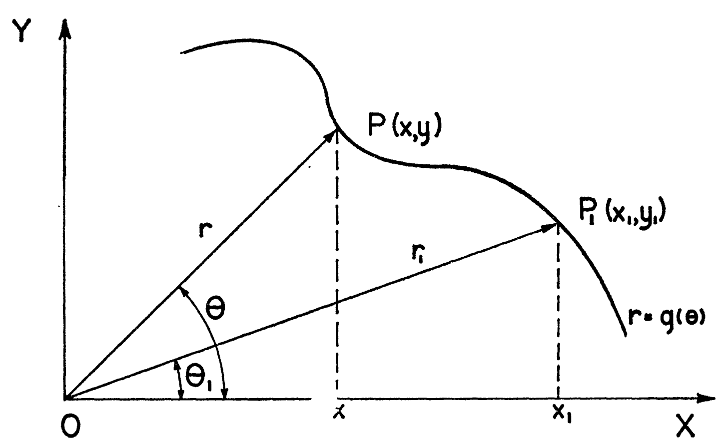

Fig. 1

Referring to Figure 1, let $A(\theta)$ be the area of the variable sector bounded by the curve $g(\theta)$ and the two radius vectors, r and r_1, *i.e.*, OP_1PO. Then

$$A(\theta) = \tfrac{1}{2}xy + \int_x^{x_1} f(x)dx - \tfrac{1}{2}x_1y_1. \tag{1}$$

Noting that x_1 and y_1 are constants, we find, differentiating both sides of (1) with respect to θ, that

$$\begin{aligned} \frac{d}{d\theta} A(\theta) &= \frac{1}{2}\left[x\frac{dy}{d\theta} + y\frac{dx}{d\theta}\right] - y\frac{dx}{d\theta} = \frac{1}{2}\left[x\frac{dy}{d\theta} - y\frac{dx}{d\theta}\right] \\ &= \frac{x^2}{2}\frac{d}{d\theta}\left(\frac{y}{x}\right) = \frac{r^2\cos^2\theta}{2}\frac{d}{d\theta}(\tan\theta) = \frac{r^2}{2}\cdot \end{aligned} \tag{2}$$

If $[r_2, \theta_2]$ specifies the point P_2 on $g(\theta)$, then it follows that

$$A(\theta_2) = \frac{1}{2}\cdot\int_{\theta_1}^{\theta_2} r^2 d\theta = \frac{1}{2}\cdot\int_{\theta_1}^{\theta_2} g^2(\theta)d\theta. \tag{3}$$

The above proof of (3) does not require knowledge of the formula, $\frac{1}{2}r^2\theta$, for the area of a circular sector; and unlike most orthodox proofs in textbooks, application of geometric intuition by the student is obviated, since the fundamental theorem of the integral calculus (*i.e.*, a limiting process involving a sum of areas of circular sectors) is not used. However, a simultaneous presentation of the fundamental theorem and the above proof of (3) would be of some academic value.

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN A. BROWN, University of Delaware, AND JOHN R. MAYOR, AAAS and University of Maryland

Contributions for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington 5, D. C.

A HIGH SPEED COMPUTER COURSE FOR HIGH SCHOOL STUDENTS AND TEACHERS

RICHARD V. ANDREE, The University of Oklahoma

Since the distribution of schools is rather sparse in Oklahoma, it has not seemed feasible to carry out a program of visiting high school lecturers without outside financial assistance. However, last year the University of Oklahoma, with the cooperation of the National High School and Junior College Mathematics Club, undertook a series of lectures in modern mathematics for high school students and teachers at the University campus in Norman. The response was so enthusiastic, with groups coming 200 miles to spend the day, that it was decided to extend the series this year to a full-semester (noncredit) course in *Programming the IBM 650 Computer and an Introduction to the Related Mathematics.*

No charge is made for the course, but participants provide their own transportation, luncheon, and text. A grant from the Oklahoma Frontiers of Science Foundation provides reimbursement for the instructor and two laboratory assistants. The University of Oklahoma furnishes classroom space and machine time.

A one-page mimeographed announcement was sent to major high schools in the area in mid-September announcing the course beginning October 4. Applications were accepted only for teacher-student combinations: no teachers without students; no students without a teacher. Classes meet both in the morning (laboratory) and in the afternoon (lecture) on alternate Saturdays during the entire semester. The announcement made it quite plain that a great deal of homework would be demanded of the participants, and that not everyone who wanted to come could be accepted. *Almost three hundred applications were received!* After some hurried consultations and changes in plans, it was decided to enlarge the group to 90 persons. Teacher-participants were chosen, and each teacher permitted to choose his own accompanying student. (In most cases only one student could be accepted per teacher, since there were so many applicants.) It proved difficult to say "no" to students eager enough to commute almost 600 miles (round trip) starting the day before, to take the course, so the class finally made up to 106 persons.

Enthusiasm is running high, and the resulting programs are well written—often ingenious. The experiment of having both teachers and students in the same class is proving highly successful. Each must "save face" by doing excellent work. The homework output is prodigious.

RESEARCH IN EDUCATION: A REPORT ON TWO CONFERENCES

R. M. WHALEY, Executive Director, Advisory Board on Education, NAS-NRC

The improvement of scientific and mathematical education is a primary concern of the Advisory Board on Education of the National Academy of Sciences-National Research Council. In working toward this objective through its various programs, the Board has sensed a need for more knowledge of the fundamental processes of learning. The increasing use of technological aids in education and of experimental programs involving changes of curriculum have emphasized this need.

A group of psychologists queried by the Board agreed that research on intellectual development and on learning in the schools has been seriously neglected. The ABE accordingly has sponsored two conferences of psychologists and educators to study the problem of research in education and recommend solutions.

Dr. Lyle Lanier, Head of the Department of Psychology at the University of Illinois, served as Chairman of the first meeting on April 24–26, 1958, at Easton, Maryland. The conference concluded that: (1) There has not been sufficient research on basic learning processes, possibly due to inadequate financial support; (2) Educational research results have not been systematically or-

ganized, codified, nor interpreted to be accessible to educators; and (3) Research on *human learning* and its application to education should be increasingly stressed.

The Easton Conference recommended consideration of the following: (1) A federal agency concerned with educational research, similar to the National Institute of Mental Health; (2) A nongovernmental agency, perhaps patterned after the National Academy of Sciences-National Research Council or the Social Science Research Council; (3) A summer institute on the psychology of learning.

A follow-up conference on July 9–11 in Madison, Wisconsin, under the chairmanship of Dr. T. R. McConnell, Director of the University of California Center for the Study of Higher Education, was requested to make recommendations for specific action. It recommended the establishment of an Organization for Research in Education to (1) Define the problems fundamental to improvement of education that might be solved by research; (2) Collect and classify existing data; (3) Promote and conduct research on the methodology of learning and of teaching; and (4) Assist in the recruitment and development of trained research personnel.

During the period when the proposed Organization for Research in Education is under consideration, the conference suggested that task force and conference groups be assembled to work on high priority education problems—problems which might be identified by national committees of scientists and mathematicians. Plans should be made this winter for a summer study in 1959.

THE HIGH SCHOOL MATHEMATICS CONTEST*

HENRY L. ALDER, UNIVERSITY OF CALIFORNIA, DAVIS

The eighty-minute contest was given on March 27, 1958, during the first two class periods. As in the past, the test was a multiple-choice examination with five choices listed for each question. The problems covered high school algebra and geometry, but not trigonometry or advanced algebra, as such. The contest examination was not confined to a specific syllabus nor to mere reproduction of classroom work.

Part I of the examination, consisting of 20 questions counting 2 credits each, tested the basic skills and techniques ordinarily associated with plane geometry and elementary and intermediate algebra. In Parts II (20 questions counting 3 credits each) and III (10 questions counting 5 credits each), however, the questions probed beyond such reproduction into the student's ability to meet new situations. The attempt was made, directly and indirectly, to convey to the students some of the fascination of mathematics, some of its broader vistas.

To discourage guessing, students were penalized for incorrect answers. The score S is calculated according to the formula $S=C-\frac{1}{4}(T-C)$, where C is the

* An official report on the contest is on page 171 of this issue.

number of points correct, T the number of points attempted.

For the purpose of awards, the United States and Canada were divided into nine regions. Every school sending in its results had a winner and received at least one award. To attempt to set up grade norms, or to standardize these tests in other ways, would have been basically contradictory to the objectives of the National Committee. However, to permit some comparable judgment of performance, the national median school-team score (obtained by adding the three highest individual scores), the median score for the first place winner, and lower and upper quartiles were published in a *Summary of Results and Awards*, which was sent to each participating school. The top three individuals in the 1958 contest were Michael Day, University High School, Urbana, Illinois (146.25 points), Jason E. Grosz, Bronx High School of Science, Bronx, New York (143.75 points), and Robert E. Kibler, Deering High School, Portland, Maine (136.3 points). The top three school-teams were from Bronx High School of Science (371.75 points), Brooklyn Technical High School (363.00 points), and Abraham Lincoln High School of Brooklyn (355.25 points).

For each region a bronze cup was awarded to the school with the highest team score. An engraved Certificate of Merit, engrossed with the name of the school and signed by the local contest chairman, was awarded on a regional basis to each of the schools with scores in the first decile of all participating schools, exclusive of the bronze-cup winner. One hundred books of mathematical tables, donated by the Chemical Rubber Publishing Company of Cleveland, Ohio, were distributed equitably to the highest-ranking students in each region. A gold pin with a facsimile of the seal of the Mathematical Association of America was awarded to the highest scorer in each participating school making a report.

While the National Committee does not make any awards other than those mentioned above, some of the local sections have built up an extensive system of local awards, consisting of scholarships, U. S. Saving Bonds, merchandise, *etc.*, which are financed by local business. Some sections used the contest as a screening device for participation in a second-stage test which is proctored at various centers. Thus, for example, the "Central Valley's Mathematics Quiz" in Northern California selected from the national contest the five top-scoring high school students in each county of that region to compete in a contest proctored at five centers. The winner of that contest, 15 year old John I. Castor of Clovis, California, received a $1000 Scholarship donated by the McClatchy newspaper chain.

The National Committee's encouragement for local administration of the contest to insure better attention to local problems has resulted in establishment of local Contest Committees in all but three of the Association's twenty-seven sections. Several large sections have organized several committees in accordance with geographical subdivisions of their regions. There are at present 32 local Contest Committees in the United States and Canada.

A MATHEMATICS-PHYSICS COURSE FOR IN-SERVICE TEACHERS

The Electric Boat Division of the General Dynamics Corporation, Groton, Connecticut, offered an intensive, integrated course in mathematics and physics for teachers at the secondary level in the summer of 1958, and is offering a second course this year.

Course description follows:

Calculus, Newtonian Mechanics, and Topics in Modern Physics

An intensive course in the mathematics necessary to the understanding of Newtonian mechanics and its descendants. The course emphasized ideas rather than techniques, but skill in differentiation and integration was suggested as a prerequisite.

Finite Mathematics, The Mathematics of Social Science

Topics include elements of logic, sets, and subsets, permutations and combinations, probability and statistics, vector and matrix algebra with applications in the social and physical sciences.

INDUSTRY EMPLOYS A SPECIAL TEACHER FOR A HIGH SCHOOL

In 1956 Olin Mathieson's Packaging Division of Monroe, Louisiana, undertook a plan to help qualify more students in the local high schools for careers in the physical sciences. The company underwrites the service of one outstanding teacher each year at one of the local high schools. The teacher is chosen to offer a course in mathematics, physics, or chemistry at a level more advanced than the normal curriculum can provide. It was believed that this plan would stimulate both the student body and the faculty alike.

An advanced chemistry class was organized at Neville High School in Monroe for the 1957–58 school session. A physics teacher has been employed for this year. The plan is made possible by a $10,000 grant used primarily to cover the teacher's salary.

JUNIOR-YEAR ENROLLEES IN SCIENCE AND MATHEMATICS

A report* of the Office of Education reveals that in November 1957 there were approximately 50,500 (from 1,104 four-year colleges and universities) junior-year enrollees in science and mathematics as a major field of study. These students are about 12.9% of all junior-year students. About 80.9% of the total number majoring in fields of science and mathematics were men.

Based on the above total enrollment figure, it is estimated that between 40 and 45 thousand bachelor's degrees in science and mathematics will be completed in the 1958–59 school year. There were 33,800 bachelor's degrees in the same fields in 1956–57.

REPORT OF THE COMMISSION ON MATHEMATICS

The report is expected to be ready for distribution about March 31. The December 1958 date (this MONTHLY, vol. 65, 1958, p. 773) was an error.

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* Junior-Year Science and Mathematics Students by Major Field of Study, Circular 520, prepared by the U. S. Department of Health, Education and Welfare, Office of Education. U. S. Government Printing Office, Washington 25, D. C. 56 pp. $0.45.

ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by Howard Eves, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1351. *Proposed by J. F. Darling, Woodstown, New Jersey*

Find the locus of the centers of the equilateral triangles whose sides a, b, c pass through three fixed points A, B, C respectively.

E 1352. *Proposed by C. W. Trigg, Los Angeles City College*

In setting the type for the multiplication $(abc)(bca)(cab)=234235286$, $a>b>c$, in which the unit's digit is 6, the remaining digits of the product became pied. Restore them to their proper order.

E 1353. *Proposed by L. A. Kenna, University of Arizona*

What is the eccentricity of an ellipse formed by a plane cutting the axis of a right circular cylinder at an angle θ?

E 1354. *Proposed by P. L. Chessin, University of Maryland*

If $(1+x)^{p-2}=1+a_1x+a_2x^2+\cdots+a_{p-2}x^{p-2}$, where p is a prime, then a_1+2, a_2-3, a_3+4, $\cdots$ are all multiples of p.

E 1355. *Proposed by A. J. Goldman, National Bureau of Standards*

If $y(x)>0$ and $y=\log[1+(\log x)/x+y/x]$ for all sufficiently large x, prove that $y\sim(\log x)/x$ as $x\to\infty$. (This is a step whose details are left to the reader in a paper by S. Chowla and F. C. Auluck, *Some properties of a function considered by Ramanujan*, J. Indian Math. Soc., vol. 4, 1940, pp. 169–173.)

SOLUTIONS

A Gathering of Six People

E 1321 [1958, 446]. *Proposed by C. W. Bostwick, Riverdale, Maryland*

Prove that at a gathering of any six people, some three of them are either mutual acquaintances or complete strangers to each other.

I. *Solution by John Rainwater, University of Washington.* Consider a fixed person A. Of the other five, there are either three whom A knows or three whom he does not. In the first instance, the three are either complete strangers, or two of them are acquaintances and form a trio of such with A. The other case is similar.

II. *Solution by J. D. Baum, Oberlin College.* If the six people are identified

with the vertices of an octahedron, and if each edge and diagonal is colored red or blue according as the two people it connects are strangers or acquaintances, then the problem reduces to Problem 2, Part I, William Lowell Putnam Prize Examination, 1953 [this MONTHLY, vol. 60, 1953, p. 541).

Also solved by W. E. F. Appuhn, Philip Bacon, D A. Breault, E. W. Brown, G. C. Bush, Fitch Cheney, R. J. Cormier, R. J. Driscoll, G. W. Erwin, Jr., Susan L. Friedman, Fred Galvin, L. D. Goldberg, Michael Goldberg, R. E. Greenwood, Cornelius Groenewoud, Werner Held, R. L. Helmbold, J. H. Hodges, R. Holt, A. R. Hyde, Irving Katz, J. D. E. Konhauser, William Kruskal, Joe Lipman, T. M. Little, D. B. Lloyd, R. F. McDermot, J. H. McKay, E. W. Marchand, D. C. B. Marsh, Helen Marston, J. B. Muskat, M. J. Pascual, R. I. Purry, Gustave Rabson, C. A. Reiher, Moses Richardson, J. T. Rosenbaum, Azriel Rosenfeld, Jack Silver, S. E. Spielberg, R. H. Sprague, and the proposer. Late solutions by L. D. Goldstone, F. W. Luttmann, and D. L. Silverman.

The Putnam problem may also be found in G. Gamov and M. Stern, *Puzzle-Math*, New York, 1958, pp. 93–95. This problem has been generalized by R. E. Greenwood and A. M. Gleason, *Combinatorial relations and chromatic graphs*, Canad. J. Math., vol. 7, 1955, pp. 1–7.

Several solvers showed that 6 is the smallest number with the stated property. Galvin showed that in any group of 18 people, there are four who are mutually acquainted or unacquainted, and that 18 is the smallest number with this property. Cheney stated that in any gathering of 6 people, there is a second trio who are either mutually acquainted or unacquainted.

Modified Harmonic Series

E 1322 [1958, 446]. *Proposed by J. W. Andrushkiw, Seton Hall University*

Multiply the first p terms of the harmonic series by $(-1)^k$, the next q terms by $(-1)^{k+1}$, the next p terms by $(-1)^k$, the next q terms by $(-1)^{k+1}$, and so on, alternately, thus forming a series denoted by $H(p, q, k)$. Show that $H(p, q, k)$ is convergent if and only if $p=q$.

I. *Solution by R. H. Sprague, University of Kentucky.* (a) Obviously $\sum_{n=r+1}^{r+p} 1/n > \sum_{n=r'+1}^{r'+p} 1/n$ whenever $r'>r$.

(b) Thus $H(p, p, k)$ acts like an alternating series whose terms approach zero, and thus converges.

(c) If $q>p$, omit the last $q-p$ terms from each q-block of terms in $H(p, q, k)$, forming a new series $H'(p, q, k)$. Because of (a), $H'(p, q, k)$ converges. The omitted terms are all of the same sign; hence we may assume them to be positive. Then each term in the nth q-block is no less than the last term in that block, $1/n(p+q)$; thus the sum of the $q-p$ omitted terms is no less than $(q-p)/n(p+q)$. But $\sum_{n=0}^{\infty} (q-p)/n(p+q)$ clearly diverges; hence $H(p, q, k)$ diverges.

(d) Finally, if $p>q$, we note that the absolute value of the last term in the nth p-block is $1/[np+(n-1)q]>1/n(p+q)$ and proceed as in (c).

II. *Solution by A. E. Danese, Union College.* If $p=q$, the series is of the form $\sum a_nb_n$ with $\sum a_n$ having bounded partial sums and the sequence $\{b_n\}$ monotonically decreasing to zero. Convergence follows by Dirichlet's test [Knopp, *Theory and Application of Infinite Series*, London, 1928, p. 315].

If $p\neq q$, divergence follows immediately by the theorem due to Cesàro [*ibid.*, p. 318, 4]: If $\sum a_n$ converges, but not absolutely, $|a_n|$ is monotonically decreasing, p_n is the number of positive terms and q_n is the number of negative

terms a_m for $m \leqq n$, then $\lim_{n\to\infty} p_n/q_n = 1$, if the limit exists.

The problem occurs as an exercise [*ibid.*, p. 150, 52].

Also solved by Fred Galvin, L. D. Goldberg, Vern Hoggatt, R. H. Hou, J. H. Jordan, Joe Lipman, D. C. B. Marsh, Douglas Maurer, D. L. Muench, M. J. Pascual, Azriel Rosenfeld, Jack Silver, W. A. Veech, Clement Winston, and the proposer. Late solution by J. B. Muskat.

Concerning the Roots of a Special Polynomial

E 1323 [1958, 446]. *Proposed by Harry Goheen, Oregon State College*

Prove that all roots but one of the equation

$$nx^n = 1 + x + x^2 + \cdots + x^{n-1}$$

have absolute value less than 1.

I. *Solution by Joe Lipman, University of Toronto.* When $|x| \geqq 1$,

$$\begin{aligned} |1 + x + x^2 + \cdots + x^{n-1}| &\leqq |1| + |x| + |x|^2 + \cdots + |x|^{n-1} \\ &\leqq n|x|^n = |nx^n|, \end{aligned}$$

with equality holding throughout if and only if $x=1$. Thus, if

$$1 + x + x^2 + \cdots + x^{n-1} = nx^n,$$

$|x|$ must be <1, except for the one-fold root $x=1$.

II. *Solution by J. B. Roberts, Reed College.* Putting $x=1/y$ yields

$$\begin{aligned} &nx^n - x^{n-1} - x^{n-2} - \cdots - x - 1 \\ &\qquad = (1/y - 1)(1/y^{n-1})(y^{n-1} + 2y^{n-2} + 3y^{n-3} + \cdots + n). \end{aligned}$$

The zeros of the last factor on the right are in absolute value >1 by a theorem of Kakeya: "*If* $0 < a_0 < a_1 < \cdots < a_n$, *then all the roots of*

$$a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$$

are outside the unit circle" (see Tôhoku Math. J., vol. 2, 1912, pp. 140–142, or Dienes, *The Taylor Series*, Ex. 17, p. 67). Thus the zeros of the left side (other than 1) are in absolute value <1.

III. *Solution by A. E. Danese, Union College.* Since $x=1$ is clearly a root of the equation, it remains to show that the zeros of

$$f(x) = 1 + 2x + 3x^2 + \cdots + nx^{n-1}$$

have absolute value <1.

A. The zeros of $f(x)$ lie in the annulus $1/2 \leqq |x| \leqq 1 - 1/n$, since if all the coefficients of the polynomial

$$p_0z^n + p_1z^{n-1} + \cdots + p_{n-1}z + p_n$$

are positive, its zeros lie in $\alpha \leqq |z| \leqq \beta$, where α is the smallest and β the largest of

$$p_1/p_0,\ p_2/p_1,\ p_3/p_2,\ \cdots,\ p_n/p_{n-1}.$$

(Pólya and Szegö, *Aufgaben und Lehrsätz aus der Analysis*, Erster Band, New York, 1945, p. 88, 23.)

B. Or, $f(x)$ is the derivative of $1+x+x^2+\cdots+x^n$, whose zeros are the nth roots of unity. The result follows with the application of the theorem (*ibid.*, p. 89, 31): The zeros of the derivative of the polynomial $P(z)$, which are not zeros of $P(z)$, lie in the interior of the smallest convex polygon which contains the zeros of $P(z)$.

Also solved by David Adorno and Joel Owen (jointly), R. G. Albert, W. E. F. Appuhn, Merrill Barnebey, D. R. Barr, D. A. Breault, B. H. Brown, A. G. Clark, Eleanor G. Dawley, E. L. Ellis, Fred Galvin, L. D. Goldberg, Michael Goldberg, A. G. Grace, Jr., Norman Greenspan, Emil Grosswald, Vern Hoggatt, R. H. Hou, Irving Katz, Jesse Krehbiel, D. C. B. Marsh, R. A. Miller, M. J. Pascual, John Rainwater, Azriel Rosenfeld, Jack Silver, R. H. Sprague, W. A. Veech, David Zeitlin, and the proposer. Late solutions by R. G. Albert, David Fink, and L. I. Lowell.

Brown pointed out that a generalization of this problem has found application in the sociological sciences. See E. D. Domar, *Depreciation, replacement and growth—and fluctuation*, Economic J., vol. 47, 1957, pp. 654–658. Also see R. M. Solow, *A note on dynamic multipliers*, Econometrica, vol. 19, 1951, pp. 306–16.

Tic-tac-toe as a Game of Chance

E 1324 [1958, 447]. *Proposed by F. E. Clark, Rutgers University*

Suppose tic-tac-toe is turned into a game of pure chance as follows. Designate the squares by $1, \cdots, 9$, numbering them from left to right by successive rows, starting with the top row. Place a set of nine chips, labeled $1, \cdots, 9$, in a bag. The first player, A, draws a chip at random and enters an X in the corresponding square. The second player, B, draws at random from the remaining chips and enters an O in the corresponding square. The game ends, of course, as soon as three like entries are obtained which lie along a line (horizontally, vertically, or diagonally). Find the probability that A will win, draw, or lose the game. Also find the probability, for each drawing, that the game will end when that chip is drawn.

Solution by T. M. Little, University of California. The number of combinations of 5 X's and 4 O's in the nine squares is $9!/5!4!=126$, since any combination of 5 X's determines the four O's and vice versa. These 126 combinations can be classified into five groups as follows: (1) 22 combinations containing 2 winning combinations for X; (2) 40 containing a single winning combination of X's; (3) 12 containing a single winning combination of O's; (4) 36 containing both a winning combination of X's and one of O's; and (5) 16 containing no winning combinations for O or X, thus resulting in a draw after the eighth chip is drawn.

For class (1) the probability is $1/5$ that A will win with the fifth chip, $3/5$ that A will win with the seventh chip, and $1/5$ that he will win with the ninth chip. For class (2) the probability is $1/10$ that A will win with the fifth chip, $3/10$ that he will win with the seventh chip, and $3/5$ that he will win with the

ninth. For class (3) the probability is 1/4 that B will win with the sixth chip and 3/4 that he will win with the eighth. For class (4) the probability is 1/10 that A will win with the fifth chip, 9/40 that B will win with the sixth, 9/40 that A will win with the seventh, and 9/20 that B will win with the eighth. For class (5) it is certain that the game will end in a draw when the eighth chip is drawn.

Multiplying these probabilities by the frequency of the corresponding class, we get the following probabilities:

A winning with 5th chip:
$(22/126)(1/5)+(40/126)(1/10)+(36/126)(1/10)=120/1260$
B winning with 6th chip:
$(12/126)(1/4)+(36/126)(9/40)=111/1260$
A winning with 7th chip:
$(22/126)(3/5)+(40/126)(3/10)+(36/126)(9/40)=333/1260$
B winning with 8th chip:
$(12/126)(3/4)+(36/126)(9/20)=252/1260$
A winning with 9th chip:
$(22/126)(1/5)+(40/126)(3/5)=284/1260$
Game ending in draw:
$16/126=160/1260$

A's total probability of winning is therefore 737/1260 and of losing is 363/1260.

Also solved by Fred Galvin, Michael Goldberg, Werner Held, A. R. Hyde, Joe Lipman, D. C. B. Marsh, J. A. O'Brien, Azriel Rosenfeld, and the proposer.

Not all these solutions agreed with the above.

The Matrix Equation $X^2-2AX+B=0$

E 1325 [1958, 447]. *Proposed by Peter Treuenfels, Brookhaven National Laboratory, Upton, L. I., New York*

Let A, B, X denote $n\times n$ matrices. Show that a sufficient condition for the existence of at least one solution X of the matrix equation

$$X^2 - 2AX + B = 0$$

is that the eigenvalues of the $2n\times 2n$ matrix

$$R = \begin{bmatrix} A & I \\ A^2 - B & A \end{bmatrix}$$

be pairwise distinct. Here I denotes the $n\times n$ identity matrix.

Solution by the proposer. Since the eigenvalues of R are pairwise distinct, there exists a $2n\times 2n$ matrix S which diagonalizes R; thus $S^{-1}RS=M$, where M is the diagonal matrix of the eigenvalues of R. Let

$$S = \begin{bmatrix} T & U \\ V & W \end{bmatrix} \quad \text{and} \quad M = \begin{bmatrix} L & 0 \\ 0 & N \end{bmatrix},$$

where T, U, V, W, L, N denote $n \times n$ matrices and where L and N are diagonal matrices. Our proof will be in two parts; we shall show that: (1) by permuting the eigenvalues of R, if necessary, we can always achieve that T is nonsingular; and (2) that $X = TLT^{-1}$ is a solution of the given matrix equation.

(1) The transforming matrix S is nonsingular; hence its $2n$ row-vectors are linearly independent, whence, *a fortiori*, its first n row-vectors are linearly independent. Hence the rank of the $n \times 2n$ submatrix $[TU]$ of S is n, and at least one $n \times n$ submatrix T^0 of $[TU]$ exists which is nonsingular. By shifting columns of the original transforming matrix we can bring this nonsingular submatrix T^0 into the upper left-hand corner of the transforming matrix. Let us denote the transforming matrix thus obtained by S^0. $(S^0)^{-1}$ is obtained from S^{-1} by a shift of the corresponding rows. Let $M^0 = S^0R^0(S^0)^{-1}$; then M^0 is obtained from M by a shifting of corresponding rows and columns. But, since M is diagonal, this is equivalent to a permutation of the eigenvalues of R. We may therefore assume that the eigenvalues of R have been ordered in such a way that T is nonsingular.

(2) Since $RS = SM$, or

$$\begin{bmatrix} A & I \\ A^2 - B & A \end{bmatrix} \begin{bmatrix} T & U \\ V & W \end{bmatrix} = \begin{bmatrix} T & U \\ V & W \end{bmatrix} \begin{bmatrix} L & 0 \\ 0 & N \end{bmatrix},$$

we obtain $AT + V = TL$, $A^2T - BT + AV = VL$. Solving the first of these equations for V, and substituting for V in the second equation, we find $TL^2 - 2ATL + BT = 0$. Multiplying by T^{-1} on the right yields $TL^2T^{-1} - 2ATLT^{-1} + B = 0$, or $X^2 - 2AX + B = 0$, for $X = TLT^{-1}$.

Also solved by Jack Silver.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscript should be typewritten, with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4828. *Proposed by M. S. Klamkin, AVCO Research, Wilmington, Mass.*

Do the sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ converge, where

$$a_{n+1} = \int_0^1 \min(x, b_n, c_n)dx,$$

$$b_{n+1} = \int_0^1 \operatorname{mid}(x, c_n, a_n)dx, \qquad c_{n+1} = \int_0^1 \max(x, a_n, b_n)dx,$$

and mid $(a, b, c) = b$ if $a \geqq b \geqq c$.

4829. *Proposed by J. de Groot, University of Amsterdam, The Netherlands*

Give a simple example of a continuum P such that, for every countable group G, there exists a group of topological transformations of P onto itself which is isomorphic to G.

4830. *Proposed by D. B. Larson, Cornell Aeronautical Laboratory, Buffalo, N. Y.*

Evaluate the integral

$$\int_{-1}^{1} e^{iAx}(1 - Bx^2)^{-1}(1 - x^2)^{-1/2}dx,$$

where A and B are positive, real constants, $B < 1$ and $i^2 = -1$.

4831. *Proposed by J. L. Massera and J. J. Schäffer, Institute of Mathematics and Statistics, Montevideo, Uruguay*

J. A. Clarkson [*Uniformly convex spaces*, Trans. Amer. Math. Soc., vol. 40, 1936, pp. 396–414] introduced the following expression for the "angle" of two nonvanishing elements of any linear normed space:

$$\alpha[x, y] = \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|.$$

Prove that if $x, y, x+y \neq 0$, then $\alpha[x+y, x] \leqq \alpha[x, y]$.

4832. *Proposed by A. Oppenheim, University of Malaya, Singapore*

Find all integral solutions of the following Diophantine equations:

$$\text{(A)} \qquad ax^2 + ay^2 + z^2 - 2axyz - 1 = 0 \qquad (a = 1, 2, \cdots),$$

$$\text{(B)} \qquad 9x^2 + 25y^2 + 49z^2 - 210xyz - 1 = 0,$$

$$\text{(C)} \qquad 9x^2 + 25y^2 + 4z^2 - 60xyz - 1 = 0.$$

[The case $a = 1$ of (A) is known.]

4833. *Proposed by F. H. Northover, Carleton University, Ottawa*

In the polynomial $A(x) = \sum_{i=0}^{n} a_i x^i$, $a_i \geqq 0$, $A(1) = 1$, and the function

$$\frac{A'(1) - xA'(x)}{(1 - x)\{1 - A(x)\}}$$

is expanded into $\sum_{i=0}^{n} b_i x^i$. Prove that $b_i \geqq A'(1)$ for all i.

SOLUTIONS

Real Functions

4756 [1956, 596; 1958, 534]. *Proposed by J. L. Massera, Institute of Mathematics and Statistics, Montevideo, Uruguay*

Let $p(x_1, \cdots, x_n)$, $q(x_1, \cdots, x_n)$ be two real functions of n real variables x_i, defined and continuous in a parallelotope $R: 0 \leqq x_i \leqq a_i < \infty$. Assume that $p(x_1, \cdots) = q(x_1, \cdots) = 0$ whenever $x_1x_2 \cdots x_n = 0$, and that $p(x_1, \cdots) > 0$, $q(x_1, \cdots) \geqq 0$ when $x_1x_2 \cdots x_n \neq 0$. Prove there exists a real function $h(u)$ of a real variable u, defined, continuous and strictly increasing for $u \geqq 0$, $h(0) = 0$, such that throughout R, except when $x_1x_2 \cdots x_n = 0$,

$$h\{q(x_1, \cdots)\} < p(x_1, \cdots).$$

II. *Solution by John Rainwater, University of Washington, Seattle.* The solution given in the September issue is fallacious as may be seen easily from the example: $n=1$, $p(x)=x^2$, $q(x)=x$. $h(x)$ can be found, however, as follows.

Let $\mathbf{x} = (x_1, \cdots, x_n)$. Define the function $i(u)$ for real numbers u between 0 and the greatest value q_0 of q by $i(u) = \min \{p(\mathbf{x}) \mid q(\mathbf{x}) \geqq u\}$; for $u > q_0$, let $i(u)$ be the maximum value p_0 of p. In general i will be discontinuous, but it is increasing (hence integrable) and strictly positive for $u > 0$. Define

$$j(u) = \frac{1}{u} \int_0^u i(t)dt \text{ for } u > 0, \qquad j(0) = 0.$$

Evidently j is continuous and $j \leqq i$; in particular $j\{q(\mathbf{x})\} \leqq p(\mathbf{x})$ for all $\mathbf{x}$. Then let $h = j/2$.

In an arbitrary compact space the result becomes: if p and q are nonnegative functions and the set of zeros of q contains the set of zeros of p, then $p \leqq hq$ for some strictly increasing continuous h vanishing at 0.

Valid solutions also by J. Horváth and by the proposer.

Polynomial Solution of a Differential System

4783 [1958, 288]. *Proposed by Leonard Carlitz, Duke University*

Find the polynomial solution of the system

$$(1) \qquad \begin{aligned} xf'(x) - g'(x) &= \mu f(x) \\ xg'(x) - f'(x) &= \nu g(x), \end{aligned}$$

where μ, ν are assigned constants.

Solution by the proposer. I. Let the degree of $f(x)$ exceed that of $g(x)$. Then the first equation in (1) implies $\deg f(x) = m = \mu$, and the second implies $\deg g(x) = m - 1$. Put

$$\begin{aligned} f(x) &= a_0x^m + a_1x^{m-1} + a_2x^{m-2} + \cdots \qquad (a_0 \neq 0), \\ g(x) &= b_1x^{m-1} + b_2x^{m-2} + \cdots. \end{aligned}$$

Then the system (1) is equivalent to the system of equations

$$(m-r)a_r-(m-r+1)b_{r-1}=ma_r,\qquad (m-r)b_r-(m-r+1)a_{r-1}=\nu b_r$$

which is the same as

$$\begin{aligned} ra_r &= -(m-r+1)b_{r-1} \\ (m-r-\nu)b_r &= (m-r+1)a_{r-1} \end{aligned}\qquad (r \geqq 1). \tag{2}$$

Since $b_1 \neq 0$ it follows that $a_2 \neq 0$, this in turn implies $b_3 \neq 0$, $a_4 \neq 0$, and so on. That is, we have generally $a_{2r} \neq 0$, $b_{2r-1} \neq 0$. Moreover the second equation in (2) then yields

$$\nu \neq m-2r+1 \qquad (1 \leqq r \leqq \tfrac{1}{2}m). \tag{3}$$

On the other hand it is clear that $a_1=0$, which implies $(m-2-\nu)b_2=0$, so that either $b_2=0$ or $\nu=m-2$. If $b_2=0$, then $a_3=0$ and $(m-4-\nu)b_4=0$, and so on. Thus if

$$\nu \neq m-2k \qquad (1 \leqq k \leqq \tfrac{1}{2}m), \tag{4}$$

then all $a_{2r-1}=b_{2r}=0$. If, however, $\nu=m-2k$ for some k in the interval $1 \leqq k \leqq \frac{1}{2}m$, we can only assert that $a_{2s-1}=b_{2s-2}=0$ $(1 \leqq s \leqq k)$.

Assume first that (4) is satisfied. By means of (2) we get

$$a_{2r}=-\frac{m-2r+1}{2r}b_{2r-1}=-\frac{(m-2r+1)(m-2r+2)}{2r(m-2r+1-\nu)}a_{2r-2},$$

from which it is clear that

$$\begin{aligned} a_{2r} &= (-1)^r\frac{m(m-1)\cdots(m-2r+1)}{2^r r!(m-\nu-2r+1)(m-\nu-2r+3)\cdots(m-\nu-1)}a_0 \\ &= (-1)^r\frac{m!a_0}{2^{2r}r!(m-2r)!(\lambda+m-r)_r}, \end{aligned}$$

where $\lambda=\frac{1}{2}(1-m-\nu)$ and $(a)_r=a(a+1)\cdots(a+m-1)$, $(a)_0=1$. Now put

$$f_1(x)=\sum_{2r\leqq m}a_{2r}x^{m-2r}=\frac{a_0 m!}{2^m}\sum_{2r\leqq m}(-1)^r\frac{(2x)^{m-2r}}{r!(m-2r)!(\lambda+m-r)_r};$$

because of (3) no factor in the denominator of a_{2r} can vanish. Alternatively we may write

$$f_1(x)=\frac{a_0}{2^m}\frac{m!}{(\lambda)_m}P_m^{(\lambda)}(x), \tag{5}$$

where $P_m^{(\lambda)}(x)$ is the ultraspherical polynomial of degree m (see for example, Szegö, *Orthogonal Polynomials*, 1939, p. 84); note however that for certain λ the representation (5) breaks down.

Turning next to the b's we find that

$$b_{2r+1} = \frac{m-2r}{m-2r-1-\nu}\, a_{2r} = \frac{(-1)^r m! a_0}{2^{2r+1} r!(m-1-2r)!(\lambda+m-1-r)_{r+1}}.$$

Hence we put

$$g_1(x) = \sum_{2r<m} b_{2r+1}x^{m-2r-1} = \frac{a_0}{2^m}\,\frac{m!}{(\lambda)_m}\, P_{m-1}^{(\lambda)}(x). \tag{6}$$

Next assume that $\nu = m - 2k$ for some k in the interval $1 \leqq k \leqq \frac{1}{2}m$. Then a_{2k+1} is not determined, but using (2) we get

$$\begin{aligned} a_{2k+2r+1} &= \frac{(\nu-1)!}{2^r r!(\nu-1-2r)!}\,\frac{a_{2k+1}}{(2k+3)(2k+5)\cdots(2k+2r+1)} \\ &= (-1)^r \frac{(\nu-1)! a_{2k+1}}{2^{2r} r!(\nu-1-2r)!(\lambda+\nu-1-r)_r} \end{aligned}$$

and accordingly put

$$f_2(x) = \sum_{2r<\nu} a_{2k+2r+1}x^{\nu-2r-1} = \frac{a_{2k+1}}{2^{\nu-1}}\,\frac{(\nu-1)!}{(\lambda)_{\nu-1}}\, P_{\nu-1}^{(\lambda)}(x).$$

Similarly we get

$$b_{2k+2r} = -\frac{2k+2r+1}{\nu-2r}\, a_{2k} = (-1)^r \frac{(\nu-1)!(\lambda)_{\nu-r}}{2^{2r-1} r!(\nu-2r)!(\lambda)_{\nu-1}}\, a_{2k+1},$$

so that

$$\sum_{2r\leqq\nu} b_{2k+2r}x^{\nu-2r} = \frac{a_{2k+1}}{2^{\nu-1}}\,\frac{(\nu-1)!}{(\lambda)_{\nu-1}} + P_\nu^{(\lambda)}(x).$$

To sum up, if (4) holds, then

$$f(x) = \frac{A}{(\lambda)_m} P_m^{(\lambda)}(x), \qquad g(x) = \frac{A}{(\lambda)_m} P_{m-1}^{(\lambda)}(x) \tag{7}$$

furnish the general polynomial solution of (1); but if $\nu = m - 2k$, where $1 \leqq k \leqq \frac{1}{2}m$, then the general polynomial solution may be written in the form

$$f(x) = AP_m^{(\lambda)}(x) + BP_{\nu-1}^{(\lambda)}(x), \qquad g(x) = AP_{m-1}^{(\lambda)}(x) + BP_\nu^{(\lambda)}(x), \tag{8}$$

where A and B are arbitrary constants. Note in particular that when $\nu = -m$, $\lambda = \frac{1}{2}$ and (7) reduces to $f(x) = CP_m(x)$, $g(x) = CP_{m-1}(x)$, where $P_m(x)$ is the Legendre polynomial. Also when $\nu = 0$, define $P_{-1}^{(\lambda)}(x) = 0$; however in this case (8) reduces to

$$f(x) = A(x^2 - 1)^k, \qquad g(x) = Am\int (x^2 - 1)^{k-1}dx + B.$$

II. An obvious interchange of roles disposes of the case where $\deg g(x) > \deg f(x)$.

III. Let $\deg f(x) = \deg g(x) = m$; thus $\mu = \nu = m$. Then (1) implies

$$(x + 1)\{f'(x) - g'(x)\} = m\{f(x) - g(x)\},$$
$$(x - 1)\{f'(x) + g'(x)\} = m\{f(x) + g(x)\},$$

from which we get $f(x) - g(x) = 2A(x+1)^m$, $f(x) + g(x) = 2B(x-1)^m$. Therefore

$$f(x) = A(x + 1)^m + B(x - 1)^m, \qquad g(x) = -A(x + 1)^m + B(x - 1)^m$$

furnish the general solution in this case.

Remark. An alternative method of solving (1) is to note that (1) implies

$$(1 - x^2)f'' - (2 - \mu - \nu)xf' + \mu(1 - \nu)f = 0,$$
$$(1 - x^2)g'' - (2 - \mu - \nu)xg' + \nu(1 - \mu)g = 0.$$

However, with this method it seems more difficult to isolate the solution (8).

Also solved, more or less exhaustively, by A. P. Boblétt, J. W. Brown, P. L. Chessin, Emil Grosswald, A. R. Hyde, N. D. Kazarinoff, A. G. Konheim, A. F. Landry, W. R. Longley, Douglas Maurer, E. P. Miles, Jr., Kyu Sam Park, R. J. Pegis, Robert Spira, Max Wyman, and J. W. Young.

Property of a Sequence

4785 [1958, 289]. *Proposed by Paul Erdös, Israel Institute of Technology.*

Let $0 \leqq n_1 < n_2 < \cdots$ be a sequence of integers. Denote by $f(N)$ the number of the n's not exceeding N. Assume that $f(2N) = f(N) < c_1$. Then there is a constant c_2, depending only on c_1, such that as $x \to 1$

$$\left| \sum_{k=1}^{\infty} x^{n_k} - f\left(\frac{1}{1 - x}\right) \right| < c_2.$$

Solution by P. T. Bateman, University of Illinois. For convenience we put $x = e^{-s}$, where s is a positive real number. We shall prove that

$$(*) \qquad -2c_1 - 1 < \sum_{k=1}^{\infty} e^{-sn_k} - f\left(\frac{1}{1 - e^{-s}}\right) < c_1$$

for all positive values of x, so that c_2 can be taken as $2c_1 + 1$.

Since $e^s - 1 > s > 1 - e^{-s}$ for all positive s, we have

$$0 < \frac{1}{1 - e^{-s}} - \frac{1}{s} < 1.$$

Hence

$$0 < f\left(\frac{1}{1-e^{-s}}\right) - f\left(\frac{1}{s}\right) < 1,$$

so that we can consider $f(1/s)$ instead of $f(1/(1-e^{-s}))$. More specifically, if we put

$$\Delta(s) = \sum_{k=1}^{\infty} e^{-snk} - f\left(\frac{1}{s}\right),$$

we have

$$\Delta(s) - 1 < \sum_{k=1}^{\infty} e^{-snk} - f\left(\frac{1}{1-e^{-s}}\right) < \Delta(s). \tag{1}$$

Since

$$\begin{aligned}\sum_{k=1}^{\infty} e^{-snk} &= \lim_{K\to+\infty} \sum_{k=1}^{K} e^{-snk} = \lim_{K\to+\infty} \int_0^K e^{-su} df(u) \\ &= \lim_{K\to+\infty} \left(e^{-sK} f(K) + s\int_0^K e^{-su} f(u)du\right) = s\int_0^{\infty} e^{-su} f(u)du,\end{aligned}$$

we obtain

$$\Delta(s) = s\int_0^{\infty} e^{-su} f(u)du - f\left(\frac{1}{s}\right) = s\int_0^{\infty} e^{-su}\left\{f(u) - f\left(\frac{1}{s}\right)\right\} du.$$

On the one hand, since f is nondecreasing, we have

$$\Delta(s) \leqq s\int_{1/s}^{\infty} e^{-su}\{f(u) - f(1/s)\} du \leqq \sum_{n=1}^{\infty} s\int_{2^{n-1}/s}^{2^n/s} e^{-su}\{f(u) - f(1/s)\} du.$$

Now by the assumption of the problem

$$f(1/s) > -c_1 + f(2/s) > -2c_1 + f(2^2/s) > \cdots > -nc_1 + f(2^n/s).$$

Hence the integrand in the nth term of the preceding sum does not **exceed** $e^{-su}\cdot nc_1$. Accordingly

$$\Delta(s) < \sum_{n=1}^{\infty} nc_1(e^{-2^{n-1}} - e^{-2^n}) = c_1 \sum_{n=1}^{\infty} e^{-2^{n-1}} < c_1. \tag{2}$$

On the other hand

$$\begin{aligned}-\Delta(s) &\leqq s\int_0^{1/s} e^{-su}\{f(1/s) - f(u)\} du \\ &\leqq \sum_{n=1}^{\infty} s\int_{2^{-n}/s}^{2^{-n+1}/s} e^{-su}\{f(1/s) - f(u)\} du.\end{aligned}$$

By the assumption of the problem

$$f(1/s) < c_1 + f(2^{-1}/s) < 2c_1 + f(2^{-2}/s) < \cdots < nc_1 + f(2^{-n}/s).$$

Thus the integrand in the nth term of the preceding sum is less than nc_1. Accordingly

$$-\Delta(s) < \sum_{n=1}^{\infty} s \cdot nc_1 \cdot 2^{-n}/s = 2c_1. \tag{3}$$

Combining (1), (2), and (3), we obtain (*).

Also solved by Oliver Aberth and Michael Barr, Robert Breusch, and Joe Lipman.

Editorial Note. The proposer remarks that the problem would have been more elegant if the condition $f(2N)-f(N)<c_1$ were necessary to the conclusion. The special case $n_k=k$ shows that this is not so. A similar proposition is the following: If $f(2N)/f(N)\to 1$, then as $x\to 1$,

$$\sum_{k=1}^{\infty} x^{n_k}/f\left(\frac{1}{1-x}\right) \to 1.$$

Generalization of Bergstrom's Inequality

4786 [1958, 289]. *Proposed by Ky Fan, Oak Ridge National Laboratory*

Let A, B be two positive definite Hermitian matrices of order n, and let $C=A+B$. For any positive integer $p<n$, let A_p denote the principal submatrix of A formed by the first p rows and columns, and let B_p, C_p have similar meaning. Prove

$$\left(\frac{\det C}{\det C_p}\right)^{1/(n-p)} \geqq \left(\frac{\det A}{\det A_p}\right)^{1/(n-p)} + \left(\frac{\det B}{\det B_p}\right)^{1/(n-p)}. \tag{1}$$

[The case $p=n-1$ is known and due to H. Bergström (See R. Bellman, this MONTHLY, vol. 62, 1955, pp. 172–173). Also the inequality of Minkowski (Hardy, Littlewood and Pólya, *Inequalities*, 1934, p. 35) may be regarded as the case $p=0$.]

Solution by the proposer. It is possible to derive (1) from Bergström's inequality, *i.e.*, to derive the general case from the special case $p=n-1$. The following proof, however, avoids the need of first establishing the special case.

The proof is based on the following minimum property: *Let H be a positive definite Hermitian linear transformation in the unitary n-space U^n. Let $e_1, \cdots, e_n$ be n orthonormal vectors in U^n and let p be a positive integer $<n$. Then*

$$\frac{\det H}{\det((He_j, e_i))_{1\leq i,j\leq p}} = \min \det ((Hx_i, x_j))_{p+1\leq i,j\leq n},$$

where the minimum is taken over all systems of $n-p$ vectors $x_{p+1}, x_{p+2}, \cdots, x_n$ in U^n subject to the biorthonormal condition

$$(x_i, e_j) = \delta_{ij} \qquad (p + 1 \leqq i, j \leqq n). \tag{2}$$

This result was proved in K. Fan, *Proc. Cambridge Philos. Soc.*, vol. 51, 1955, pp. 414–421.

To prove (1), let $e_1, \cdots, e_n$ denote the unit vectors of our coordinate system in U^n. Then

$$A_p = ((Ae_j, e_i))_{1 \leq i,j \leq p}, \qquad B_p = ((Be_j, e_i))_{1 \leq i,j \leq p},$$
$$C_p = ((Ce_j, e_i))_{1 \leq i,j \leq p}.$$

According to the minimum property stated above, there exist $n-p$ vectors $x_{p+1}, \cdots, x_n$ satisfying (2) and

$$\frac{\det C}{\det C_p} = \det ((Cx_i, x_j))_{p+1 \leq i,j \leq n}. \tag{3}$$

Let D, E, F denote the matrices of order $n-p$:

$$D = ((Ax_i, x_j))_{p+1 \leq i,j \leq n}, \qquad E = ((Bx_i, x_j))_{p+1 \leq i,j \leq n},$$
$$F = ((Cx_i, x_j))_{p+1 \leq i,j \leq n}.$$

Again by the minimum property, we have

$$\det D \geqq \frac{\det A}{\det A_p}, \qquad \det E \geqq \frac{\det B}{\det B_p} \cdot \tag{4}$$

The matrices D, E, F are clearly Hermitian and positive definite. Now, applying Minkowski's inequality to $F = D + E$, we get

$$(\det F)^{1/(n-p)} \geqq (\det D)^{1/(n-p)} + (\det E)^{1/(n-p)}. \tag{5}$$

From (3), (5), and (4), (1) now follows immediately.

An Incorrect Proposal

4787 [1958, 289]. *Proposed by M. S. Klamkin, AVCO Research Division, Wilmington, Mass.*

Express as a single definite integral

$$\sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{r=0}^{\infty} \frac{(m + n + r)!m^n n^r}{m!n!r!(r + 1)^{m+n+r+1}} \cdot$$

Editorial Note. As several readers were quick to point out, the stated series is divergent. Attempts to rework the original idea into a well-posed problem have not been satisfactory. The editor and the proposer apologize for a careless oversight.

Omitted Values of Entire Functions

4788 [1958, 370]. *Proposed by D. J. Newman, AVCO Research, Wilmington, Mass.*

Let $f(z)$ be an entire function and consider the two statements: (1) If f, f', f'' never vanish, then $f(z) = e^{az+b}$. (2) If f, f', f'' each miss a value, then $f(z) = a + be^{cz}$. (1) is a known theorem. (2) is apparently stronger than (1). Show, however, that (2) is an elementary consequence of (1).

I. *Solution by the proposer.* We assume (1) and proceed to prove (2). Suppose then f is given where f misses the value a, f' misses ξ, f'' misses η. We distinguish three cases:

Case I. $\xi \neq 0$. Form the function

$$F = e^{\xi \int 1/(f-a)}.$$

We find that

$$F' = \frac{\xi}{f-a} e^{\xi \int 1/(f-a)}, \qquad F'' = \frac{\xi(\xi - f')}{(f-a)^2} e^{\xi \int 1/(f-a)},$$

and that no one of F, F', F'' is ever zero. (1) applied to F tells us that $F = e^{\alpha z + \beta}$ and hence f must be a constant. (2) is trivially true in this case.

Case II. $\eta \neq 0$. A similar argument with

$$F = e^{\eta \int 1/(f' - \xi)}$$

proves that f' is a constant so that f is linear. Since f misses a value, however, it must be constant and again (2) follows.

Case III. $\xi = 0$, $\eta = 0$. (2) follows upon applying (1) to $f - a$.

II. *Solution by D. Gaier, Technische Hochschule, Stuttgart.* Saxer proved (1923) that if $f(z)$ is an entire function and $f(z)$ and $f'(z)$ miss the values A and B, respectively, then $B = 0$. Hence, if $f(z)$, $f'(z)$, $f''(z)$ miss the values A, B, C, respectively, and $f(z)$ is not constant, then $B = C = 0$ and by proposition (1), $f(z) - A$ is of the form e^{az+b}.

It is possible to improve the stated result. If $f(z)$ is an entire function and $f(z)$ and $f''(z)$ miss the values A and C, then $f(z) = A + e^{az+b}$. In fact, Saxer's theorem has been generalized by Bureau (1931) who replaced $f'(z) \neq B$ by $f^{(k)}(z) \neq B$, $(k \geqq 1)$ with the same conclusion, $B = 0$. Hence $f(z) - A$ and $f''(z)$ each miss the value zero, so that by a recent result of Hayman (1958), $f(z) - A$ is of the form e^{az+b}.

Also solved by James Clunie.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

The regulations recently adopted by the Office of Education on the administration of Title III of the National Defense Education Act of 1958 seem to spell out quite clearly that *these funds may be used to purchase books in mathematics for school libraries.* Apparently many administrators are unaware that "the acquisition of supplementary reading materials to strengthen the mathematical portion of school libraries" is an eligible and important project under the Act. If mathematics teachers are to take advantage of this vital opportunity, they must act vigorously and swiftly at both the local and the state levels. Please do call this provision to the attention of *your* administrative officers. The high school book list prepared by the National High School and Junior College Math Club is still available to persons sending a stamped self-addressed envelope (number 10 size) to HIGH SCHOOL BOOK LIST, c/o Richard V. Andree, The University of Oklahoma, Norman, Oklahoma.

Convexity. By H. G. Eggleston. Tracts in Mathematics and Mathematical Physics No. 47, Cambridge, 1958. viii+136 pp. $4.00.

In this tract the author presents an introduction to convex sets in finite dimensional euclidean space. Since there had been nothing previously available in book form in English on convexity, this book will be very useful. All the topics chosen, except those in chapters 6 and 7, are virtually necessary to the subject. Included are Helly's theorem, mixed volumes, Blaschke's compactness theorem, the Brun-Minkowski inequality, supporting functions, gauge functions and so on. Chapter 6 is devoted to special problems and chapter seven to sets of constant width.

The author has performed a valuable service to the mathematical community in making the early results on convex sets available. However, in doing this, one might object that some items of value have been developed since 1930 and might have been included. For example, the semispaces the reviewer introduced some years ago are more important to the basic theory of convex sets than are the halfspaces. None the less, an introduction to convex sets is now made accessible to undergraduate students by virtue of this publication.

PRESTON C. HAMMER
University of Wisconsin

Modern Business Statistics. By John E. Freund and Frank J. Williams. Prentice-Hall, Englewood Cliffs, N. J., 1958. xv+539 pp. $10.00.

One school of thought insists that basic statistics is basic statistics and that there is little to be gained, usually much to be lost, by offering introductory

courses in educational statistics, agricultural statistics, *etc.* The reviewer accepts this philosophy and accordingly sees little merit in this volume. He objects to the word "modern" in the title, for the material is traditional, and we even find those twin Ulyssean perils of ancient statistics, platykurtic and leptokurtic, introduced on page 107.

For the unenlightened (those who take the opposite point of view), it might be admitted that this book is better than most of its competitors in the "applied" class. Indeed, chapters VI–XIV constitute a brief but entirely satisfactory introduction to basic statistical concepts, and it is gratifying to find mention of such ideas as conditional probability. The most serious omission here is a chart for testing the significance of the sample correlation coefficient.

Unfortunately, the remaining half of the book is not of the same quality. Thus 120 pages are devoted to a routine discussion of index numbers and time series, and in the first 42 pages the student will essentially only learn the meanings of $\sum$, histogram, and class frequency. There are also some minor slips such as the phrase "absolute certain" on page 112 and the implication that the customary definition of s is unbiased on page 86.

The authors do not offer the teacher overly much choice in problem material. Generally speaking, the problems are not very stimulating although some of the examples have interesting business pedigrees.

K. A. BUSH
University of Idaho

Preparatory Mathematics, Vols. I and II. By Georges, Sunko, Eulenberg, Piety. Edwards, Ann Arbor, 1957 and 1958. 208 and 209 pp. $4.00 and $4.50.

These volumes are "designed to meet the mathematical needs" of college freshmen who either "have had neither algebra nor geometry in high school" or "who score low on the placement test administered by the Department of Mathematics."

The subjects listed in the Contents range from elementary arithmetic through numerical trigonometry.

The first half of each volume consists primarily of many declarative sentences. Practically no attempt is made to have the student understand what he is doing. The entire discussion of the problem of finding the area of a circle consists of the following: "$A=\pi r^2$, where $\pi=3\frac{1}{7}$, 3.14, 3.1416, *etc.*" At another point we find the statement "a power of 10 which is a multiple of 3 is a perfect cube."

The second half of each volume consists of tear sheets on which the student, following the directions given in the corresponding section of the first half, inserts numbers in the proper boxes.

The student who uses these volumes will have much practice in writing neatly. This reviewer doubts that he will learn any mathematics.

LLOYD L. LOWENSTEIN
Arizona State University

Selections from Modern Abstract Algebra. By Richard V. Andree. Holt, New York, 1958. xii+212 pp. $6.50.

This book differs from recent texts which are introductions to modern algebra in several respects. It is intended for an early place in the mathematics curriculum, namely at the sophomore level for mathematics majors, and possibly somewhat later for science and engineering students. It makes no pretense at completeness for the topics included and does not emphasize any one facet of modern algebra, but rather offers short selections from various parts of the subject. It does not presuppose much mathematical maturity and, in fact, one of the basic purposes of the book is to develop this maturity.

In Chapter I, the postulates for an integral domain are given as an abstraction of the arithmetical properties of the integers. Undefined terms, postulates, definitions, and the nature of proof are discussed briefly. Examples of finite number systems are given, and some elementary number theory is included.

Chapter II is devoted to equivalence relations and integral congruences, and Chapter III discusses binary Boolean arithmetic with its application to circuit network theory.

Chapter IV introduces the concept of a group and numerous examples are given. The elementary properties of a group are derived up to the notions of invariant subgroup and quotient group.

Chapter V contains the basic algebra of matrices, and Chapter VI applies this material to the solution of systems of linear equations. Chapter VII gives a fairly standard account of the properties of determinants.

Chapter VIII gives the ring and field postulates. Ideals and residue class rings are defined.

The final Chapter IX makes a selection from further topics in matric theory, discussing briefly the characteristic equation, characteristic roots and vectors, the minimum function of a matrix, infinite series with matric elements, and derivatives and integrals of matrices with elements which are functions of one variable.

The misprints which this reviewer noted are not of serious nature and should be easily spotted by a careful reader. There are adequate problem sets after each section, and an interesting feature of the book is that many problems consist of suggested reading from the AMERICAN MATHEMATICAL MONTHLY.

R. A. BEAUMONT
University of Washington

Editorial Note. Since Professor Andree is the editor of this section, the editor-in-chief has assumed responsibility for the editing of this review. R.D.J.

Intermediate Analysis. By John M. H. Olmsted. Appleton-Century-Crofts, New York, 1956. xii+305 pp. $6.00.

The title of this book does not suggest its content; the area treated is better described by its subtitle, *An Introduction to the Theory of Functions of One Real*

Variable. The author's stated purpose is "to present the basic ideas and techniques of analysis, for functions of a single real variable, in such a way that students who have studied calculus can proceed at whatever pace and intensity are considered suitable," and he has quite definitely accomplished his aim.

The author presents in a highly readable and extremely careful fashion *all* of the purely mathematical topics concerning functions of a single real variable usually found in an American calculus text, as well as material on the most elementary point-set topology, uniform continuity, functions of bounded variation, Cauchy criteria for the convergence of sequences and functions, some special convergence tests for series, the doppelreihensatz, the Cauchy product of series, and uniform convergence. Many topics not treated in the main body of the book are covered in a large collection of exceedingly good, graduated exercises: The relation between Riemann integrability and continuity almost everywhere and some of its consequences (dominated and bounded convergence theorems for Riemann integrals), uniform approximation of continuous functions by polynomials, the partial fractions decomposition theorem, and the Moore-Osgood theorem on interchange of limits, to mention but a very few.

Some flexibility in the use of the book has been considered by the author. He has organized the material so that certain starred portions may be omitted without affecting the continuity of the book as a whole, but the reviewer feels that some caution should be exercised in this direction. To cite but one case in point, omission of the axiom of completeness for the real number system seems to defeat the very purpose of the book as a deeper study of the calculus, since all of the latter is based on that axiom!

In the reviewer's opinion, this book would be a superb text for a one-quarter, five-credit (one-semester, three-credit) course at the junior or senior level to complement a course in differential equations and thereby afford a good year-sequence in elementary analysis. Further, every math major (including high school teachers and teacher majors), whose advanced mathematical training leans away from the field of analysis, would gain immeasurably by the presence of this book in his personal library.

The reviewer could find only one feature about which to complain: The author's definition of a function is the time-honored one of a relation.

ARTHUR E. LIVINGSTON
University of Washington

Mathematical Foundations of Information Theory. By A. I. Khinchin, translated by R. A. Silverman and M. D. Friedman. Dover, New York, 1957. vii+120 pp. $1.35.

The subject of information theory is very new and has been developed primarily by mathematicians in this country. The basic ideas were set forth by C. Shannon in 1948, and extended in fundamental papers by B. McMillian and A. Feinstein.

There have been a number of expository treatments of information theory, but these have usually been based on ill-defined concepts and accompanied by rather exaggerated claims about the universal applicability of the subject.

The book under consideration is a translation of two papers written by the Russian mathematician, A. I. Khinchin, for the expository journal *Uspekhi.* These papers present the mathematical foundations of information theory. While completely rigorous, the flavor of the engineering applications which led to the theory runs throughout and very much helps the intuition. Khinchin has here reformulated basic concepts and presents for the first time rigorous proofs of certain fundamental theorems in the subject.

The first paper discusses the concept of entropy and gives one major application to coding. The only stochastic processes used are Markov chains. This paper would serve as a valuable supplement to an introductory probability course.

The second and longer paper uses more advanced topics from probability theory—for example, stationary processes and martingales. However, the treatment is quite complete and the nonspecialist would not suffer, thanks to Khinchin's amazing expository ability.

It is a tribute to Shannon's theory that a rigorous treatment only enhances the elegance of the basic theorems.

J. LAURIE SNELL
Dartmouth College

An Introduction to Euclidean Geometry. By J. C. Eaves and A. J. Robinson. Addison-Wesley, Reading, Massachusetts, 1957. xii+327 pp. $4.25.

This interesting workbook-style text (with soft cover and $8\frac{1}{2}$ by 11 inch pages) is designed primarily to meet the needs of students who enter college with inadequate high school preparation in geometry. In the thirty-seven lessons into which the book is divided, the authors have presented those concepts of plane and solid geometry which will give adequate background to the student who intends to major in science or engineering. The material can be covered in a semester three- or four-hour course.

The book is well written and is very attractive, with its many excellent drawings and reproductions of photographs. Students are carefully led to the point where they can fill in many of the details of the proofs in the text portions of the lessons. At the end of each lesson is a "summary for easy reference," which the student is encouraged to amplify and use for review purposes. Sets of exercises are on perforated sheets which can be torn out and handed in.

Plane geometry is covered in the first twenty-five lessons. In these lessons are found twenty-two postulates, followed by ten constructions and forty-eight theorems. Other theorems are included in the 150 exercises. The pace seems a little faster in the twelve lessons on solid geometry. These lessons introduce four postulates, sixty-eight theorems, and eighty-five exercises. Many of the

theorems of three-space are stated without proof, and it is not intended that the student prove all of them.

This book is highly recommended to those who find it necessary to offer an elementary geometry course for college freshmen.

VIOLET HACHMEISTER LARNEY
State University of New York
College for Teachers, Albany

Linear Algebras. Publication 502, National Academy of Sciences—National Research Council. Washington, D. C., 1957. v+59 pp. $1.50.

This is a report of a Conference on Linear Algebras held on Long Island in June 1957. A preface by A. A. Albert briefly describes the Conference and mentions several papers given at the Conference but published elsewhere. A list of the 23 participants is given at the end of the booklet. The following seven papers are published in the booklet: Irving Kaplansky, *Problems in the theory of rings;* R. Brauer, *Some remarks on associative rings and algebras;* N. Jacobson, *Jordan algebras;* Erwin Kleinfeld, *On alternative and right alternative rings;* George B. Seligman, *A survey of Lie algebras of characteristic p;* Reinhold Baer, *Meta ideals;* David Buchsbaum, *A Survey of homological algebra.*

As the titles indicate, several of the papers survey the literature and have fairly extensive bibliographies. On the other hand, Kaplansky states twelve outstanding problems; Brauer generalizes the Artin-Wedderburn structure theory; Kleinfeld disposes of simple alternative rings; and Baer introduces a topic new to ring theory. Since detailed individual reviews of the papers presumably will appear elsewhere, we shall not peer more closely. Let us say merely that the booklet seems well worth owning.

R. H. BRUCK
University of Wisconsin

Vector Spaces and Matrices. By Robert M. Thrall and Leonard Tornheim. Wiley, New York, 1957. xii+318 pp. $6.75.

The past few decades have witnessed a growing interest in vector spaces and matrices, with the juxtaposition of these topics indicating that matrices have come to be recognized for what they are. Even more noteworthy, perhaps, is the fact that these topics have descended from the level of esoterica to that area which is traditionally the citadel of reaction and conservatism: the undergraduate mathematics curriculum. The book under review is the latest in an ever-lengthening series of texts treating vectors and matrices in the modern flavor and is, apparently, intended for the undergraduate market.

There are nice things to be said about this compendium. Noteworthy is the authors' announced intention to proceed on two levels—one concrete and the other abstract—the one via matrices and the other via linear transformations. This program is maintained, more or less, for the first six chapters, and serves to

clarify the theory. An item deserving more attention still is the beautiful development of chapters 8 through 10: this is not easy reading, but the result is the complete canonical form under similarity in the general case, and the reader can feel satisfied that he has seen the whole exposition before his very eyes.

The reviewer's main criticism is that it is difficult to decide for whom the book was written, since the level of exposition varies markedly from page to page. For instance, on page 25 the following lemma is proved: Let W be a subspace of a finite-dimensional vector space V over F. Then $W=V$ if and only if dim $W=$dim V. The authors, without further discussion of cardinality, remark that the lemma is false for infinite-dimensional spaces, "as can be seen by taking $F\equiv$rationals, $W\equiv$reals and $V\equiv$complexes." As the beginning student recovers from this remark, he gets to page 37, where the summation symbol is patiently explained, and to page 38, where three illustrations of the meaning of the symbol are written out. But probably the most startling page in the text is page 61. For hereon there are defined the concepts of group, abelian group, ring, commutative ring with identity, algebra over a field, and total matric algebra of degree $\boldsymbol{v}$.

Other such examples of the hill-and-valley school of writing could be listed, but the point has undoubtedly been made. In the reviewer's opinion the book is worth the price for chapters 8–10, for use as one of many supplementary texts, by beginning graduate students. This class of reader might also be interested in the last chapter, Linear Programming and Game Theory, although it is only tenuously related to the rest of the book and seems isolated and out of place.

As an undergraduate text the book is uneven and certainly not appealing to this reviewer. However, intrepid souls will certainly want to try it and, for these, the reviewer can but paraphrase the old aphorism: Let the prospective teacher beware!

R. L. SAN SOUCIE
Sylvania Electronic Systems
Buffalo, New York

Applied Differential Equations. By Murray R. Spiegel. Prentice-Hall, Englewood Cliffs, N. J., 1958. xv+381 pp. $6.75.

This book is an excellent one for students who are studying differential equations for the first time and who are primarily interested in applications to engineering, physics and chemistry.

Well-chosen examples are used to motivate each basic idea as it is introduced. Then the theory underlying the idea is developed, with due regard to rigor. Finally the idea is exploited through a variety of applications; the treatment of the applications emphasizes the formulation of the problem, the solving, and the interpretation of the solution. The A exercises are straightforward, the B exercises are more difficult, while the C exercises are intended to challenge the student and at the same time to open up a number of new possibilities to him.

The book has been well conceived and should effectively achieve its purposes. Besides using it with a regular class, one could recommend it to anyone who wants to study, or restudy, elementary differential equations and their applications.

About a third of the book is devoted to first-order and simple higher-order equations and their applications. Then follows a chapter on linear differential equations with constant coefficients, and another on their applications. There is then one chapter each on simultaneous differential equations and their applications, series solutions, numerical methods, how partial differential equations arise, and Fourier-series solutions of boundary value problems.

The chapter on numerical methods is not quite as well presented as the others. My only other criticism is that the book contains a few minor inaccuracies and a number of statements which need to be either modified or clarified.

T. E. HULL
The University of British Columbia

BRIEF MENTION

Functions of Real and Complex Variables. By William F. Osgood. Chelsea, New York, 1958. xii+407 pp. and viii+262 pp. $4.95.

It is welcome news for many to find Osgood's 1935 and 1936 books bound together in one volume at this low price. In spite of the many books which have been written on these two subjects during the last two decades, Osgood's works still have much to recommend them to the student emerging from a course in advanced calculus to take his first real look at analysis.

Elementary Number Theory. By Edmund Landau. Chelsea, New York, 1958. 256 pp. $4.95.

This is an English translation of Landau's *Elementare Zahlentheorie* by J. E. Goodman, which has been supplemented by exercises contributed by Professors Bateman and Kohlbecker. The material is, of course, in no sense modern, but still is worthy of a student's time. Part I contains the foundations of elementary number theory including quadratic residues and Pell's equation. Part II deals with the theorems of Brun and Dirichlet. Part III discusses decomposition into squares while the fourth portion is devoted to binary quadratic forms and their class numbers.

Logic Machines and Diagrams. By Martin Gardner. McGraw-Hill, New York, 1957. ix+157 pp. $5.00.

The skilled pen of Martin Gardner has now turned to a discussion of logic, mainly using geometric forms. This interesting blend of history, mathematics, and just plain fun is combined with an occasional rather sharp barb. (See for example page 59, number 5.) Certainly the book deserves a place on the shelf of anyone interested in elementary logic machines.

The Teaching of Modern School Mathematics. By E. J. James. Oxford University Press, New York, 1958. viii+275 pp. $3.40.

A pedagogical type of book, dealing with the instruction in the English modern secondary school. The majority of the work discussed would have sounded quite familiar to a seventh or eighth grade mathematics teacher in the American school system of 15 or 20 years ago.

Science in a Tavern. By Charles S. Slichter. University of Wisconsin Press, 1958. ix+206 pp. $1.00.

A reprint of Professor Slichter's delightful essays on science-in-the-making which first appeared some twenty years ago. This inexpensive little volume belongs on the bookshelf of anyone who habitually thinks of a scientist in terms of a large well-equipped modern laboratory.

Dilogarithms and Associated Functions. By L. Lewin. Macdonald, London, 1958. xvi+353 pp. 65s.

A book which requires, essentially, only elementary calculus as a prerequisite. This is probably the first extensive publication on dilogarithms since they were introduced almost a hundred and fifty years ago.

Linear Programming and Associated Techniques. By Vera Riley and Saul I. Gass. Johns Hopkins, Baltimore, 1958. x+613 pp. $6.00.

This is not a text but a comprehensive bibliography on linear programming, nonlinear programming and dynamic programming. Each publication contains its complete title, source and a brief descriptive paragraph. This volume contains over twice as many items as that of its 1954 predecessor and should be welcomed indeed by mathematicians interested in linear programming.

Hammond's Ambassador World Atlas. C. S. Hammond and Co., Maplewood, New Jersey, 1954. 416 pp. $12.50.

An excellent collection of large maps of both physical and human geography. A mathematician, naturally, wishes that somewhere among the more than 400 pages it had been possible to include half a page describing the various projections used throughout the book.

Sound Pulses. By F. G. Friedlander. Cambridge University Press, New York, 1958. xi+202 pp. $7.50.

A text on aperiodic disturbances with clearly-defined fronts, based on linear partial differential equations mainly of hyperbolic type.

Engineering Systems Analysis. By Robert L. Sutherland. Addison-Wesley, Reading, Mass. 1958. xii+223. $7.50.

An undergraduate text of interest primarily because it points up the simi-

larity between topics in different fields of engineering. A calculus background is assumed.

The Principles of Science (A Treatise on Logic and Scientific Method). By W. Stanley Jevons. Dover, New York, 1958. liii+786 pp. $2.98.

Many of Jevons' ideas on logic are still of more than historical interest even though this volume was first published in 1873.

A Mathematics Dictionary. By Bristow (Oklahoma) High School chapter of Mu Alpha Theta. 46 (8½ by 11) pp. $1.00 postpaid.

Our sincere congratulations to this group of high school students and their teacher, Neva Gurley, who have constructed this carefully-compiled dictionary of more than 600 mathematical terms likely to be encountered in grades 9–12. From *abacus* and *Abelian group* to *zero* and *zone* they have done an excellent job of giving understandable definitions that do not seem stilted. Their definitions are more accurate than those in the commercially published mathematical dictionary recently reviewed in this journal. It is a pleasure to receive publications of this type.

How to Study, How to Solve. Second Edition. By H. M. Dadourian. Addison-Wesley, Reading, Mass., 1958. iv+43 pp. $0.50.

A slight revision of Dadourian's 1947 book of the same title. A welcome reference for any college or high school student who is having difficulty in mathematics.

Tables of Modified Quotients of Bessel Functions of the First Kind for Real and Imaginary Arguments. By Morio Onoe. Columbia University Press, 1958. 338 pp. $12.50.

Computations performed chiefly on the I.B.M. 607 with auxiliary computations on the I.B.M. 602 and 650 were tabulated on the I.B.M. 407 and processed by photo-offset for this publication. From eight to ten significant digits have been kept in the tables. An introductory section includes a collection of formulae and the general methods used in the preparation of the tables.

Handbook of Automation, Computation and Control, Volume I: *Control Fundamentals*. Eugene M. Grabbe, Simon Ramo and Dean E. Wooldridge, Editors. Wiley, New York, 1958. xx+972 pp. $17.00.

This amazing handbook has been prepared by a staff of specialists. The section on general mathematics, for example, was edited by R. M. Thrall and W. Kaplan. However, individual chapters have been written by R. M. Thrall, R. C. Lyndon, G. E. Hay, W. Kaplan, E. H. Rothe, A. H. Copeland, Sr., and A. B. Clarke, and include the following topics: Sets and Relations, Algebraic Equations, Matrix Theory, Finite Difference Equations, Differential Equations, Integral Equations, Complex Variables, Operational Mathematics, Laplace Transforms, Conformal Mapping, Boolean Algebra, Probability, and Statistics. The second portion is devoted to Numerical Analysis, with such well-known authors as Bernard Dimsdale, M. Mannos, J. M. Cameron, R. F. Clippinger, J. B. Diaz, Bernard Friedman, Eugene Isaacson, and Robert Richtmeyer, and is edited by Richard F. Clippinger and Joseph H. Levin. The remainder of the book is devoted to what appear to be equally competent presentations of Operations Research, Information Theory and Transmission and Feedback Control, including an excellent discussion of noise problems. All in all, this handbook certainly belongs on the shelf of every computation center and in the library of any mathematician seriously interested in computation.

pointed Member of the Technical Staff at Space Technology Laboratories, El Segundo, California.

Mr. L. R. Stewart, Jr., I.B.M. Corporation, Poughkeepsie, New York, has been appointed Senior Applied Science Representative for the I.B.M. Corporation, Washington, D. C.

Mr. R. W. Stewart, University of Arizona, has accepted the position of Electrical Engineer at Nortronics, Anaheim, California.

Dr. D. L. Thomsen, Jr., I.B.M. Corporation, Yorktown Heights, New York, has been appointed Data Processing Manager with the I.B.M. Corporation, Watson Scientific Computing Laboratory at Columbia University.

Dr. D. B. J. Tomiuk, Catholic University of America, has been appointed Assistant Professor at Fordham University.

Mr. J. R. VanAndel, Burroughs Corporation, Paoli, Pennsylvania, has accepted the position of Senior Engineer with Bendix Research Laboratories, Detroit, Michigan.

Assistant Professor S. I. Vrooman, University of Pittsburgh, has been appointed Associate Professor at the Pratt Institute.

Associate Professor J. H. Wahab, Georgia Institute of Technology, has been appointed Professor at Louisiana State University, New Orleans.

Dr. F. J. Weyl, Naval Analysis Group, has been appointed Research Director of the Office of Naval Research, Washington, D. C.

Dr. R. E. Wild, Douglas Aircraft Corporation, Santa Monica, California, has been appointed Assistant Professor at the University of Arizona.

Dr. J. E. Wilkins, Jr., Nuclear Development Corporation of America, White Plains, New York, has been promoted to Assistant Manager.

Mr. G. K. Williams, University of Kentucky, has been appointed Assistant Professor at Madison College.

Mr. H. C. Arnold, Federal Enamel & Stamping Company, McKees Rocks, Pennsylvania, died on September 27, 1958. He was a member of the Association for sixteen years.

Associate Professor J. P. Brewster, Clemson Agricultural College, died on October 14, 1958. He was a member of the Association for ten years.

Mr. W. J. Ettinger, Lombard, Illinois, died on January 21, 1958. He was a member of the Association for twenty-eight years.

Professor K. P. Williams, Indiana University, died on September 25, 1958. He was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

ITINERARIES OF VISITING LECTURERS, 1958–59

Tom Apostol

New Mexico Institute of Mining and Technology	Socorro, N. M.	Mar. 16–17
New Mexico College of Agriculture and Mechanical Arts	State College, N. M.	Mar. 19–20
Univ. of Arizona	Tucson, Ariz.	Mar. 23–24
Arizona State University	Tempe, Ariz.	Mar. 26–27

San Diego State College	San Diego, Calif.	Mar. 30–31
Fresno State College	Fresno, Calif.	Apr. 2–3
San Jose State College	San Jose, Calif.	Apr. 6–7
San Francisco State College	San Francisco, Calif.	Apr. 8–9
Chico State College	Chico, Calif.	Apr. 10
University of Oregon	Eugene, Ore.	Apr. 13
Willamette University	Salem, Ore.	Apr. 14–15
Linfield College	McMinnville, Ore.	Apr. 16–17
Seattle University	Seattle, Wash.	Apr. 20–24
University of Washington	Seattle, Wash.	Apr. 20–24
Western Washington College of Education	Bellingham, Wash.	Apr. 20–24
University of Idaho	Moscow, Idaho	Apr. 27–29
State College of Washington	Pullman, Wash.	Apr. 27–29
Montana State University	Missoula, Mont.	May 1–5
Utah State University	Logan, Utah	May 8–11
Colorado State University	Fort Collins, Colo.	May 14–15
University of Colorado	Boulder, Colo.	May 18–20
Los Angeles State College	Los Angeles, Calif.	To be arranged
Pomona College	Claremont, Calif.	To be arranged
Whittier College	Whittier, Calif.	To be arranged

Robert E. Gaskell

University of Alberta	Edmonton, Canada	Jan. 5–6
University of Minnesota	Duluth 11, Minn.	Jan. 7–8
Wisconsin State College	Superior, Wis.	Jan. 9
University of North Dakota	Grand Forks, N. D.	Jan. 12–13
North Dakota Agricultural College	Fargo, N. D.	Jan. 14–15
Concordia College	Moorhead, Minn.	Jan. 16
Grinnell College	Grinnell, Iowa	Jan. 19–20
Baldwin-Wallace College	Berea, Ohio	Jan. 28–30
Antioch College	Yellow Springs, Ohio	Feb. 2–4
University of Dayton	Dayton, Ohio	Feb. 4–6
University of Buffalo	Buffalo 14, N. Y.	Feb. 9–10
University of Vermont	Burlington, Vt.	Feb. 11–12
Norwich University	Northfield, Vt.	Feb. 12–13
Hamilton College	Clinton, N. Y.	Feb. 16–17
Alfred University	Alfred, N. Y.	Feb. 18–19
University of Detroit	Detroit 21, Mich.	Feb. 19–20
University of Kansas	Lawrence, Kan.	Feb. 23–24
Ottawa University	Ottawa, Kan.	Feb. 25
St. Benedict's College and Mount St. Scholastica College	Atchison, Kan.	Feb. 26–27
Iowa State Teachers College	Cedar Falls, Iowa	Mar. 2–3
Carroll College	Waukesha, Wis.	Mar. 5–6
Purdue University	Indianapolis, Ind.	Mar. 9
Butler University	Indianapolis 7, Ind.	Mar. 10–11
Valparaiso University	Valparaiso, Ind.	Mar. 12–13
Albion College	Albion, Mich.	Mar. 16
North Central College	Naperville, Ill.	Mar. 18
University of Wisconsin	Milwaukee, Wis.	Mar. 19–20
University of Omaha	Omaha, Neb.	Mar. 23–24
Indiana Technical College	Fort Wayne, Ind.	Mar. 26–27
Drake University	Des Moines, Iowa	Mar. 30–31

R. E. Johnson

Randolph-Macon Women's Coll. and Lynchburg Coll.	Lynchburg, Va.	Feb. 3–4
Sweet Briar College	Sweet Briar, Va.	Feb. 3–4
University of Richmond	Richmond, Va.	Feb. 5–6
College of William and Mary	Norfolk, Va.	Feb. 9–10
East Carolina College	Greenville, N. C.	Feb. 12–13
University of North Carolina	Chapel Hill, N. C.	Feb. 16–17
Carolina State College and Duke University	Durham, N. C.	Feb. 16–17
North Georgia College	Dahlonega, Ga.	Feb. 19–20
Alabama Polytechnic Institute	Auburn, Ala.	Mar. 2–3
Mississippi State University	State College, Miss.	Mar. 5–6
University of Mississippi	University, Miss.	Mar. 9–10
Southwestern Louisiana Institute	Lafayette, La.	Mar. 12–13
Centenary College	Shreveport, La.	Mar. 16–17
University of Arkansas	Fayetteville, Ark.	Mar. 19–21
Washington University	St. Louis, Mo.	Mar. 30–31
University of Wichita	Wichita, Kan.	Apr. 2–3
Southwestern at Memphis	Memphis, Tenn.	Apr. 8–10
University of Kentucky	Lexington, Ky.	Apr. 13–15
Wilson College	Chambersburg, Pa.	Apr. 17
State Teachers College	Shippensburg, Pa.	Apr. 20–24
Dickinson College	Carlisle, Pa.	Apr. 20–24
Bucknell University	Lewisburg, Pa.	May 4–6
Hartwick College and State University Teachers College	Oneonta, N. Y.	May 7–8
Upper New York Section of M.A.A.	Oneonta, N. Y. (Hartwick Coll.)	May 9
Montclair State College	Upper Montclair, N. J.	May 12–13
St. John's University	Jamaica, N. Y.	May 14–15

John L. Kelley

University of Miami	Coral Gables, Fla.	Apr. 1–3
University of Florida	Gainesville, Fla.	Apr. 6–7
Florida A. and M. University and Florida State University	Tallahassee, Fla.	Apr. 13–17
Georgia Institute of Technology	Atlanta, Ga.	Apr. 20–24

Stanislaw Ulam

California Institute of Technology	Pasadena, Calif.	Jan. 7
Occidental College	Los Angeles, Calif.	Jan. 8–9
Univ. of California at Santa Barbara	Goleta, Calif.	Jan. 12–13
Oregon State College	Corvallis, Ore.	Jan. 15–16
Reed College	Portland, Ore.	Jan. 19–20
University of Utah	Salt Lake City, Utah	Jan. 22
Brigham Young University	Provo, Utah	Jan. 23
Univ. of New Mexico	Albuquerque, N. M.	To be arranged

S. S. Wilks

East Texas State College	Commerce, Tex.	Oct. 30–31
Oklahoma State University	Stillwater, Okla.	Nov. 3–7
Oklahoma Central State College	Edmond, Okla.	Nov. 3–7
North Texas State College	Denton, Tex.	Apr. 8–18

Southern Methodist University	Dallas, Tex.	Apr. 8–18
Texas Christian University	Fort Worth, Tex.	Apr. 8–18
Rice Institute	Houston, Tex.	Apr. 8–18

THE HIGH SCHOOL MATHEMATICS CONTEST

The Mathematical Association of America, in cooperation with the Society of Actuaries, this past spring assumed control of the high school contest examination which had been administered for the last few years by the Committee on Contests and Awards of the Metropolitan New York Section of the Mathematical Association of America.

On March 27, 1958, an eighty-minute test, consisting of fifty questions, was given to over 80,000 students in 2889 high schools. These figures compare with 43,500 students in 1469 high schools in 1957 and show this year's truly explosive growth in both number of participating schools and contestants. Nearly all of the states of the United States were represented, many in large numbers, as were also a number of provinces of Canada, and Alaska, Hawaii, American Samoa, and the Canal Zone.

The contest for 1959 will be held on Thursday morning, March 5, 1959, during the first two class periods. The examination will be of the same type as was used for the 1958 contest. High schools who have not as yet received their registration form for this contest should contact the contest chairman of their section of the M.A.A. or Professor W. H. Fagerstrom, Executive Director of the Contest, Pan American College, Edinburg, Texas.

THE NOVEMBER MEETING OF THE NEW JERSEY SECTION

The third annual meeting of the New Jersey Section of the Mathematical Association of America was held at Rutgers, The State University, New Brunswick, New Jersey, on November 1, 1958. Professor B. E. Meserve, Chairman of the Section, presided at the morning session, and Dr. H. O. Pollak, Senior-Member-at-Large of the Executive Committee, presided at the afternoon session. There were 135 persons present, including 100 members of the Association.

The following new officers were elected: Chairman, Professor S. S. Wilks, Princeton University; Member-at-Large of the Executive Committee, Professor E. P. Starke, Rutgers, The State University. It was noted that Dr. H. O. Pollak will be Chairman of the Program Committee during the coming year, and that Dr. J. D. Daugherty, Eastside High School, Paterson, continues as a Member-at-Large of the Executive Committee.

Dr. George Cherlin, Secretary of the Contest Committee, reported that the contest had been very successful, with 127 schools enrolled and about 3500 students participating.

The following invited addresses of one hour each were presented at the morning session:

1. *The mathematical theory of strictly competitive games*, by Dr. Harlan Mills, Market Research Corporation of America and Princeton University.

Strictly competitive games, exemplified by matching pennies, and poker, provide two mathematical problems: 1) restating their rules in "normalized" form—given a matrix $G=[g_{ij}]$, players I, II simultaneously pick strategies s, t and II pays g_{st} to I; 2) finding probability mixtures of strategies which guarantee the players as favorable expectations as possible, solving problems $\max_x \min_y xGy$ and $\min_y \max_x xGy$, x and y probability vectors—these can be restated as dual linear programs. These best guarantees are identical, and this supplies persuasive rationale for adopting the probability mixtures—the simplex method places them within computational reach.

2. *Linear programming*, by Dr. A. J. Hoffman, General Electric Company, New York, New York.

The main part of the address was a discussion of three applications of linear programming

The first application was the determination of the most profitable product mix of a boot factory. The second discussed the award of contracts at least cost to the awarder. The third application was the exploitation of the duality theorem of linear programming to yield a combinatorial theorem. The address concluded with an explanation of two geometric interpretations of the simplex method.

The program at the afternoon session consisted of a panel discussion on

3. *Teacher education for modern mathematics.*

Dean A. E. Meder, Jr., Rutgers, The State University, who had arranged this part of the program, presented the members of the panel, and introduced the subject. Professor A. W. Tucker, Princeton University, presided.

Each member of the panel discussed one facet of the subject: Professor F. G. Fender, Rutgers, The State University, spoke on computers; Professor E. R. Lorch, Columbia University, spoke on algebra; Professor B. E. Meserve, New Jersey State College, Upper Montclair, spoke on geometry; Professor R. M. Walter, Douglass College, Rutgers, The State University, spoke on probability and statistical inference; and Professor Tucker concluded the formal presentations with a discussion of analysis. There was then discussion from the floor.

Since the formal presentations were recorded and will appear under Mathematical Education Notes in this MONTHLY, the abstracts are not included here.

I. L. BATTIN, *Secretary*

CALENDAR OF FUTURE MEETINGS

Fortieth Summer Meeting, University of Utah, Salt Lake City, Utah, August 31–September 3, 1959.

Forty-third Annual Meeting, Hotel Conrad Hilton, Chicago, Illinois, January 28–30, 1960.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, University of Pittsburgh, May 2, 1959.

ILLINOIS, Millikin University, Decatur, May 8–9, 1959.

INDIANA, Valparaiso University, May 2, 1959.

IOWA, Iowa Wesleyan University, Mount Pleasant, April 17, 1959.

KANSAS, Marymount College, Salina, April 11, 1959.

KENTUCKY, Centre College of Kentucky, Danville, April, 1959.

LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 13–14, 1959.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Goucher College, Towson, Maryland, May 2, 1959.

METROPOLITAN NEW YORK, Polytechnic Institute of Brooklyn, April 18, 1959.

MICHIGAN, Michigan State University, East Lansing, March 28, 1959.

MINNESOTA, University of Minnesota, Minneapolis, April 25, 1959.

MISSOURI, Lindenwood College, St. Charles, April 25, 1959.

NEBRASKA, University of Nebraska, Lincoln, April 18, 1959.

NEW JERSEY, Princeton University, November 7, 1959.

NORTHEASTERN

NORTHERN CALIFORNIA

OHIO, Miami University, Oxford, May 9, 1959.

OKLAHOMA, Tulsa University, Spring, 1959.

PACIFIC NORTHWEST, University of Oregon, Eugene, June 19, 1959.

PHILADELPHIA

ROCKY MOUNTAIN, Utah State University, Logan, May 8–9, 1959.

SOUTHEASTERN, East Tennessee State College, Johnson City, March 20–21, 1959.

SOUTHERN CALIFORNIA, University of Redlands, March 14, 1959.

SOUTHWESTERN, Arizona State University, Tempe, April 10–11, 1959.

TEXAS, University of Texas, Austin, April, 1959.

UPPER NEW YORK STATE, Hartwick College, Oneonta, May 9, 1959.

WISCONSIN, Wisconsin State College, Platteville, May 2, 1959.

HAVE YOU REGISTERED WITH THE MATHEMATICAL SCIENCES SECTION

of the

NATIONAL REGISTER OF SCIENTIFIC AND TECHNICAL PERSONNEL?

On behalf of the Mathematical Association of America and various other mathematical organizations, the American Mathematical Society is assembling and maintaining a register of mathematicians and mathematical scientists. The mathematics register is a section of the National Register of Scientific and Technical Personnel, which is an official responsibility of the National Science Foundation. The purpose of the Register is to provide up-to-date information on the scientific manpower resources of the United States.

As a result of the splendid cooperation accorded to the project by most of the mathematicians and mathematical scientists who have received questionnaires to fill in, the mathematical section of the Register is now remarkably complete. However, there are still a few gaps to be filled in.

If you have received a National Register questionnaire from the American Mathematical Society, won't you please fill it in now and send it to the Headquarters Offices of the Society at 190 Hope Street, Providence 6, Rhode Island?

If you have never received a questionnaire and feel that you are qualified for inclusion in the Register, please drop a note to that effect to the above address.

The Mathematical Association of America

The Association is a national organization of persons interested in mathematics at the college level. It was organized at Columbus, Ohio, in December 1915 with 1045 individual charter members and was incorporated in the State of Illinois on September 8, 1920. Its present membership is over 7700, including more than 200 members residing in foreign countries.

Any person interested in the field of mathematics is eligible for election to membership. Annual dues of $5.00 includes a subscription to the *American Mathematical Monthly.* Members are also entitled to reduced rates for purchases of the Carus Mathematical Monographs and for subscriptions to several journals.

Further information about the Association, its publications and its activities may be obtained by writing to:

H. M. GEHMAN, *Secretary-Treasurer*
THE MATHEMATICAL ASSOCIATION OF AMERICA
University of Buffalo
Buffalo 14, New York

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MARCH 1959

The AMERICAN MATHEMATICAL MONTHLY

(Founded in 1894 by Benjamin F. Finkel)

Ralph D. James, *Editor*

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RECENT RESULTS IN SURFACE AREA THEORY

LAMBERTO CESARI, Purdue University

There is a growing interest in a general theory concerning the analytical properties of transformations, or mappings

$$(1) \qquad (T, A): p = p(w),\ w \in A \subset X,\ p \in Y, \text{ or } T: A \to Y,$$

from a set A of a "space X" into a "space Y." Let us say explicitly that T is meant to be single valued but not necessarily one-one; that is, each $w \in A$ is mapped into one and only one $p = p(w) \in Y$ (image of w), that all these $p = p(w)$, $w \in A$, form a set $T(A) \subset Y$ (graph) of T, but each $p \in T(A)$ may be the image of more than one $w \in A$, even infinitely many $w \in A$ (counter images of p). Real functions of one real variable, parametric curves, surfaces, *etc.*, are examples of such mappings, and the analytical entities attached to (T, A) may be called total variation, length, area, *etc.*, or, more generally, line integrals, surface integrals, *etc.*

The last concepts are usually introduced under very restrictive conditions on T, but recently it has been recognized that length, area, *etc.*, can be introduced under the mere hypothesis of continuity of T (and even this may not be required), and that the finiteness of the length [area, *etc.*] assures the existence of a line integral [surface integral, *etc.*].

This viewpoint has been illustrated for curves in a previous article [II].* Here we discuss the analogous approach for surfaces, which entails a much deeper connection with topology and measure theory than for curves. Total variation of real functions of one real variable and related concepts have been illustrated in another article [I].†

1. The concept of surfaces. We shall denote by E_2 the real Euclidean w-plane, $w = (u, v)$, by E_N any real Euclidean space (for $N = 3$ let E_3 be the p-space with $p = (x, y, z)$), by $\overline{M}$, M^*, M^0 the closure, the boundary, the set of the interior points of a set M inone such space, by $|p| = (x^2 + y^2 + z^2)^{1/2}$ the Euclidean norm, and by $|p - q|$ the Euclidean distance of two points p, q.

By a *surface* S we shall mean a mapping (1) from some set $A \subset E_2$ into E_3 (or E_N), where A could be, for instance, a square, a circle, a polygonal region, or a (closed) simple Jordan region J, or more generally, a (closed) Jordan region of finite connectivity $\nu \geq 0$, say $J = J_0 - (J_1 + \cdots + J_\nu)^0$, $J_i \subset J_0^0$, $J_i J_j = 0$, $i \neq j$, $i = 1, \cdots, \nu$, where all J_i and J_0 are closed simple Jordan regions. It has been found convenient to take for A any "admissible" set, *i.e.*, either any Jordan region J as above, or a finite sum of disjoint Jordan regions, or any set $G \subset E_2$, open in E_2, or any set $G \subset J$, open in J (further generalizations have been and

* Lamberto Cesari, Rectifiable curves and the Weierstrass integral, this MONTHLY, vol. 65, 1958, pp. 485–500. This article will be quoted as [II].

† Lamberto Cesari, Variation, multiplicity, and semicontinuity, this MONTHLY, vol. 65, 1958, pp. 317–332. This article will be quoted as [I].

are being studied). We will suppose $N=3$. Thus (T, A) is defined by a continuous vector function $T(w)$, $w \in A$, say

$$(1.1)\quad \begin{aligned} S &= (T, A): p = T(w),\ w \in A,\ T(w) = [x(w), y(w), x(w)], \text{ or} \\ S &= (T, A): x = x(u, v),\ y = y(u, v),\ z = z(u, v),\ (u, v) \in A. \end{aligned}$$

The set $T(A) \subset E_3$ is the graph of $S=(T, A)$, but in no way defines (T, A). For instance the mappings $T: x=u,\ y=v,\ z=0,\ (u, v) \in Q$, and $T': x=\sin \frac{1}{2}k\pi u$, $y=v, z=0, (u, v) \in Q$, where Q is the square $Q=[0 \leqq u, v \leqq 1]$ and $k>1$ an integer, have both the same graph $T(Q)=T'(Q)=U$, the unit square $U=[0 \leqq x,\ y \leqq 1$, $z=0]$ in the xy-plane, but T covers U just once, while T' covers U exactly k times. Finally, if $P: x=\phi(t),\ y=\psi(t),\ 0 \leqq t \leqq 1$, denotes the well-known Peano curve covering U, then $T'': x=\phi(u),\ y=\psi(u),\ z=0,\ (u, v) \in Q$, has the same graph as J and T, but is a completely different surface. Indeed we shall mention that the "areas" of T, T', T'' are different numbers, namely 1, k, 0, respectively.

Actually there are cases where our intuition associates to different mappings (or surfaces) the same entity (as for instance T and T' above with $k=1$). Indeed, various concepts of "equivalence" have been taken into consideration, as the Lebesgue, Fréchet equivalences (for instance T and T' above for $k=1$ are Lebesgue equivalent). Classes of equivalent mappings are then denoted as Lebesgue surfaces, Fréchet surfaces. The definitions are similar to the ones which hold for curves [II] and we prefer here not to insist on these concepts.* Surfaces, or mappings $S=(T, A)$ for which each point $p \in T(A)$ has only one counterimage are said to be *simple* (and if T^{-1} is continuous then T is a *homeomorphism* between A and $T(A)$). If, for every $p \in T(A)$, the set $T^{-1}(p) \subset A$ is connected, then T is said to be *monotone*; if $T^{-1}(p) \subset A$ is totally disconnected, then T is said to be *light*. For instance, surfaces of the type

$$T:\ x = u,\ y = v,\ z = z(u, v),\ (u, v) \in Q,$$

that is, $T: x=z(x, y),\ (x, y) \in Q$, are said, slightly improperly, to be "nonparametric surfaces," and they are certainly simple. The mapping $T: x=u,\ y=v$, $z=(1-u^2-v^2)^{1/2},\ (u, v) \in Q=[u^2+v^2 \leqq 1]$, whose graph is a "halfsphere" is nonparametric and simple. The mapping $T: x=2(r-r^2)^{1/2} \cos \theta,\ y=2(r-r^2)^{1/2} \sin \theta$, $z=2r-1$, where $(u, v) \in Q,\ r \cos \theta=u,\ r \sin \theta=v$, whose graph is the whole sphere $x^2+y^2+z^2=1$, is monotone, but not light since the point $p=(0, 0, 1)$ is the image of all points $(u, v) \in Q^*$. The mapping $T: x=1-u^2,\ y=u-u^3,\ z=v,\ (u, v) \in A$ $=[-2 \leqq u \leqq 2, 0 \leqq v \leqq 1]$, whose graph is a portion of a cylinder with generatrices parallel to the z-axis, is light, but not monotone.

Of particular interest are the "flat surfaces" or "plane mappings," *i.e.*, those mappings whose graph is contained in a plane. For instance, if $\tau_r,\ r=1, 2, 3$, denote the projections of E_3 onto the yz, zx, xy coordinate planes, say E_{21}, E_{22}, E_{23}, then $T_r=\tau_r T,\ r=1, 2, 3$, are plane mappings, namely,

* Lamberto Cesari, Surface Area, Princeton, 1956. This book will be quoted as [SA].

$$(1.2)\qquad \begin{array}{llll} T_1\colon x = 0 & y = y(u, v), & z = z(u, v), & (u, v) \in A, \\ T_2\colon x = x(u, v), & y = 0, & z = z(u, v), & (u, v) \in A, \\ T_3\colon x = x(u, v), & y = y(u, v), & z = 0, & (u, v) \in A. \end{array}$$

It was pointed out already by S. Banach and G. Vitali that properties of a mapping T (for instance the finiteness of the "area") have none, or very little bearing on the properties of the single functions $x(u, v)$, $y(u, v)$, $z(u, v)$, but they have an essential bearing on the properties of the pairs of functions (y, z), (z, x), (x, y), *i.e.*, on the plane mappings T_r, $r=1, 2, 3$, above. For instance, no matter which continuous function $\phi(u, v)$, $(u, v)\in A$, we consider, the mapping $T\colon x=y=z=\phi(u, v)$, $(u, v)\in A$, has a graph completely contained in the straight line $x=y=z$ in E_3, and its "area" is zero. For nonparametric surfaces $T\colon x=u$, $y=v$, $z=z(u, v)$, all properties of T are, of course, reflected into properties of the single function $z(u, v)$.

2. Lebesgue area. For surfaces (mappings) $S=(T, A)\colon p=p(w)$, $w\in A$, $p=(x, y, z)$, $w=(u, v)$, defined by functions $x(u, v)$, $y(u, v)$, $z(u, v)$ which are continuous in A with their first partial derivatives, it is usual to assume for the area of S the value of the integral (area integral)

$$(2.1)\qquad I(T, A) = (A^0)\int |J|\,dudv = (A^0)\int (J_1^2 + J_2^2 + J_3^2)^{1/2} dudv,$$

and for surface integral the value of

$$(2.2)\qquad I(T, A, f) = (A^0)\int f[p(w), J(w)]dudv,$$

where $J=J(w)$ is the vector Jacobian $J=(J_1, J_2, J_3)$, $J_1=y_uz_v-z_uy_v$, *etc.*, and where $f(p, t)$ is any given function of (p, t) continuous in (p, t) for all $p=(x, y, z)\in T(A)$, and all $t=(t_1, t_2, t_3)$. In order to assure that $I(T, A, f)$ is invariant with respect to Lebesgue, or Fréchet equivalences the further condition is added

$$(h)\colon f(p, kt) = kf(p, t) \text{ for all } k > 0,\ t \text{ and } p \in T(A).$$

Thus for $f=|t|$, $I(T, A, f)$ is the area integral $I(T, A)$ and condition (h) is satisfied. The definitions (2.1) and (2.2) are adequate under the restrictive conditions mentioned above but not in general. To see this let us denote by $\phi(t)$, $0\leqq t\leqq 1$, $\phi(0)=0$, $\phi(1)=1$, the well-known monotone nondecreasing function which is constant on each complementary interval I_j of the ternary Cantor set in $[0, 1]$. Thus $\sum|I_j|=1$ and $\phi'(t)=0$ almost everywhere (a.e.) in $[0, 1]$.* Then the nonparametric light mapping (surface) $S'\colon x=u$, $y=v$, $z=\phi(u)$, $(u, v)\in Q=[0, 1, 0, 1]$ should have "area" $\geqq\sqrt{2}$, while the integral $I(T, A)$ has the value 1. Analogously, the monotone mapping (surface) $S''\colon x=\phi(u), y=v, z=0$ (a mono-

* E. W. Hobson, The Theory of Functions of a Real Variable, Cambridge University Press, 1927, vol. I, p. 368.

tone mapping from Q into the square $U=[0\leqq x, y\leqq 1, z=0]$) should have "area" 1 (or at least 1), while the integral $I(T, A)$ has the value zero.

A definition, which has been shown to be quite adequate is the one proposed in 1900 by H. Lebesgue. In very simple words it could be said that the Lebesgue area of a surface S is the minimum limit of the elementary areas of the polyhedral surfaces approaching S. To make this a precise definition a few more words are necessary.

We shall denote as a figure F any finite sum of disjoint closed polygonal regions in E_2 (for instance, a square, a polyhedral region). A mapping $(P, F): x = p(w)$, $w\in F$, from a figure F is said to be *quasilinear*, or a *polyhedral surface* if (a) $p(w)$ is single valued and continuous in F; (b) there exists some subdivision D of F into nonoverlapping triangles t such that each component $x(u, v)$, $y(u, v)$, $z(u, v)$ of $p(w)$ is linear in each $t\in D$, *i.e.*, of the form $au+bv+c$, a, b, c constants for each t. Then P maps each $t\in D$ into a triangle $\Delta\subset E_3$, which may be degenerated into a segment, or a single point. Then by the elementary area $a(P, F)$ of (P, F) is meant the sum $a(P, F)=\sum a(\Delta)$, where $a(\Delta)$ is the usual area of the triangle $\Delta=P(t)$ and $\sum$ ranges over all $t\in D$. We shall say that a sequence $[A_n]$ of admissible sets A_n *invades* an admissible set A, if $A_n\subset A$, $A_n\subset A_{n+1}$, $A_n^0\rightarrow A^0$ as $n\rightarrow\infty$. Finally, a sequence (T_n, A_n), $n=1, 2, \cdots$, is said to be *convergent* toward (T, A) if (a) $[A_n]$ invades A; (b) $d_n\rightarrow 0$ as $n\rightarrow\infty$ where $d_n = \sup |T_n(w)-T(w)|$ for all $w\in A_n$. Thus, if $A_n\equiv A$, $n=1, 2, \cdots$, the convergence of (T_n, A) toward (T, A) is the uniform convergence in A of $T_n(w)$ toward $T(w)$, $w\in A$.

Suppose now that (T, A) is a given continuous mapping from an admissible set A and denote by γ the class of all sequences $[(P_n, F_n), n=1, 2, \cdots]$ of quasilinear mappings convergent toward (T, A). Then the *Lebesgue area* $L(T, A)$ of (T, A) is defined by

$$L(S) = L(T, A) = \inf_{\gamma} \liminf_{n\rightarrow\infty} a(P_n, F_n). \tag{2.3}$$

Of course some reader may ask why this definition is chosen instead of considering simply "the supremum, or the limit of the elementary areas of the polyhedral surfaces inscribed in S." This is due to the fact discovered by H. Schwarz and G. Peano in 1890 that such a supremum may be $+\infty$ and such a limit may not exist even with as simple a surface as a portion of "circular cylinder." ([SA], 4.2, p. 24). The definition (2.3), proposed by H. Lebesgue in 1900, is the analogue of one of the alternative definitions of Jordan length for a curve [11]. It can be proved ([SA], 5.9, p. 37) that there exists some sequence (P_n, F_n), $n=1, 2, \cdots$, convergent toward (T, A) with $\lim a_n(P_n, F_n)=L(T, A)$ as $n\rightarrow\infty$.

In case A is itself a figure (for instance a square) it is not restrictive to suppose ([SA], 6.2, p. 61) that $F_n=A$ for all n. Then γ is the class of all sequences $[(P_n, A), n=1, 2, \cdots]$ of quasilinear mappings convergent uniformly in A toward (T, A).

3. The lower semicontinuity property of Lebesgue area. The definition (2.3) has, among others, the advantage that it makes it easy to prove the lower semicontinuity of $L(S)$. This can be expressed by the following statement:

(3.i) *If* (T_n, A_n), $n=1, 2, \cdots$, *is convergent toward* (T, A), *then* $L(T, A) \leqq \liminf L(T_n, A_n)$ *as* $n \to \infty$.

We shall prove (3.i) in the case A is a given figure, $A_n = A$ for all n, and by assuming in the definition (2.3) $F_n = A$ for all n, according to the remark at the end of Section 2. Also we may assume $\lambda = \liminf L(T_n, A_n) < +\infty$, and $L(T_n, A) < +\infty$ for all n. Then, if $d_n = \max |T_n(w) - T(w)|$, we have $d_n \to 0$ as $n \to +\infty$. By the definition of $L(T, A)$ there exists some sequence (P_{nm}, A), $m = 1, 2, \cdots$, of quasilinear mappings convergent toward (T_n, A) as $m \to \infty$ with $a(P_{nm}, A) \to L(T_n, A)$. Thus we have $\delta_{nm} \to 0$ as $m \to \infty$ where $\delta_{nm} = \max |P_{nm}(w) - T_n(w)|$ for all $w \in A$. For each n, there exists an integer $m = m(n)$ such that $\delta_{nm} < 1/n$, $|a(P_{nm}, A) - L(T_n, A)| < 1/n$. Now we have $|P_{nm}(w) - T(w)| \leqq |P_{nm} - T_n| + |T_n - T| \leqq \delta_{nm} + d_n < d_n + 1/n$ for all $w \in A$. If $P_n' = P_{nm}$, then the sequence (P_n', A), $n = 1, 2, \cdots$, converges toward (T, A), that is, belongs to the class γ relative to (T, A), and thus, by (2.3), we have

$$L(T, A) \leqq \liminf_{n \to \infty} a(P_n', A) \leqq \liminf_{n \to \infty} [L(T_n, A) + 1/n] = \lambda.$$

4. Plane mappings, their total variation, and absolute continuity. Let (T, A): $p = T(w)$, $w \in A$, $w = (u, v)$, $p = (x, y)$, be any continuous mapping from an admissible set A of the oriented uv-plane E_2 into the oriented xy-plane E_2', *i.e.*,

$$(T, A): x = x(u, v), \ y = y(u, v), \ (u, v) \in A.$$

For every simple closed polygonal region $\pi \subset A$, let us consider the oriented boundary π^* and the image C: (T, π^*) of π^*, that is, the continuous mapping (T, π^*) defined by T on π^* (restriction of T on π^*). This is a closed oriented

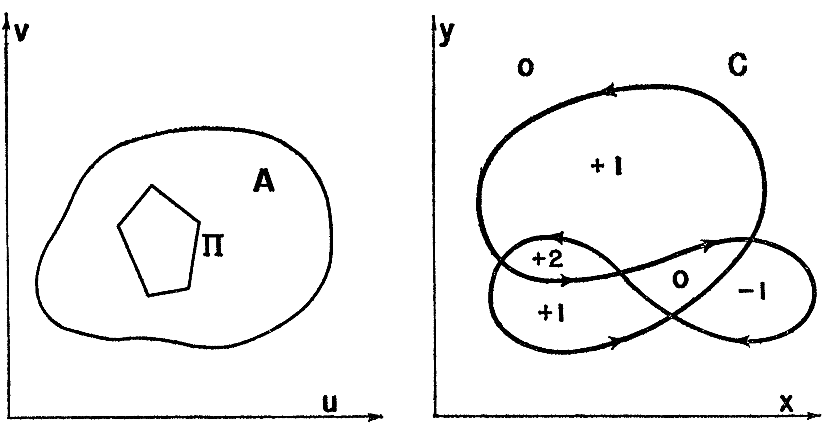

FIG. 1

curve C of the xy-plane E_2 (Fig. 1). For each point $p_0=(x_0, y_0)$ not on the graph $[C]$ of C, we can define the *topological index* $O(p_0; C)$ of C with respect to the point $p_0 \in E_2' - [C]$. Roughly speaking, $O(p_0; C)$ is the integral number of times ($\gtreqless 0$) in which C "links" the point p_0 in the positive direction. Suppose we assume on π^* a parameter, say s, $0 \leqq s \leqq a$. Suppose we use polar coordinates (ρ, θ) of center p_0 in E_2, then as s describes $[0, a]$, *i.e.*, w describes π^* once in the positive sense, then $p=T(w)$, describes C. The modulus $\rho=\rho(s)$ of p, *i.e.*, $\rho=\rho(s)=|p-p_0|$, is a single-valued continuous function of s on $[0, a]$ and $\rho(a)=\rho(0)$. The argument $\theta(s)$ of p is defined only mod 2π. If we fix any one of its values, say $\bar{\theta}=\theta(0)$, for $s=0$, then by continuity, $\theta(s)$ is defined on $[0, a]$ as a single-valued continuous function of s, and $\theta(a) \equiv \theta(0)=\bar{\theta}$ (mod 2π). Then, by definition, $O(p_0; C)=(2\pi)^{-1}[\theta(a)-\theta(0)]$, and it is easy to prove that $O(p_0; C)$ does not depend on the parametrization of π^*, and on the choice of $\bar{\theta}$. This purely analytical definition* of $O(p_0; C)$ is certainly the most elementary one. Other equivalent and purely topological definitions are given in topology. If C is rectifiable, and we think of E_2' as the plane of the complex variable $\zeta=x+iy$, with $\zeta_0=x_0+iy_0$, then $O(p_0; C)$ is given by the line integral

$$O(p; C) = (2\pi)^{-1}\int_C (\zeta - \zeta_0)^{-1}d\zeta.$$

The topological index $O(p; C)$ has a number of analytical and topological properties ([SA], 8.3–11, p. 85). Let us mention here that $O(p; C)$, $p \in E_2' - [C]$, is always finite, [but not necessarily bounded in E_2']; $O(p; C)$ is constant on each complementary component of the graph $[C]$ of C and is zero in the unbounded one. In the case of the illustration (Fig. 1), the values of $O(p, C)$ are given. If we assume $O(p, C)=0$ at all points $p \in [C]$, then $O(p; C)$, $p \in E_2'$, is a single-valued, integral-valued ($\gtreqless 0$) function of p in E_2', and $O(p; C)$ is B-measurable. Thus the L-integral

$$v(T; \pi) = (E_2')\int |O(p; C)| dxdy$$

exists (finite, or $+\infty$). Analogously, we may consider the numbers

$$v_+(T, \pi) = (E_2')\int O^+(p; C)dxdy \geqq 0, \qquad v_-(T, \pi) = (E_2')\int O^-(p; C)dxdy \geqq 0,$$

where $O^+=\frac{1}{2}[|O|+O]$, $O^-=\frac{1}{2}[|O|-O]$, and $v_++v_-=v$. Finally, if $O(p; C)$ is L-integrable, *i.e.*, $v(T, \pi)<+\infty$, then also the number

$$u(T, \pi) = (E_2')\int O(p; C)dxdy \gtreqless 0,$$

exists and $|u| \leqq v$.

* P. Alexandroff and H. Hopf, Topologie, Berlin, 1935, p. 462.

If D denotes any finite system of closed nonoverlapping simple polygonal regions $\pi \subset A$, let

$$V(T, A) = \sup_{D} \sum_{\pi \in D} v(T, \pi),$$

and, for every point $p \in E_2'$, also

$$N(p; T, A) = \sup_{D} \sum_{\pi \in D} | O(p; C) | .$$

Then $N(p; T, A)$, $0 \leqq N \leqq +\infty$, is an integral-valued, single-valued function of p in E_2', and $N(p)$ is lower semicontinuous in E_2, *i.e.*, $N(p_0) \leqq \liminf N(p)$ as $p \rightarrow p_0$, for every $p_0 \in E_2'$. The function $N(p, T, A)$, $p \in E_2'$, is said to be the multiplicity function of (T, A). (It is similar to the corrected multiplicity of a real function of one real variable considered in [I], p. 239). Finally, let

$$W(T, A) = (E_2') \int N(p; T, A) dxdy.$$

Both V and W can be considered as total variations of (T, A). Indeed, V is of the type of the Jordan total variation, and W of the type of the Banach total variation of a real function of one real variable ([I], p. 321). The following basic theorem extends Banach theorem for real functions ([I], p. 321):

(4.i) *For every continuous plane mapping* (T, A) *we have* $V(T, A) = W(T, A) = L(T, A)$.

For a proof see [SA], p. 186 and p. 390. Then the common value (finite, or $+\infty$) of V, W, L can be assumed as the total variation of the plane mapping (T, A), and (T, A) is said to be of *bounded variation* (BV) if $V(T, A) = W(T, A) < +\infty$. Analogously let us put

$$V_+(T, A) = \sup_{D} \sum_{\pi \in D} v_+(T, \pi), \qquad V_-(T, A) = \sup_{D} \sum_{\pi \in D} v_-(T, \pi),$$

$$N_+(p; T, A) = \sup_{D} \sum_{\pi \in D} O^+(p; C), \qquad N_-(p; T, A) = \sup_{D} \sum_{\pi \in D} O^-(p; C),$$

$$W_+(T, A) = (E_2') \int N_+(p; T, A) dxdy, \qquad W_-(T, A) = (E_2') \int N_-(p, T, A) dxdy.$$

(4.ii) *For every continuous plane mapping* (T, A) *we have* ([SA], p. 187)

$$V_+(T, A) = W_+(T, A), \; V_-(T, A) = W_-(T, A), \; V_+ + V_- = V, \; W_+ + W_- = W.$$

Thus the common values $V_+ = W_+$, $V_- = W_-$ can be assumed as the *positive* and *negative total variations* of (T, A), and the difference $\mathfrak{V}(T, A) = V_+(T, A) - V_-(T, A)$ as the *signed*, or *relative*, total variation of (T, A).

Finally if (T, A) is BV, *i.e.*, $V(T, A) < +\infty$, then $v(T, \pi) < +\infty$ for every $\pi \in A$, and thus also the following number is defined

$$U(T, A) = \sup_{D} \sum_{\pi \in D} |u(T, \pi)|.$$

(4.iii) *For every continuous BV plane mapping* (T, A) *we have* $U(T, A) = V(T, A)$.

Let us observe finally that V is "overadditive;" *i.e.*, if $[A']$ is any finite subdivision of A into nonoverlapping admissible sets, or, more generally, $[A']$ is any finite system of nonoverlapping admissible sets $A' \subset A$, then $V(T, A) \geqq \sum V(T, A')$, and the sign $>$ may hold even in the apparently elementary case where A' and A are polygonal regions.

The considerations above show how the concept of "plane mapping of bounded variation" (BV) can be founded on topological and measure theoretical considerations. On the same basis we can introduce the corresponding concept of absolute continuity (AC).

A continuous (plane) mapping $(T, A): p = T(w), w \in A, w = (u, v), p = (x, y, z)$ from the uv-plane E_2 into the xy-plane E_2', is said to be *absolutely continuous* (AC) if both the following conditions hold:

(a) *given* $\epsilon > 0$ *there is a* $\delta = \delta(T, A, \epsilon) > 0$ *such that for each finite system* $D = [\pi]$ *of nonoverlapping simple closed polygonal regions* $\pi \subset A$ *with* $\sum |\pi| < \delta$ *we have* $\sum v(T, \pi) < \epsilon$;

(b) *for every simple closed polygonal region* $\pi \subset A$ *and finite subdivision* $[\pi']$ *of* π *into nonoverlapping simple polygonal regions* π' *we have* $V(T, \pi) = \sum V(T, \pi')$.

In (a), $|\pi|$ denotes the Lebesgue measure of the set π in E_2 (area). Condition (a) is familiar, and essentially requires that v (and thus V) is "an absolutely continuous set function." Condition (b) simply requires that V is "additive" (at least in the class of the simple polygonal regions $\pi \subset A$). Conditions (a) and (b) are independent, as has been shown by examples, and it is just their logical sum (a)+(b) which is assumed here as a definition of absolute continuity (AC). There are a number of interesting properties, each of which is a necessary and sufficient condition for a mapping (T, A) to be AC.

5. A characterization of surfaces with finite area. The basic concepts of area of a mapping (T, A), $T: A \to E_3$, (Sec. 2) and of total variation of the plane mappings (T_r, A) defined by (1.2) (Sec. 4) are connected by a basic theorem, which extends formally to Lebesgue area a known Jordan theorem for Jordan length ([II], p. 489).

(5.i) *For every continuous mapping* $(T, A): p = T(w), w \in A, w = (u, v)$ $p = (x, y, z)$, *we have*

$$(5.1) \quad V(T_r, A) \leqq L(T, A) \leqq V(T_1, A) + V(T_2, A) + V(T_3, A), \qquad r = 1, 2, 3.$$

Thus $L < +\infty$ *if and only if all plane mappings* (T_r, A), $r = 1, 2, 3$, *are* BV [L. Cesari, 1942].

This theorem, in spite of its analogy with the elementary Jordan theorem for curves (see II), has been shown to have a deep topological and measure theoretical basis. The proof is given in [SA], p. 295 and consists in the process of stretching and smoothing the continuous surface $S=(T, A)$ into continuous polyhedral surfaces $S_n=(P_n, F_n)$, $n=1, 2, \cdots$, with $P_n \to T$ as $n \to \infty$, and $a(P_n, F_n) \leqq V_1+V_2+V_3$, $V_r=V(T_r, A)$, $r=1, 2, 3$.

6. Peano and Geöcze areas. Let (T, A): $p=p(w)$, $w \in A$, be any continuous mapping from $A \subset E_2$ into E_3, and let us denote by T_1, T_2, T_3 the plane mappings (1.2), which are the projections $T_r=\tau_r T$ of T on the yz, zx, xy coordinate planes E_{21}, E_{22}, E_{23}. If π denotes any simple polygonal region $\pi \subset A$, π^* the oriented boundary of π, C: (T, π^*), C_r: (T_r, π^*), $r=1, 2, 3$, the continuous oriented closed curves which are the images of π^* under T and T_r. Thus $[C] \subset E_3$, $[C_r] \subset E_{2r}$, and C_r is the "projection" of C on E_{2r}. According to Section 4 we put

$$v_r = v(T_r, \pi) = (E_{2r}) \int \left| O(p; C_r) \right| dp, \qquad r = 1, 2, 3,$$

$$v = (v_1, v_2, v_3), \qquad |v| = (v_1^2 + v_2^2 + v_3^2)^{1/2},$$

$$V(T, A) = \sup_D \sum_{\pi \in D} \left| v(T, \pi) \right|,$$

where the supremum is taken with respect to all finite systems D of nonoverlapping simple polygonal regions $\pi \subset A$. Thus $v(T_r, \pi) \leqq v(T, \pi)$, $V(T_r, A) \leqq V(T, A) \leqq +\infty$, $r=1, 2, 3$. The number V can be thought of as an "area" of the surface $S=(T, A)$. A variant of this definition is the following one. Denote by α any plane in E_3, τ_α the projection of E_3 onto α, $T_\alpha=\tau_\alpha T$ the projection of T on α, C_α: (T_α, π^*) the oriented continuous closed curve which is the image of π^* under T_α, $[C_\alpha] \subset \alpha$. Then put $v(T_\alpha, \pi)=(\alpha)\int |O(p; C_\alpha)| dp$, $v^*(T, \pi) = \sup_\alpha v(T_\alpha, \pi)$,

$$P(T, A) = \sup_D \sum_{\pi \in D} v^*(T, \pi) = \sup_D \sum_{\pi \in D} \sup_\alpha v(T_\alpha, \pi).$$

Also, $P(T, A)$, like $V(T, A)$ and $L(T, A)$, is an "area" of T, in the sense that all three definitions (just as many others) are precise mathematical formulations of concepts, which all strongly appeal to our intuition as areas. As a matter of fact the following theorem has been proved:

(6.i) *For every continuous mapping (surface) (T, A) we have $L(T, A) = V(T, A) = P(T, A)$* [C. B. Morrey, T. Rado, L. Cesari, J. Cecconi].

The common value of the three numbers, L, V, P, is defined as the *area* of (T, A), and is usually called the Lebesgue area of (T, A). The present definitions of V and P are the final result of successive refinements of a concept which was first proposed by G. Peano (1890). The process of successive refinements just mentioned was necessary in order to reach the basic identity (6.i) which did not hold for the previous somewhat crude concepts. A number of authors, among

them Z. de Geöcze, S. Banach, L. Tonelli, R. Caccioppoli, T. Rado, E. J. McShane, C. B. Morrey, J. Cecconi, L. Cesari have contributed to these refinements. By common agreement the area V is usually designated as the Geöcze area and P as the Peano area of $S=(T, A)$, so as to honor two mathematicians who in retrospect proved to have so deep an insight in the theory to come. For a direct proof of (6.i) see [SA] p. 390. If $V(T, A)<+\infty$, then $V_r(T, A)<+\infty$, $r=1, 2, 3$, and for every $\pi\subset A$, also the following numbers exist

$$u_r = u(T_r, \pi) = (E_{2r})\int O(p; C_r)dp, \qquad r = 1, 2, 3,$$

$$u = (u_1, u_2, u_3), \qquad |u| = (u_1^2 + u_2^2 + u_3^2)^{1/2},$$

$$U(T, A) = \sup_D \sum_{\pi\in D} |u(T, \pi)|,$$

where the same conventions are used as above. Then we have obviously

$$0 \leqq |u(T, \pi)| \leqq |v(T, \pi)| < +\infty, \qquad 0 \leqq U(T, A) \leqq V(T, A) \leqq +\infty$$

and it is clear that for some π and T we may well have $|u|<|v|$. Nevertheless the following theorem holds:

(6.ii) *For every mapping* (T, A) *with* $V(T, A)<+\infty$, *we have* $U(T, A)=V(T, A)$ [L. Cesari].

7. A Weierstrass-type integral. By the same blending of topological and analytical considerations, by means of which we have defined Geöcze and Peano areas, we may now define the concept of surface integral through a Weierstrass-type limit process.

Let $S=(T, A): p=T(w)$, $w\in A$, $w=(u, v)$, $p=(x, y, z)$, be a given mapping of finite area and, for simplicity of exposition, let us suppose that A is closed finitely-connected Jordan region. Let $f(p, t)$ be a continuous function of (p, t), $p=(x, y, z)$, $t=(t_1, t_2, t_3)$, for all $p\in T(A)$ and real vector t satisfying the usual condition $f(p, kt)=kf(p, t)$ for all $k\geqq 0$, $p\in T(A)$, and t. For every simple polygonal region $\pi\subset A$ let us consider the vector

$$u = u(T, \pi) = (u_1, u_2, u_3), \qquad u_r = u(T_r, \pi), \qquad r = 1, 2, 3,$$

of norm $|u|=(u_1^2+u_2^2+u_3^2)^{1/2}$. If $|u|>0$ then the unit vector

$$a = a(T, \pi) = (a_1, a_2, a_3), \qquad a_r = u_r/|u|, \qquad r = 1, 2, 3, \qquad a_1^2 + a_2^2 + a_3^2 = 1,$$

can be thought of as the vector of the direction cosines of an "average normal" n to the piece of the surface S defined by T on π. If D is any finite system $D=[\pi]$ of nonoverlapping simple polygonal regions $\pi\subset A$, and for each $\pi\in D$, we denote by $\tilde{p}$ any point of $T(\pi)$, we may consider the sum

$$S = \sum_{\pi\in D} f(\tilde{p}, u) = \sum_{\pi\in D} f(\tilde{p}, a)\,|u(\pi, T)|$$

as an "approximate expression" of a Weierstrass-type integral $J(T, A, f)$ of f on the surface $S=(T, f)$. An index $\delta=\delta(D)$ measuring the "fineness" of D can be introduced in various ways. Then it can be proved that the following limit exists and is finite

$$J(T, A, f) = \lim_{\delta\to 0} S = \lim_{\delta\to 0} \sum_{\pi\in D} f(p, a) \mid u(\pi, T) \mid,$$

using only the general hypotheses listed above (Cesari). The integral J is invariant with respect to both Lebesgue and Fréchet equivalences; *i.e.*, $J(T, A, f)$ has the same value in correspondence with Lebesgue or Fréchet equivalent mappings.

An index δ of fineness of a system D is, for instance, defined by the maximum of all following numbers: diam $T(\pi)$ for $\pi\in D$; $\left|\sum T_r(\pi)\right|$, $r=1, 2, 3$, and

$$U(T, A) - \sum_{\pi\in D} \mid u(T, \pi) \mid, \qquad U(T_r, A) - \sum_{\pi\in D} \mid u(T_r, \pi) \mid, \qquad r = 1, 2, 3.$$

8. Relation between area and area integral. Now the question presents itself: What relation is there between area and the area integral of Section 2, and between the Weierstrass-type integral J and the classical surface integral I of Section 2? As mentioned in Section 2 the finiteness of the area does not imply the existence of the partial derivatives of $x(u, v)$, $y(u, v)$, $z(u, v)$, and hence of the ordinary Jacobians $J_1=y_u z_v - y_v z_u, \cdots$. The example $S=(T, A)$: $x=y=z=\phi(u, v)$, where $\phi(u, v)$ is any continuous function with partial derivatives at no point, is typical since here $L(S)=0$ is certainly finite. Thus it is clear that $L(S)<+\infty$ implies the existence of J, but neither area integral, nor the classical integral I need exist.

Nevertheless, under the sole hypothesis of finiteness of the area, certain "generalized Jacobians" $\mathcal{J}_r(w)$, $w\in A^0$, can be defined a.e. in A^0, and for which we have $\mathcal{J}_r(w)=J_r(w)$, $r=1, 2, 3$, a.e. in A^0, whenever the functions x, y, z have ordinary partial derivatives a.e. in A^0. Of the various equivalent definitions of generalized Jacobians the following is certainly the simplest one, and holds for almost all $w_0\in A^0$:

$$\mathcal{J}_r(w_0) = \lim_{\sigma\to 0} \mathfrak{V}(T_r, q)/\mid q \mid, \qquad r = 1, 2, 3,$$

where $w_0\in A^0$, q denotes any square with sides parallel to the u and v axes, with $w_0\in q$, $q\subset A^0$, and $\sigma=\text{diam } q$. With these definitions the following theorem can be proved, which corresponds formally to a Tonelli's theorem for curves ([II], p. 492):

(8.i) *For any continuous mapping $S=(T, A)$ with $L(T, A)<+\infty$, we have*

$$L(T, A) \geqq (A^0)\int \mid \mathcal{J} \mid dudv, \tag{8.1}$$

where $\mathcal{J}=(\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3)$. The equality sign holds if and only if the plane mappings (T_r, A), $r=1, 2, 3$, are AC.

Suppose now that $S=(T, A)$ and $f(p, t)$ are given as in Section 7. Then we have

(8.ii) *If* $L(S)<+\infty$ *and the plane mappings* (T_r, A), $r=1, 2, 3$, *are* AC, *then*

$$J(T, A, f) = (A^0)\int f[p(w), \mathfrak{g}(w)]dudv. \tag{8.2}$$

In other words, the integral J *is given as an ordinary surface integral (with generalized Jacobians) whenever the area is given by the corresponding area integral.*

In both (8.1) and (8.2) the integrals are L-integrals.

The question finally presents itself, whether any continuous surface $S=(T, A): x=x(u, v),\ y=y(u, v),\ z=z(u, v),\ (u, v)\in A$, has a "representation" $(T^*, A): x=X(u, v),\ y=Y(u, v),\ z=Z(u, v),\ (u, v)\in A$, for which the partial derivatives $X_u, \cdots, Z_v$ exist a.e. in A^0, and for which the area is given by the classical area integral. In other words, we ask whether another continuous mapping (T^*, A) exists which is Lebesgue, or Fréchet equivalent to (T, A), for which the plane mappings (T^*_r, A), $r=1, 2, 3$, are AC and for which $X_u, \cdots, Z_v$ exist a.e. in A^0. The answer is affirmative when Fréchet equivalence is considered (Cesari).

9. Fine-cyclic elements. For the sake of simplicity we shall suppose, as in Section 7, that A is a closed finitely connected Jordan region, say $A=J_0-(J_1+\cdots+J_\nu)^0$, (Sec. 1), where $0\leqq\nu<+\infty$ is the *order of connectivity* of A. Thus, for $\nu=0$, A is called a *disc*, for $\nu=1$, A is called an *annular region*. We have already mentioned in Section 1 the concepts of monotone mappings and of light mappings. We should now recall a precise characterization of the topological structure of any continuous mapping. For the purpose let us mention here that if we have a mapping $f: y=f(x)$ from a "set" A onto a set B, and a mapping $g: z=g(y)$ from B into a space C, then the composition mapping $F=gf$: $z=g[f(x)]$ from A into C, is said to be the product of f and g (in this order) and denoted by gf. Also, $F=gf$ is said to be a factorization of F into the two factors f and g. With this convention, we may state at once the factorization theorem of analytic topology, namely, that every continuous mapping, say (T, A): $p=p(w), w\in A$, has a factorization $T=lm$ into two factors, a monotone mapping m, followed by a light mapping l (monotone-light factorization).

In order to understand this in the terms which are needed here, let us consider for every point $p\in T(A)$ the inverse set $T^{-1}(p)\subset A$. Since A is compact and T is continuous, the set $T^{-1}(p)$ is closed, hence its components g are continua $g\subset A$. By a "continuum" is meant, as usual, a bounded, closed, connected set, and hence even single points may be considered as "continua." Note that, if T is monotone (Sec. 1), then, for every $p\in T(A)$, the inverse set $g=T^{-1}(p)\subset A$ is just one continuum; if T is light (Sec. 1), then, for every $p\in T(A)$, all components g of $T^{-1}(p)\subset A$ are single points. In any case, for any continuous mapping (T, A), the collection $\Gamma=\{g\}$ of all components g of the set $T^{-1}(p)$,

$p \subset T(A)$, is a decomposition of A into disjoint continua $g \subset A$ (which may well be all single points of A). Γ is the collection of all maximal continua of constancy for T in A.

Also we may consider the family $\mathcal{G}_0$ of all sets $G \subset A$ which are (1) open in A; (2) the union of continua $g \in \Gamma$, say $G = \sum g$ (*i.e.*, have the property that $g \in \Gamma$, $gG \neq 0$ implies $g \subset G$).

We may now consider the elements g of Γ as "points," say $\tilde{g}$, and Γ as the "space," say $\tilde{\Gamma}$, whose "points" are $\tilde{g}$. In order to consider $\tilde{\Gamma}$ as a space we actually should define a "topology" on $\tilde{\Gamma}$, which turns out to be equivalent as defining in $\tilde{\Gamma}$ the collection $\tilde{\mathcal{G}}$ of the open sets $\tilde{G}$. This can be done easily by considering each set $G = \sum g$, $G \in \mathcal{G}_0$, as the union $\tilde{G} = \sum \tilde{g}$ of the corresponding elements $\tilde{g}$. Then $\tilde{\mathcal{G}}$ is just the family of all sets $\tilde{G}$.

By means of these quite natural definitions $\tilde{\Gamma}$ can be proved to be not only a topological space, but a "Peano space" (in particular compact, connected, locally connected). Now if $m: \tilde{g} = m(w)$, $w \in A$, is the mapping from A onto $\tilde{\Gamma}$ which maps each point $w \in g$, $g \in \Gamma$ into the point $\tilde{g} \in \tilde{\Gamma}$, then m is obviously monotone since for each $\tilde{g} \in \tilde{\Gamma}$, the set $m^{-1}(\tilde{g}) = g$ is exactly the continuum $g \in \Gamma$, $g \subset A$. Finally, if $l: p = p(\tilde{g})$, $\tilde{g} \in \tilde{\Gamma}$, is the mapping from $\tilde{\Gamma}$ onto $T(A) \subset E_3$ defined by $l = Tm^{-1}$, then l is "light" since for each $p \in T(A)$, the components g of the set $l^{-1}(p)$ are all single points $\tilde{g} \in \tilde{\Gamma}$. While we refer to the usual expositions for more formal proofs, we note that $T = lm$ is a monotone-light decomposition of T, that $m: A \rightarrow \tilde{\Gamma}$, $l: \tilde{\Gamma} \rightarrow T(A)$, and that $\tilde{\Gamma} = m(A)$ is the "middle space," or "hyperspace" of the decomposition (G. T. Whyburn, Amer Math. Soc. Colloq. Publ., vol. 28, 1942).

Obviously any "space" M which is homeomorphic to $\tilde{\Gamma}$, may be considered as a middle space, or hyperspace, for T, since, if h is a homeomorphism of $\tilde{\Gamma}$ onto M, and $m' = hm$, $l' = lh^{-1}$, then $T = l'm'$, $m': A \rightarrow M$, $l': M \rightarrow T(A)$, and m' is monotone and l' is light. Thus M is the middle space, and M can be called a *model* of $\tilde{\Gamma}$. Also, for every monotone-light factorization $T = lm$ of T, $m: A \rightarrow M$, $l: M \rightarrow T(A)$, M is homeomorphic to $\tilde{\Gamma}$, *i.e.*, M is a model of $\tilde{\Gamma}$ (G. T. Whyburn).

If for some readers the previous considerations appear somewhat abstract, the following remark may be of help: A model M can be built in the Euclidean space E_3. If T is light, we may take for m the identity mapping and then $M = A$ is the middle space; if T is monotone, then we may take for l the identity mapping and then $M = T(A)$, *i.e.*, the graph of (T, A) is the middle space M.

For instance for the monotone mapping $T: x = u$, $y = 0$, $z = 0$, $(u, v) \in A = [0 \leqq u, v \leqq 1]$, M is a segment (an arc); for the monotone mapping $T: x = \sin \pi r \cos \theta$, $y = \sin \pi r \sin \theta$, $z = \cos \pi r$, $(u, v) \in A = [u^2 + v^2 \leqq 1]$, where $r \cos \theta = u$, $r \sin \theta = v$, $M = T(A)$ is the unit sphere in E_3; for the light mapping $T: x = u$, $y = v$, $z = z(u, v)$, $(u, v) \in A$, where $z(u, v)$ is a continuous function constant on no proper subcontinuum of A, the middle space is $M = A$.

Let $(T, A): p = p(w)$, $w \in A = [u^2 + v^2 \leqq 1]$ be the monotone mapping from the disc A into E_3 defined by $x = (2r - 1) \cos \theta$, $y = (2r - 1) \sin \theta$, $z = 0$, if $\frac{1}{2} \leqq r \leqq 1$, and by $x = y = 0$, $z = 1 - 2r$, if $0 \leqq r \leqq \frac{1}{2}$ where $r \cos \theta = u$, $r \sin \theta = v$. Then Γ is the

collection of all circles $u^2+v^2=r^2$ with $0<r\leqq\frac{1}{2}$, and of all single points $(u, v)\in A$, with $u=v=0$, or $\frac{1}{4}<u^2+v^2\leqq 1$ (Fig. 2). The middle space $M=T(A)$ is made up of the unit circle of the xy-plane and of the unit segment of the z-axis issuing from its center (a disc and a thread).

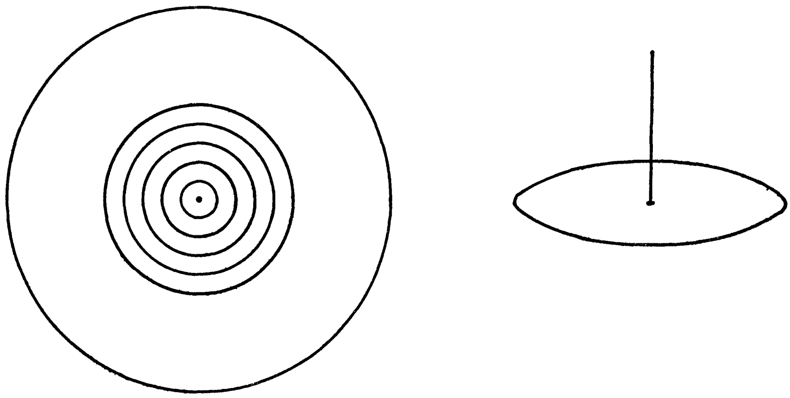

Fig. 2

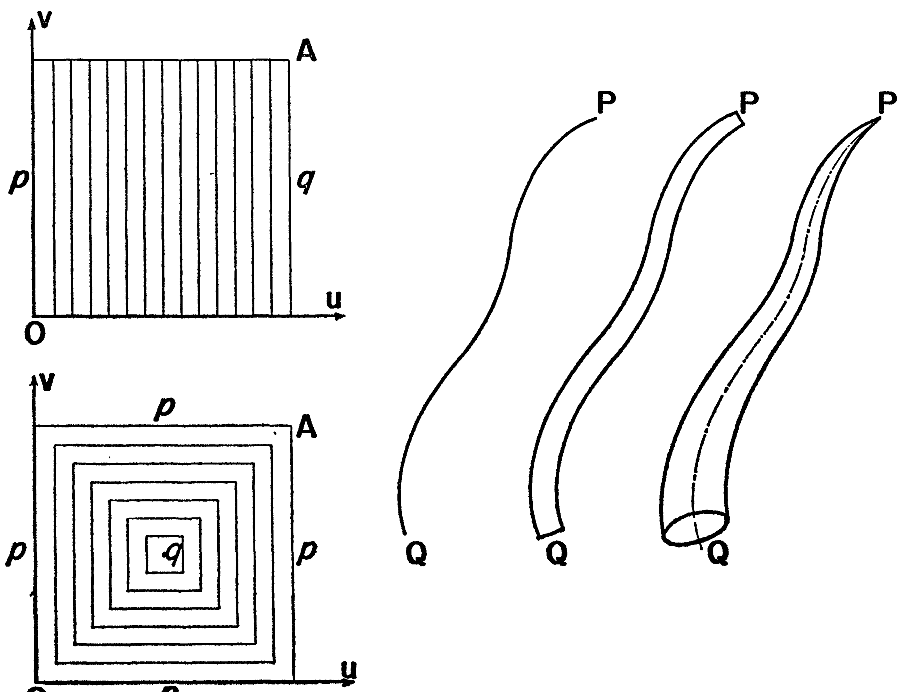

Fig. 3

Let us consider now mappings $(T, A): p = T(w)$, $w \in A = [0 \leqq u, v \leqq 1]$, from the unit square into E_3 (Fig. 3), where either Γ is the collection of all segments $g = [0 \leqq v \leqq 1, u = t]$, $0 \leqq t \leqq 1$, (surfaces $S_1 = (T, A)$), or Γ is the collection of the boundaries $g = [\max(|2u-1|, |2v-1|) = t]$, $0 \leqq t \leqq 1$, of all squares contained in A, concentric and similar to A (surfaces $S_2 = (T, A)$). In either case we may assume $T(w) = f(t)$, where $f(t)$ is a continuous vector function of t in $0 \leqq t \leqq 1$, constant on no subinterval of $[0, 1]$. If we suppose that $f(t)$ never takes twice the same value $f = (x, y, z)$, then T is monotone, and $M = T(A)$ is the arc PQ. The two types of surfaces S_1, S_2 apparently identical in E_3, are quite different. Surfaces S_1 can be thought as limit cases of thin strips, surfaces S_2 as limiting cases of thin cones. Two surfaces S_1, S_2, even defined by the same vector function f, are neither Lebesgue nor Fréchet equivalent.

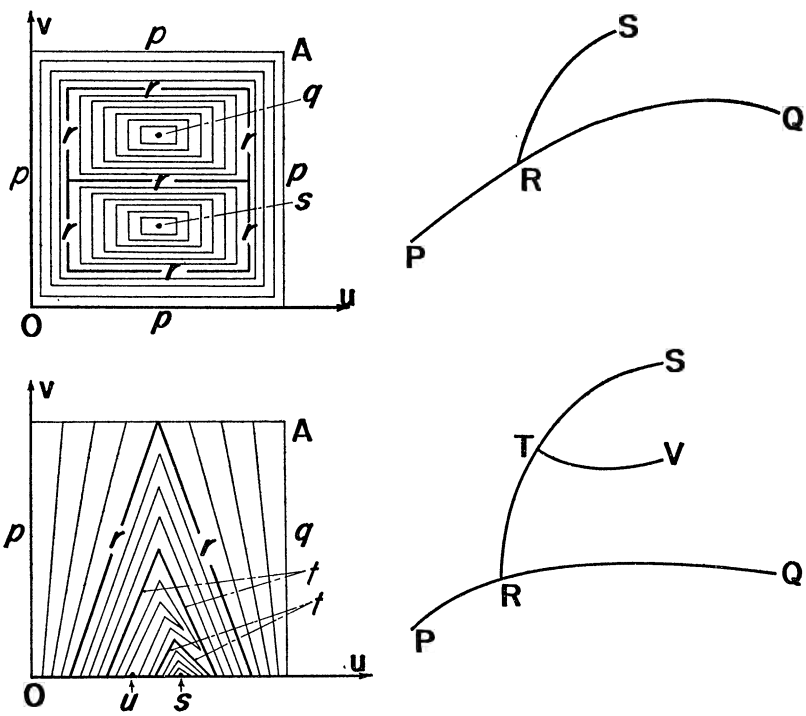

FIG. 4

Mappings (T, A) with $L(T, A) = 0$ can be characterized topologically, namely, M is a space of dimension $\leqq 1$ (T. Rado, 1945, for $\nu = 0$; R. F. Williams, 1958, for $\nu \geqq 0$). In the case A is a disc $(\nu = 0)$, M has been further characterized (as a dendrite of analytic topology). In simple words, we may expect M to be a ramified system of threads, as for the simple examples given in Figure 4 for $\nu = 0$.

Mappings $(T, A): p=T(w)$, $w \in A$, with $L(T, A)>0$ must therefore possess a middle space M with some parts of dimension 2 (see, for instance, the example of Figure 2 where M is made up of a disc and a thread). These parts are important and, in the case A is a disc ($\nu=0$), they are the *cyclic elements* Σ of M. A subset Σ of M is said to be a cyclic element of M if (1) Σ is a proper continuum; (2) Σ is not disconnected by suppressing any one of the points of Σ. Any two cyclic elements Σ of M are not overlapping (they may have in common at most one point), and the collection $\{\Sigma\}$ of all cyclic elements of M is at most countable. Thus in the example above the only cyclic element Σ of M is the disc. Again, for $\nu=0$, the cyclic elements Σ of M have been fully characterized: each of them is either a disc, or a sphere (G. T. Whyburn). The illustrations (Figs. 5, 6, 7, 8) give examples of decompositions Γ of A and of the corresponding middle spaces M of mappings (T, A) from a disc ($\nu=0$). For some only the middle space is given.

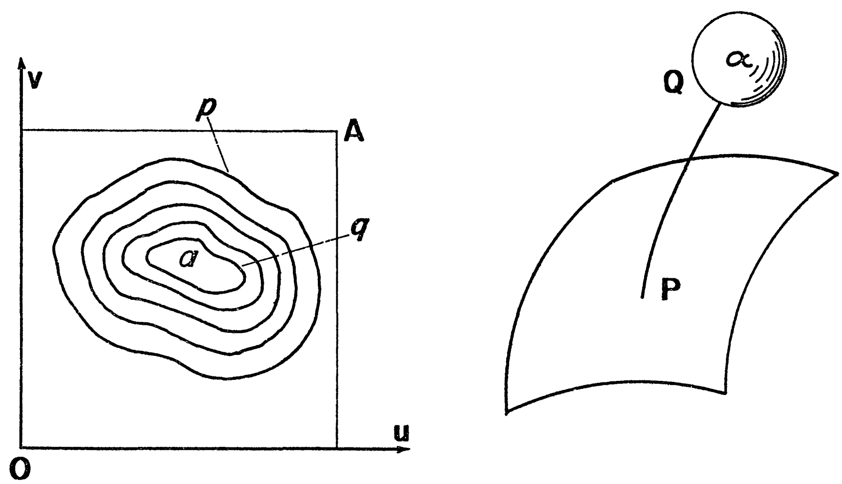

FIG. 5

For $\nu>0$ a further decomposition of M may be necessary. The parts σ of M of dimension 2 may actually be finer than the cyclic elements Σ of M, and they are denoted as the *fine-cyclic elements* σ of M. A subset σ of M is said to be a fine cyclic element of M if (1) σ is a proper continuum; (2) σ is not disconnected by suppressing any finite system of points of σ (L. Cesari, Riv. Mat. Univ. Parma, vol. 7, 1956, pp. 149–185; C. J. Neugebauer, Trans. Amer. Math. Soc., vol. 88, 1958, pp. 121–136). Any two fine-cyclic elements σ of M are not overlapping (they may have in common at most finitely many points), and the collection $\{\sigma\}$ of all fine-cyclic elements σ of M is at most countable. The illustration given here (Fig. 9) shows the decomposition Γ and the corresponding middle space M of a mapping (T, A) from an annular region ($\nu=1$). Here M constitutes a unique cyclic element $\Sigma=M$, while M contains two fine-cyclic

elements (two discs) joined by two threads. In the remaining illustrations (Figs. 10, 11, 12) other examples are given of decompositions Γ and corresponding middle spaces for mappings (T, A) from an annular region $A(\nu=1)$. For some only the middle space is given.

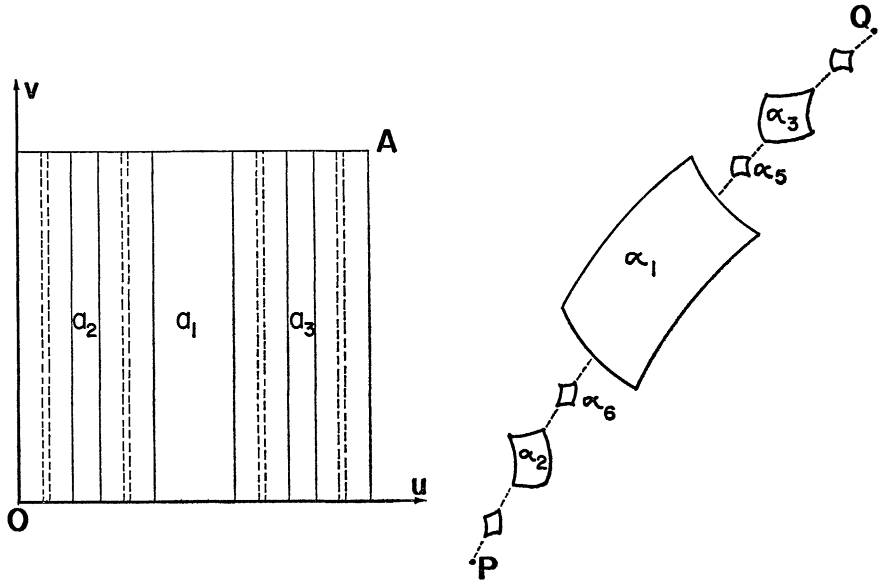

FIG. 6

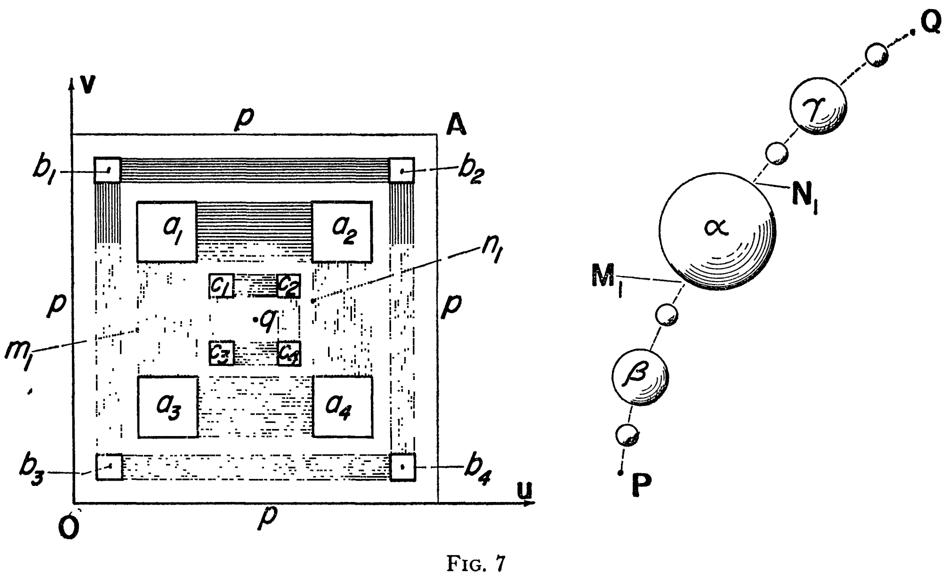

FIG. 7

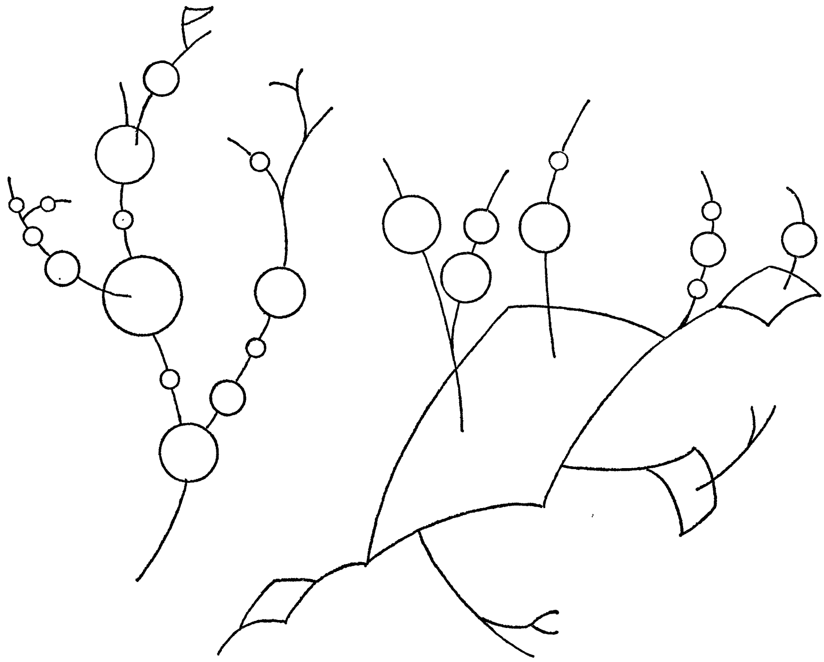

FIG. 8

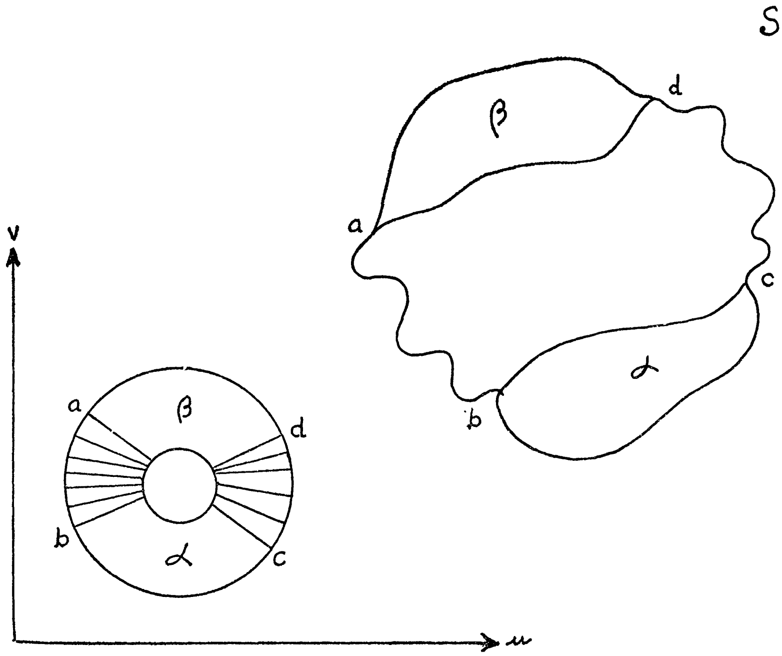

FIG. 9

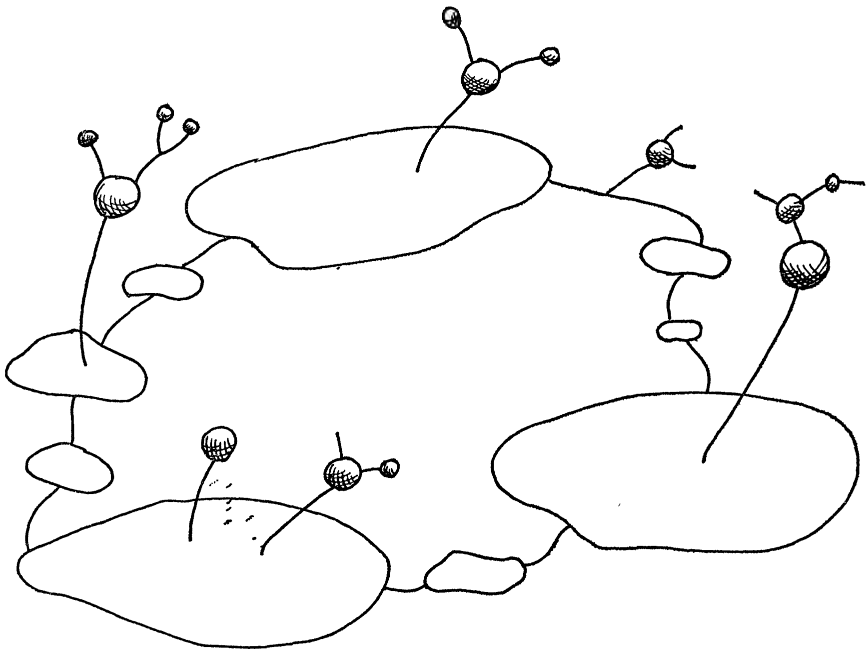

FIG. 12

On the general topic of the present article, the reader may consult [SA] and T. Rado, *Length and Area*, Amer. Math. Soc. Colloq. Publ., vol. 30, 1948.

ON THE SOLUTION OF THE DIFFERENTIAL EQUATION $f(x, y, y')=0$

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In this paper a procedure is given for applying the method of successive approximations to the solution of a differential equation of the type $f(x, y, y')=0$, without solving it explicitly for y'. The theorem below asserts the existence and uniqueness of the solution under conditions weaker than those usually imposed. The theorem also contains the outline of the procedure for constructing the solution. In a corollary following the proof of the theorem, two appraisals of the remainder error at the nth stage of approximation are given; one in terms of the difference of the derivatives of the nth and $(n-1)$st approximating functions, the other in terms of the original data. The paper ends with an illustrative example.

THEOREM. *Let $f(x, y, z)$ be a continuous real-valued function defined on the closed region N determined by the relations*

$$|x-x_0| \leqq H_1, \qquad |y-y_0| \leqq H_2, \qquad |z-z_0| \leqq h_3,$$

where H_1, H_2, h_3 *are positive constants, and let there be three positive constants* D_1, D_2, D_3 *such that, for* $(x, y, z) \in N$,

$$(1) \qquad |f(x, y_2, z) - f(x, y_1, z)| \leqq D_1 |y_2 - y_1)|,$$

$$(2) \qquad |f(x_0, y_0, z_0)| < D_2 h_3,$$

and, if $z_1 \neq z_2$,

$$(3) \qquad D_2 \leqq \frac{f(x, y, z_2) - f(x, y, z_1)}{z_2 - z_1} \leqq D_3.$$

Then there exist two positive constants $h_1 \leqq H_1$ *and* $h_2 \leqq H_2$ *such that the differential equation* $f(x, y, y')=0$ *has a unique solution* $y = Y(x)$ *in the interval* $|x-x_0| \leqq h_1$, *with* $Y(x_0)=y_0$ *and such that in that interval,* $|Y(x)-y_0| \leqq h_2$, *and* $|Y'(x) - z_0| \leqq h_3$.

Furthermore, if we define k *to be any constant such that*

$$(4) \qquad 0 < k(h_3 D_3 + |f(x_0, y_0, z_0)|) < 2h_3,$$

and if we define

$$(5) \qquad F(x, y, z) = z - kf(x, y, z)$$

for $(x, y, z) \in N$, *then the function*

$$(6) \qquad Y_1(x) = y_0 + \int_{x_0}^{x} F(t, y_0, z_0)dt$$

as well as the function

$$(7) \qquad Y_{n+1}(x) = y_0 + \int_{x_0}^{x} F[t, Y_n(t), Y_n'(t)]dt,$$

for every $n \geqq 1$, *is well defined for* $|x-x_0| \leqq h_1$, *and*

$$(8) \qquad Y(x) = \lim_{n\to\infty} Y_n(x), \qquad |x - x_0| \leqq h_1.$$

We shall prove the theorem with the aid of the following three lemmas.

LEMMA 1. *If*

$$(9) \qquad B = \max(|1 - kD_3|, |1 - kD_2|) \quad \text{and} \quad A = kD_1,$$

then

$$(10) \qquad 0 \leqq B < 1,$$

$$(11) \qquad (1 - B)h_3 - k|f(x_0, y_0, z_0)| > 0,$$

and, for $(x, y, z) \in N$,

$$|F(x, y_2, z_2) - F(x, y_1, z_1)| \leqq A|y_2 - y_1| + B|z_2 - z_1|. \tag{12}$$

Proof. From (4) we see that $0<kD_3<2$ and since, by (3), $D_2 \leqq D_3$, it follows that both $1-kD_3$ and $1-kD_2$ are less than 1 in absolute value. Hence B, as defined in (9), satisfies (10).

To prove (11), we first note that, since $D_2 \leqq D_3$, the only possible values for B are $1-kD_2$ and kD_3-1. If $B=1-kD_2$, (11) follows from (2). If $B=kD_3-1$, then from (4) we have $k|f(x_0, y_0, z_0)| < h_3(2-kD_3) = h_3(1-B)$, so that again (11) is satisfied.

We now turn to the proof of (12). If (x, y, z_1) and (x, y, z_2) are two distinct points of N, then, since $k>0$, we infer from (3) that

$$1 - kD_3 \leqq 1 - k\frac{f(x, y, z_2) - f(x, y, z_1)}{z_2 - z_1} \leqq 1 - kD_2. \tag{13}$$

From (13) and (9) we obtain for $z_1 \neq z_2$

$$|(z_2 - z_1) - k[f(x, y, z_2) - f(x, y, z_1)]| \leqq B|z_2 - z_1|, \tag{14}$$

a relation which is true also when $z_1 = z_2$.

Now, for $(x, y, z) \in N$,

$$\begin{aligned} F(x, y_2, z_2) - F(x, y_1, z_1) = {} & F(x, y_2, z_2) - F(x, y_1, z_2) \\ & + F(x, y_1, z_2) - F(x, y_1, z_1), \end{aligned} \tag{15}$$

which, by (5), is equal to

$$\{-k[f(x, y_2, z_2) - f(x, y_1, z_2)]\} + \{(z_2 - z_1) - k[f(x, y_1, z_2) - f(x, y_1, z_1)]\}.$$

Relation (12) is now seen to follow from (15), (1), (9) and (14). This completes the proof of Lemma 1.

LEMMA 2. *There exist two positive constants* $h_1 \leqq H_1$ *and* $h_2 \leqq H_2$ *such that*

$$0 < Ah_1 + B < 1, \tag{16}$$

$$h_1(|z_0| + h_3) \leqq h_2, \tag{17}$$

$$|F(x, y, z) - z_0| \leqq h_3, \tag{18}$$

for every point of the closed region $N' \subset N$ *defined by*

$$|x - x_0| \leqq h_1, \qquad |y - y_0| \leqq h_2, \qquad |z - z_0| \leqq h_3. \tag{19}$$

Proof. In view of (11) and the continuity of $F(x, y, z)$ we see that positive constants $h_1 \leqq H_1$ and $h_2 \leqq H_2$ exist such that for $|x-x_0| \leqq h_1$, $|y-y_0| \leqq h_2$,

$$|F(x, y, z_0) - F(x_0, y_0, z_0)| \leqq (1 - B)h_3 - k|f(x_0, y_0, z_0)|. \tag{20}$$

It is clear from (10) that h_1 can then be decreased in value, if necessary, so that

(16) and (17) are satisfied, and that (20) will remain valid. Further, if (x, y, z) satisfies (19), from the obvious inequality

$$\begin{aligned} | F(x, y, z) - z_0 | \leqq & \ | F(x, y, z) - F(x, y, z_0) | \\ & + | F(x, y, z_0) - F(x_0, y_0, z_0) | + | F(x_0, y_0, z_0) - z_0 |, \end{aligned}$$

and from (12), (19), (20) and (5), we infer that

$$| F(x, y, z) - z_0 | \leqq Bh_3 + (1 - B)h_3 = h_3.$$

Hence (18) is satisfied, and Lemma 2 is proved.

LEMMA 3. *If $U(x)$ and $V(x)$ are of class C^1 on $|x-x_0| \leqq h_1$, with*

$$U(x_0) = y_0, \qquad | U(x) - y_0 | \leqq h_2, \qquad | U'(x) - z_0 | \leqq h_3, \tag{21}$$

$$V(x_0) = y_0, \qquad | V(x) - y_0 | \leqq h_2, \qquad | V'(x) - z_0 | \leqq h_3, \tag{22}$$

then

$$\begin{aligned} & | F[x, U(x), U'(x)] - F[x, V(x), V'(x)] | \\ & \qquad \leqq (A | x - x_0 | + B) \left(\max_{t \in [x, x_0]} | U'(t) - V'(t) | \right). \end{aligned} \tag{23}$$

Proof. From (12) we infer that for $|x-x_0| \leqq h_1$,

$$\begin{aligned} & | F[x, U(x), U'(x)] - F[x, V(x), V'(x)] | \\ & \qquad \leqq A | U(x) - V(x) | + B \left(\max_{t \in [x, x_0]} | U'(t) - V'(t) | \right). \end{aligned} \tag{24}$$

Since $U(x_0) = V(x_0)$ we have, obviously,

$$U(x) - V(x) = \int_{x_0}^{x} [U'(t) - V'(t)] dt,$$

which, in view of (24), yields (23). Hence Lemma 3 is valid.

With the aid of the lemmas proved above, we can now prove the theorem. As a first step in establishing the existence of a solution, we introduce (6) and (7), and shall show that for $n \geqq 1$ and for $|x-x_0| \leqq h_1$, $Y_n(x)$ is well defined and of class C^1 and that

$$Y_n(x_0) = y_0, \qquad | Y_n(x) - y_0 | \leqq h_2, \qquad | Y_n'(x) - z_0 | \leqq h_3. \tag{25}$$

From (6) we infer that for $|x-x_0| \leqq h_1$, $Y_1(x)$ is well defined and of class C^1 and that $Y_1(x_0) = y_0$. Moreover from (6), (18) and (17) we see that for $|x-x_0| \leqq h_1$,

$$| Y_1(x) - y_0 | \leqq h_1(| z_0 | + h_3) \leqq h_2,$$

$$| Y_1'(x) - z_0 | = | F(x, y_0, z_0) - z_0 | \leqq h_3,$$

which implies the truth of (25) for $n=1$. The inductive proof for $n=m+1$, with $Y_m(x)$ assumed to be of class C^1 for $|x-x_0|\leqq h_1$, and properties (25) assumed for $n=m$, is exactly like that just given for $n=1$, but using (7) instead of (6). Therefore we conclude that for $|x-x_0|\leqq h_1$, $Y_n(x)$ is well defined and of class C^1, and that the relations (25) are true, for all $n\geqq 1$.

As the next step in the existence proof we shall show that, for $n\geqq 2$ and $|x-x_0|\leqq h_1$,

$$\max_{t\in[x,x_0]} |Y_n'(t) - Y_{n-1}'(t)| \leqq (Ah_2+Bh_3)(A|x-x_0|+B)^{n-2}, \tag{26}$$

$$|Y_n(x) - Y_{n-1}(x)| \leqq |x-x_0|(Ah_2+Bh_3)(A|x-x_0|+B)^{n-2}. \tag{27}$$

From (6) and (7), by differentiation with respect to x, we obtain

$$|Y_2'(x) - Y_1'(x)| = |F[x, Y_1(x), Y_1'(x)] - F(x, y_0, z_0)|,$$

the right member of which, in view of (12) and (25), is $\leqq Ah_2+Bh_3$. Hence (26) is true for $n=2$. Assuming (26) true for $n=m$, we obtain from (7), for $|x-x_0|\leqq h_1$, $|Y_{m+1}'(x)-Y_m'(x)|=|F[x, Y_m(x), Y_m'(x)]-F[x, Y_{m-1}(x), Y_{m-1}'(x)]|$, and comparing (21) and (22) with (25), we thus infer from Lemma 3 that

$$|Y_{m+1}'(x) - Y_m'(x)| \leqq (A|x-x_0|+B)\left(\max_{t\in[x,x_0]}|Y_m'(t) - Y_{m-1}'(t)|\right), \tag{28}$$

which, with (26) for $n=m$, implies

$$|Y_{m+1}'(x) - Y_m'(x)| \leqq (Ah_2+Bh_3)(A|x-x_0|+B)^{m-1}.$$

Since the right member of this inequality is a monotone increasing function of $|x-x_0|$, we infer that (26) is true with $n=m+1$. Hence (26) is valid on $|x-x_0|\leqq h_1$ for $n\geqq 2$. Since $Y_n(x_0)=Y_{n-1}(x_0)$, relation (27) is obtained easily from (26) by integration of $Y_n'(t)-Y_{n-1}'(t)$ with respect to t from x_0 to x.

With (26) and (27) thus proved, we are in a position to complete the existence proof. From (16), (27) and (26) we see that the sequences $\{Y_n(x)\}$ and $\{Y_n'(x)\}$ converge uniformly on $|x-x_0|\leqq h_1$. Letting, as in (8), $Y(x)=\lim_{n\to\infty} Y_n(x)$, we have by a well-known theorem,

$$Y'(x) = \lim_{n\to\infty} Y_n'(x).$$

Since $Y_n(x)$ satisfies (25) we thus have

$$Y(x_0) = y_0, \qquad |Y(x) - y_0| \leqq h_2, \qquad |Y'(x) - z_0| \leqq h_3. \tag{29}$$

By differentiating each side of (7) with respect to x and letting $n\to\infty$, we obtain, in view of the continuity of $F(x, y, z)$,

$$Y'(x) \equiv F[x, Y(x), Y'(x)]. \tag{30}$$

From (5) we then infer that

$$f[x, Y(x), Y'(x)] \equiv 0, \tag{31}$$

so that $y = Y(x)$ is a solution of the differential equation $f(x, y, y')=0$, valid on $|x-x_0| \leqq h_1$ and satisfying (29).

To prove the uniqueness, let $U(x)$ be any function of class C^1 on $|x-x_0| \leqq h_1$ satisfying (21) and such that $f[x, U(x), U'(x)] \equiv 0$. Then by (5)

$$U'(x) \equiv F[x, U(x), U'(x)]. \tag{32}$$

In view of (30) and (32) we obtain from Lemma 3 the relation

$$|U'(x) - Y'(x)| \leqq (A|x - x_0| + B)\left(\max_{t\in[x,x_0]} |U'(t) - Y'(t)|\right)$$

and hence

$$\max_{|x-x_0|\leqq h_1} |U'(x) - Y'(x)| \leqq (Ah_1 + B)\left(\max_{|x-x_0|\leqq h_1} |U'(x) - Y'(x)|\right).$$

Since $0 < Ah_1 + B < 1$ we infer that $U'(x) - Y'(x) \equiv 0$, and since $U(x_0) = Y(x_0)$, we have $U(x) \equiv Y(x)$. Hence $Y(x)$ is the only function of class C^1 on $|x-x_0| \leqq h_1$ which satisfies (29) and (31). This completes the proof of the theorem.

We now give two appraisals of the remainder error.

COROLLARY. *With* $|x-x_0| \leqq h_1$,

$$|Y_n(x) - Y(x)| \leqq \frac{|x - x_0|(A|x - x_0| + B)}{1 - A|x - x_0| - B}\left(\max_{t\in[x,x_0]} |Y_n'(t) - Y_{n-1}'(t)|\right), \tag{33}$$

$$|Y_n(x) - Y(x)| \leqq \frac{|x - x_0|(Ah_2 + Bh_3)(A|x - x_0| + B)^{n-1}}{1 - A|x - x_0| - B}, \tag{34}$$

where A and B are given by (9).

Proof. Since, for $|\xi-x_0| \leqq h_1$,

$$Y'(\xi) - Y_n'(\xi) = [Y_{n+1}'(\xi) - Y_n'(\xi)] + [Y_{n+2}'(\xi) - Y_{n+1}'(\xi)] + \cdots,$$

and since the right member of (28) is a monotone increasing function of $|x-x_0|$, we infer from (28) and the formula for the sum of a geometric series, the relation

$$|Y'(\xi) - Y_n'(\xi)| \leqq \frac{A|\xi - x_0| + B}{1 - A|\xi - x_0| - B}\left[\max_{t\in[\xi,x_0]} |Y_n'(t) - Y_{n-1}'(t)|\right].$$

Since $Y(x_0) = Y_n(x_0)$, (33) then follows at once by integration of $Y'(\xi) - Y_n'(\xi)$ with respect to ξ from x_0 to x. Relation (34) follows easily from (26) and (33).

Remark. If the hypotheses of the theorem are satisfied except that (3) is replaced by a similar relation with D_2 and D_3 both negative, we can obtain a relation of the form (3) by changing the sign of $f(x, y, z)$. Furthermore, a relation of the form (3) can always be obtained if f is of class C^1 and $\partial f(x_0, y_0, z_0)/\partial z \neq 0$.

Example. Consider the problem of finding a solution of the differential equation

$$f(x, y, y') \equiv -(y')^2 + y' + 4y - 2x - \frac{7}{64} = 0 \tag{35}$$

passing through the point (0, 0). Here $f(x, y, z) = -2x+4y-z^2+z-a$, where for convenience we have written a for 7/64. We take the closed region N as defined by $|x| \leqq H_1 = 1/384$, $|y| \leqq H_2 = 1/384$, $|z| \leqq h_3 = 1/4$, so that $x_0 = y_0 = z_0 = 0$. By taking $D_1 = 4$, $D_2 = 1/2$, $D_3 = 3/2$, and $k=1$, the inequalities (1), (2), (3), and (4) will be satisfied.

By (5) we have $F(x, y, z) = 2x - 4y + z^2 + a$. With $h_1 = H_1$ and $h_2 = H_2$ we verify that (16), (17), and (20) are satisfied so that $N' = N$. Since $F(t, 0, 0) = 2t + a$, by (6) we find

$$Y_1(x) = \int_0^x (2t + a)dt = x^2 + ax$$

and $Y_1'(x) = 2x + a$. Next we find $Y_2'(x) = F(x, Y_1, Y_1') = 2x + a^2 + a$ and consequently

$$Y_2(x) = \int_0^x (2t + a^2 + a)dt = x^2 + (a^2 + a)x.$$

Proceeding by induction, we obtain

$$Y_n(x) = x^2 + [(\cdots ((a^2 + a)^2 + a)^2 \cdots + a)^2 + a]x.$$

Remembering that $a = 7/64$, it is easy to verify that, as $n \to \infty$, the coefficient of x in the above expression for $Y_n(x)$ tends to 1/8, and therefore $Y(x) = x^2 + x/8$ is the unique solution of the differential equation $(y')^2 - y' - 4y + 2x + 7/64 = 0$ in the interval $|x| \leqq 1/384$, satisfying $Y(0) = 0$ and such that in that interval $|Y(x)| \leqq 1/384$ and $|Y'(x)| \leqq 1/4$.

Discussion. Setting $x = y = 0$ in (35) and solving for y', we obtain the values $y' = 1/8$ and $y' = 7/8$, indicating the possibility of two solutions through the point (0, 0) with different slopes at that point. It would have been natural, in applying our iteration method, to choose $z_0 = 1/8$ to get the solution for which $y' = 1/8$ at (0, 0). However, in our illustration we have deliberately avoided $z_0 = 1/8$ and instead have taken $z_0 = 0$ in order better to illustrate our method. The same solution could have been obtained by choosing any other value for z_0 sufficiently close to 1/8.

Similarly (with appropriate changes in the domain of definition of $F(x, y, z)$), if we had chosen $z_0=7/8$ (or a value sufficiently close to $7/8$) we would have obtained the other solution, $y=x^2+7x/8$.

Reference

T. H. Hildebrandt and Lawrence M. Graves, Implicit functions and their differentials in general analysis, Trans. Amer. Math. Soc., vol. 29, 1927, pp. 127–153 (for related ideas in a more general setting).

PROUHET'S 1851 SOLUTION OF THE TARRY-ESCOTT PROBLEM OF 1910

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In what follows all small latin letters denote rational integers. We take $k \geqq 1$ and $j \geqq 2$ and consider the $k(j-1)$ simultaneous equations

$$\sum_{i=1}^{s} a_{i1}^{h} = \sum_{i=1}^{s} a_{i2}^{h} = \cdots = \sum_{i=1}^{s} a_{ij}^{h} \qquad (1 \leqq h \leqq k). \tag{1}$$

A solution of the equations is called nontrivial if no set $\{a_{iu}\}$ $(1 \leqq i \leqq s)$ is a permutation of another set $\{a_{iv}\}$. We may consider (i) the problem of determining $P(k, j)$, the least value of s for which a nontrivial solution of (1) exists and (ii) that of finding an actual solution of (1) for given k, j and s. Any solution of the second problem provides an upper bound for $P(k, j)$ and so may contribute towards a solution of the first. It is known [**1**] that $P(k, j) \geqq k+1$ and [**7**] that

$$P(k,j) \leqq \tfrac{1}{2}(k^2+3) \ (k \text{ odd}), \qquad P(k,j) \leqq \tfrac{1}{2}(k^2+k+2) \ (k \text{ even}).$$

These upper bounds were obtained by an enumerative argument and not by finding an actual solution. From actual solutions, it is known [**2, 3**] that

$$P(k, 2) = k+1 \ (2 \leqq k \leqq 9), \qquad P(k,j) = k+1 \ (k = 2, 3, 5).$$

The theorem that $P(k, 2)$ exists and that $P(k, 2) \leqq 2^k$ is usually ascribed to Tarry and Escott, who published papers on the subject *circa* 1910–12 (see [**2**]). In fact, Prouhet in 1851 gave a solution of (1) for general k and j with $s=j^k$ and so showed that $P(k, j) \leqq j^k$. It is, of course, notorious that the names given to many mathematical problems and theorems are not those of their discoverers. But Prouhet seems to have been particularly unlucky. He discovered the first general result in this field in 1851. Although sufficient papers were written on the topic to oblige Dickson in 1920 to devote to it a whole chapter of his History [**2**], Prouhet's result was not completely rediscovered until 1948 [**4**]. But the whole subject is usually given the name of two mathematicians who rediscovered

the special case $j=2$ some 60 years after Prouhet discovered the general case. Of course, both made other contributions to the subject.

Prouhet's note [5] is no more than an "Extrait par l'auteur" of a "Mémoire présenté" to the Academy. The secretaire-archiviste of the Academy was kind enough to inform me that the memoir was returned to the author in 1852. But there appears to be no trace in the literature of its publication. The extract gives a correct rule for constructing a solution to (1) from any consecutive j^{k+1} members of an arithmetic progression. If we ignore the trivial generalisation to a progression, the following is equivalent to Prouhet's result:

Express each n $(0 \leqq n \leqq j^{k+1}-1)$ as a "decimal" in the scale of j. If the least positive residue to modulus j of the sum of the digits of n in this scale is v, assign n to the set S_v. Then each of the sets S_v contains just j^k members, which may be taken as $a_{1v}, a_{2v}, \cdots$ to satisfy (1) *with $s=j^k$.*

Prouhet did not publish a proof. But the achievement lies in finding the result and it is difficult to see how it could be found except by proof. Certainly the more general parametric result found independently by Lehmer [4] in 1948 is easy to prove (see, for example [8]) once one knows the result.

The reference to Prouhet in [2] gives his numerical example for $k=2, j=3$ and then goes on:—"As a generalisation, it is stated that there are numbers separable into j sets each of j^k terms, such that the sum of the hth powers of the terms is the same for all the sets when $h \leqq k$." This understatement appears to have misled subsequent writers.

The ideas of a recent article in this MONTHLY [6] can be extended to give a direct proof of Prouhet's result. As in [6], for any nonnegative integer b, we write $E(b)$ for the operator which maps any polynomial $\phi(x)$ into $\phi(x+b)$. The $E(b)$ form a subset of a commutative ring with unity; also $E(a)E(b)=E(a+b)$. For any positive integer n, let $v=v(n)$ be the least positive residue (mod j) of the sum of the digits of n, when n is expressed in the scale of j. Let ρ be any jth root of unity other than 1. If we write

$$\Lambda_t = 1 + \rho E(j^t) + \rho^2 E(2j^t) + \cdots + \rho^{j-i} E(j^t(j-1)),$$

we have

$$\prod_{t=0}^{k} \Lambda_t = \sum_{n=0}^{j^{k+1}-1} \rho^{v(n)} E(n). \tag{2}$$

If $\phi(x)$ is a polynomial whose term of highest degree is αx^m, the coefficient of x^m in $\Lambda_t \phi(x)$ is

$$\alpha(1 + \rho + \rho^2 + \cdots + \rho^{j-1}) = 0.$$

Hence Λ_t reduces the degree of any polynomial by one and the left-hand side of (2) maps any polynomial of degree not greater than k onto 0. Thus, if we operate with (1) on x^h $(0 \leqq h \leqq k)$ and put $x=0$, we have

$$\sum_{n=0}^{j^{k+1}-1} \rho^{v(n)} n^h = 0, \text{ i.e., } \sum_{v=1}^{j} \rho^v H_v = 0, \tag{3}$$

where $0^0=1$, $H_v=\sum a_{iv}^h$, and $\{a_{iv}\}$ is Prouhet's set S_v of those n which satisfy $0\leqq n\leqq j^{k+1}-1$, $v(n)=v$.

If we write $G_v=H_v-H_j$ $(1\leqq v\leqq j-1)$ and observe that $\rho+\rho^2+\cdots+\rho^j=0$, we have

$$\sum_{v=1}^{j-1} \rho^v G_v = 0 \tag{4}$$

by (3). If τ is a primitive jth root of unity, (4) is true for $\rho=\tau^m$ $(1\leqq m\leqq j-1)$ and so we have

$$\sum_{v=1}^{j-1} \tau^{mv} G_v = 0 \qquad (1 \leqq m \leqq j-1).$$

The determinant of the coefficients of the G_v in these equations is of Vandermonde type in the distinct numbers $\tau, \tau^2, \cdots, \tau^{j-1}$. Hence this determinant does not vanish and $G_1=G_2=\cdots=G_{j-1}=0$. This is Prouhet's result.

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A LOOK AT MATHEMATICAL COMPETITIONS

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1. Introduction. On a Thursday morning last spring, some 80,000 high school students in the United States and Canada sat down to match themselves against the First National Contest in High School Mathematics, sponsored by the Mathematical Association of America and the Society of Actuaries. Such a contest is not without precedent. In some countries, such competitions have a long tradition. One need only recall, for example, the Eötvös Competition held annually in Hungary from 1894 until 1928, and the extensive system of Mathematical Olympiads in the Soviet Union. In the United States, there have been many contests that operate on a regional basis, some sponsored by an individual

college (*e.g.*, Stanford, Mt. Mary), others growing out of the activities of a particular section of the M.A.A. (*e.g.*, Metropolitan New York, Maryland-D.C.-Virginia, Wisconsin).

The present paper is chiefly a report of the experience of the Wisconsin Section, which initiated an annual high school contest in 1956. It seemed appropriate to preface this with a discussion of some of the basic principles underlying mathematical competitions, and to include some data on other contests, here and abroad. The latter is not meant to be an exhaustive survey; I have selected only a few representative contests with an eye to variety and interest.

At the heart of the expanded program of the Association is the need to reach out to students in our high schools and colleges, and bring them a glimpse of the excitement and power of mathematics. It is my belief that mathematical competitions provide one such avenue. May I hope that we will see a wider support for the contest program of the M.A.A. among those mathematicians whose central interest lies in research? This was certainly the case in Hungary; it is presently the case in the USSR. A well-designed and exploited competition, supported both by the M.A.A. and other interested organizations, could have a beneficial effect upon the mathematical atmosphere in our secondary schools. The number of participants should not be the sole criterion of success; quality must not be sacrificed to quantity, nor must awards go only to those students who show ability to handle routine calculations with fantastic speed. We are looking for creative minds, and they are difficult to trap with the best of weapons.

2. Basic principles. In the past, many arguments have been offered in support of high school contests in mathematics. The following seem to be most representative:

(a) *To give official recognition by awards, presented in the name of the M.A.A., to some of the better students now taking mathematics in high schools.*

Most of those concerned with the administration of contests have found that teachers and students alike welcome the prestige value of awards for excellence in mathematics. Cash prizes, books, pins, certificates, all are effective.

(b) *To discover and encourage talented students who might otherwise escape attention.*

(c) *To lend additional motivation for some students to take more mathematics in high school.*

(d) *To encourage some students, especially those who are highly gifted and whose talents extend in many directions, to consider mathematics as a career.*

There is some indication that a properly designed examination, carrying the authority of official sponsorship by a national mathematical organization, can

in fact stimulate the imagination of students and show them unexpected directions of mathematical study. It is also possible that a hitherto unspectacular student may show surprising insight under the prod of a contest; an alert teacher may then be able to direct his energies toward higher goals.

(e) *To give a certain amount of tactful guidance to the high school curriculum by indicating the level of competence and maturity of viewpoint that can be expected from the better students.*

It is well known that the College Board examinations have influenced high school curricula; that this responsibility has been fully realized is made clear by the creation and activity of the Commission on Mathematics, under Dean Meder. A successful National Mathematics Contest may also exert considerable force; accordingly, great care must be given to the selection of questions.

In addition to these reasons, we have recently seen a further endorsement; the following is a quotation from a report of the Joint Congressional Committee on Atomic Energy: [1]

"*Listed below are a few of the major recommendations made to the committee that illustrate the types of approach which might assist in meeting these requirements:*

(1) *Establishment of a Federal mathematics scholarship award program to provide substantial cash awards to any high school graduate who passes an examination, at college-entrance level, in mathematics. . . . Chief objective of the award would be to arouse greater interest in high school mathematics as an important part of general education and an indispensable base for college-level study in science and engineering. . . .*

(2) *Earlier identification of potentially ablest students. . . . In order to recoup this loss of potential talent, it is proposed that statewide testing start at the 8th or 9th grade level, rather than the 12th. . . . Increased emphasis should also be placed on intellectual rigor for the ablest students and challenging programs provided.*"

There is, of course, some objection to the whole idea of competitive mathematical contests. One objection that has been raised to all tests of this sort, including the College Board's, is that in coaching the ablest students for such a contest, a teacher will neglect the education of his less able students, who are the very ones who need him the most. Even for the better students, drilling on sample test questions may not be the best form of mathematics education. Others have felt that a contest that does not emphasize the individual contestant, and which pits school against school and teacher against teacher, will emerge in the end only as a device for rating teacher efficiency. It has also been stated that success, especially where a large cash prize is at stake, is harmful both to the winners and the losers. Finally, some see a possible threat to intellectual freedom in the creation of one central committee to administer such an

influential instrument as a National Contest, tied for example into a scholarship program, and would prefer to see a larger degree of local autonomy.

Many of these criticisms have considerable merit, and are difficult to answer. The responsibility for meeting them rests with the M.A.A. and its administrative committees.

Let us now see what the five basic objectives listed above suggest about the nature of a mathematical competition. (Not unnaturally, the end product will bear some resemblance to the Wisconsin Contest!) We must first decide the level to which the test is to be directed. Several reasonable possibilities are open, and a choice between them may be determined by local conditions. Should the test be designed so that only those students who have completed $3\frac{1}{2}$ years of high school mathematics stand a chance of winning? This might be entirely defensible in a large metropolitan area, but does not seem wise if we are to think of the country as a whole. What of the gifted student who has the misfortune to attend one of the 40% of our high schools offering only two years of mathematics? It would thus seem to run contrary to objectives (b) and (c). One alternative is to give the contest at several different grade levels. There might be one test for students in grades 9 and 10, and another for students in grades 11 and 12. This indeed is the procedure followed in the Mathematical Olympiads in the Soviet Union. (See Sec. 3.) If we do not want to resort to this complexity (again a factor that may be decided by local conditions) the best solution would seem to be a single test for all grade levels, based upon the first $2\frac{1}{2}$ years work, which would have questions of sufficient originality so that it would severely tax the average senior, but would still be within the reach of a very able sophomore. (Our experience in Wisconsin has shown that these two goals are not contradictory; in last year's contest, the top 20 winners of almost 10,000 entrants consisted of 5 sophomores, 6 juniors, and 9 seniors.)

This brings up another aspect of test design. Broadly speaking an examination can be designed to test either achievement or aptitude. The first can be characterized as a timed multiple choice test which attempts to measure a wide sampling of the basic skills and concepts that make up the subject matter to be covered. Its questions are of varying difficulty, and are drawn from the more or less standard subject matter of the appropriate grade levels. Speed of performance is an important factor. ETS has designed many very effective testing instruments of this type, such as the familiar College Board Examinations and the STEP series. The 1958 M.A.A. National Contest was also of this type. At the other extreme, we find the type of test represented by the Stanford-Sylvania Competition. Here, we have an emphasis upon originality and insight rather than routine competence. The student is confronted by a handful of questions, uniformly difficult, and allowed to puzzle over them for several hours. A typical question might call for specific knowledge within the reach of those being tested, but would call for the employment of this in unusual ways requiring a high degree of ingenuity. The question may in fact introduce certain concepts which are quite unfamiliar to the student. In short, the winning student is asked to dem-

onstrate research ability. The original model for this was the Eötvös contest in Hungary. (See Sec. 4 and [6].) Here, one additional feature was noteworthy. Each test included one question that contained within it the germ of a further generalization. The contestant who discovered this, who posed for himself the more general question, and then proceeded to investigate it, was given bonus points; it is especially significant when a student does this without the guidance of a specific command on the instruction sheet. In the examinations used in the Russian Olympiads, half of the questions are of this "perceptive" or aptitude type.

Both types of tests have worthwhile features, as well as drawbacks. It is probably impossible to administer a test of the perceptive type on a large scale. For one thing, it is probably not possible to design a multiple choice type question which will allow a contestant to demonstrate the depth of insight and originality that is required; this would then impose a severe grading problem for the committee in charge. It should be noted that the Eötvös contest involved only several hundred students, that the Stanford Competition has had at most 800 entrants, and that the Olympiads seldom run as high as 700 in any one district. Our experience in grading the Wisconsin Final (1958), which was a test of the perceptive type, shows that it is feasible to handle 1000 test papers, but that there must be rather elaborate scoring procedures to ensure fairness.

In 1956, when a statewide contest was initiated, these considerations led the Contest Committee of the Wisconsin Section to adopt a two-stage examination. Originally, this consisted of two separate tests administered to the contestants on the same day, the first being chiefly multiple choice questions of the achievement type, and the second a much smaller number of questions of the perceptive type. Later, due to the increased enrollment, the two tests were given on different days, and the results on the first used as a guide in recommending entrance to the Final. (For details, see Sec. 6.) Although we have made mistakes that a more experienced group would have avoided, we have been reasonably satisfied with the results of our experience. It is natural to ask if our system can be extended to a national system. In some respects, Wisconsin reflects in miniature the problems of the national picture; there is wide variation among both its rural and urban secondary schools, accompanied by equally wide variation in ethnic and cultural backgrounds. It is my own personal belief that the pattern we have used could be adopted successfully in other Sections of the M.A.A., and integrated into the current National Contest. A participating Section would assume full responsibility for devising, administering, and grading its Final Contest; it might do the same for the more extensive Preliminary Contest, or it might find it possible to make use of a nationally prepared test for this purpose. A detailed proposal of this sort was presented at the recent Washington Conference, and has been discussed by the Committee on Contests at the summer meeting at M.I.T. However, the responsibility for implementing this, or indeed any extension of the present National Contest, must rest with the local sections; no small central committee can carry the burden alone.

The tasks that face the mathematical community are many and varied. We are asked to join in the development and implementation of a long range program to revise elementary and secondary school curricula. As college instructors, we must also help the present and future teachers of the nation to meet the challenge which this revision poses by offering them a training program suited to their needs. The colleges are also called upon to produce trained mathematicians in ever increasing numbers, and on a wide range of levels of accomplishment; this in turn requires the development of new courses and texts, both on the undergraduate and graduate level. The necessary and sufficient conditions for attaining all of these are easily stated: (1) manpower (2) money. Ultimately, leadership for much of this undertaking may come from within the active sections of the M.A.A.; the present National Contest owes its existence to one such section, that of Metropolitan New York.

3. The Mathematical Olympiads of the U.S.S.R. In the last year, a considerable amount of reliable information has been published about the educational system of the Soviet Union, and in particular about its program for the mathematically gifted. (See especially the authoritative book by A. G. Korol [5], and the article by B. V. Gnedenko in the MONTHLY [3]). Although there are other equally interesting aspects to the Olympiad system, we shall concentrate only on the Contest. Described briefly, this is a two-stage competitive examination sponsored in a local district for its secondary school pupils by an Administrative Committee from the local University or Technical Institute; only those students completing the first round are admitted to the second, and prizes are awarded to the winners of the second round. The nature of the questions used seems to vary somewhat with the particular district, but more often than not are of the perceptive, rather than achievement type. Descriptions of the Olympiads, often accompanied by the test questions, are frequently reported in *Uspehi Matematicheski Nauk*, and for the interest of readers, I have included the following free translations of several of these from Volume 11; it is not yet clear whether the very recent changes in Soviet educational policy will alter the pattern which had been set, starting with the first Olympiad in Leningrad (1934).

The Mathematical Olympiads in Lvov during 1955 and 1956

A. S. KOVANKO

During the school year 1954/55, a Mathematical Olympiad was held in Lvov for school children of the 7th, 8th, 9th, and 10th grades, based on somewhat different principles from those used in previous years. The first round of the tournament was held in the schools and the second round at the University, as before; however, the contest questions in both rounds were more of a technical nature, and thus were not similar to those usually asked on Olympiad tests in previous years. The attendance was high. 800 students took part in the first round, and 375 were admitted to the second round. Of these, 90 won prizes or certificates.

During the school year 1955/56, an Olympiad was held on March 25 for the students in the city of Lvov, sponsored by the University and by the Society for the Promulgation of Political and Scientific Knowledge. The contest was given separately for students in the 8th, 9th, and 10th grades, and consisted of only one examination. There were 114 entrants, of whom 10 were selected

as winners. The best of these was a student in the 9th grade (school No. 35) B. L. Zgelskii. The group also had certain preparatory work to do in the form of lessons and exercises. The following are some of the more interesting problems.

8th grade

1. Solve the equation $\sqrt[3]{(8-x)^2}+\sqrt[3]{(27+x)^2}=\sqrt[3]{\{(8-x)(27+x)\}}+7$.

2. Through two given points on the circumference of a circle, draw two parallel chords having a specified sum.

9th grade

1. Solve the equation

$$\sin\left(\frac{\pi}{10}+\frac{3x}{2}\right)=2\sin\left(\frac{3\pi}{10}-\frac{x}{2}\right)$$

2. Given an equilateral triangle ABC, mark a point A_1 which is $\frac{1}{3}$ of the way from B toward C, a point B_1, $\frac{1}{3}$ of the way from C toward A, and a point C_1, similarly placed on side AB. Show that the lines AA_1, BB_1, CC_1 form the sides of a triangle KLM whose area is exactly $\frac{1}{7}$ of the area of triangle ABC.

3. Take 121 terms of each of the arithmetic progressions $2, 7, 12, \cdots$ and $2, 5, 8, \cdots$. How many numbers will there be in common?

10th grade

1. Prove that

$$\sin x+\sin 2x+\cdots+\sin nx=\frac{\sin\frac{1}{2}nx\sin\frac{1}{2}(n+1)x}{\sin\frac{1}{2}x}\,.$$

2. Solve the equation $4^x-(13)6^{x-1}+9^x=0$.

3. For a cone with altitude 12 and base of radius 4, find the inscribed cylinder of maximum surface area. (Uspehi Mat. N. 11(1956) 255–256)

The Fourth Mathematical Olympiad at Ordžonikidze

F. S. CHURIKOV

The Olympiad was held during the school vacation of the school year 1955/56, for children in the 8th, 9th, and 10th grades. It was arranged by the North Osetinsk State Pedagogical Institute and with the assistance of the City Department of National Education. Preparation for the Olympiad was conducted in the Mathematics Circles of the Schools . . . [*Here, there follows a description of the supplementary lectures that often form a feature of the Olympiads.*] The first round of the Tournament took place on March 28. As in previous years, it was conducted in the schools, and supervised by teachers who also marked the papers. The tests, which had been prepared by the Organizing Committee, were sent to the schools on the eve of the first competition. It was held in 15 schools within the city, and 2 country schools. The second round was held on April 15 in the Institute, and was supervised by the Organizing Committee. Students were admitted to the second round who had scored at least 12 of 22 points on the first round test. The numbers taking part are given below:

	8th grade	9th grade	10th grade
round 1	36	53	92
round 2	7	15	49

Prizes were awarded to five students. In addition, six students were awarded honorable mention. [*At this point, Professor Churikov gives what seems to be the complete set of examinations used in the*

contest; I have selected a few representative questions from each part to show the range of difficulty. At each grade level, the tests in both rounds consisted of 5 questions apiece.]

8th grade, First Round

2. Construct the graph of $y=1/(x^2-4)$.
4. Given a circle, and two radii, construct a chord that is trisected by the radii.

8th grade, Second Round

1. What must be the values of p and q if the roots of $x^2+px+q=0$ satisfy the two equations

$$x_1 - 2x_2 = 2, \; 2x_1 - 3x_2 = 5?$$

5. What is the value of the sum

$$\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{99\cdot 100}?$$

9th grade, First Round

2. Construct the graph of $y=1/(\sqrt{x}-1)$.
4. Find the sum of the numbers

$$1 + 11 + 111 + \cdots + \underbrace{111\cdots 111}_{n \text{ times}}.$$

9th grade, Second Round

1. Let a, b, and c be consecutive terms of an arithmetic progression. Show that this is also true of the numbers a^2+ab+b^2, b^2+bc+c^2, and c^2+ac+a^2.

3. Show that $(\log 2)(\log 5)=.2104$, without using tables to find log 2 or log 5.

10th grade, First Round

1. Using the method of mathematical induction, prove that

$$(1!)1 + (2!)2 + (3!)3 + \cdots + (n!)n = (n+1)! - 1.$$

3. Through vertex A of the triangle ABC construct the angle bisector, the median, and the altitude. If these lines divide the angle A into four equal parts, prove that A is a right angle.

10th grade, Second Round

2. Solve the system of equations

$$\begin{aligned}(x+y+z)(ax+y+z) &= k^2,\\ (x+y+z)(x+ay+z) &= l^2,\\ (x+y+z)(x+y+az) &= m^2.\end{aligned}$$

3. Find the maximum and minimum values of the function $y=2\sin x-\cos 2x$.

(Uspehi Mat. N. v. 11 (1956) 251–253)

Before leaving this subject, I should like to add two remarks. Because of the objectivity of a multiple choice test, there can be valid comparisons; "passing grade is 70%" has an unambiguous meaning. However, in evaluating the effectiveness of a test such as those described above, variation can arise because of different standards of success. In particular, it would be interesting to have some valid measure of the difficulty which these questions have for Soviet students; comparative test scores or other data have not, so far as I know, been

published. (In this connection, Dr. Robert Kalin of Florida State University has made some tentative studies of the entrance examinations for admission to the mathematics department of Moscow University [4]). The second remark bears on enrollments. Using the published figures for admission to the first round, one finds totals ranging from less than 200 in smaller districts such as Ivanova and Ordžonikidze to over 1000 in Moscow and Leningrad. From the relatively small size of these, compared with estimated school enrollments, it is likely that there is considerable preselection of contestants.

4. The Eötvös prize competition in Hungary. The most complete description of this mathematical contest is to be found in the book by Jozsef Kurschak, published the year after its termination in 1928. This contains problem lists, representative solutions by contestants, and names of the winners [6]. More accessible, perhaps, is the article by Rádo published in the MONTHLY in 1932 [9]. This contest, held annually for over thirty years, was open to high school graduates who were enrolled in their first semester in college. It was held in October, and administered by college faculty locally. Each test consisted of three questions, usually selected from the areas of arithmetic, algebra and geometry, and the students were allowed to use books for reference; stress was thus laid on innate ability and insight rather than upon memorization and speed. As mentioned above, additional credit was given to those who were able to go further than the posed question. It is interesting to speculate upon the relationship between this contest and the mathematical fertility of Hungary. Among the winners of the Eötvös prize, one might mention Fejér, Karman, Haar, Pólya, Riesz, Szegö, Rádo, Rédei, and others. The following examples, taken from the Kurschak book, will illustrate the type of problem used.

1. Given a right triangle ABC, with right angle at C. Is there at least one point P inside the triangle such that $\angle PAB = \angle PBC = PCA$?

2. Find all possible whole numbers x, y, and z for which

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$$

is a whole number.

3. Suppose that x and y are whole numbers such that $2x+3y$ is a multiple of 17. Show that $9x+5y$ is also a multiple of 17.

4. Show that any closed curve that encloses a triangle must be longer than the perimeter of the triangle.

5. In a triangle ABC, show that $\sin(A/2)\sin(B/2)\sin(C/2)\leq 1/4$.

5. The Stanford University competitive examination. Now sponsored jointly with Sylvania Corporation, this contest has been held annually for a decade. Under the guidance of Professor Pólya, this has followed closely the pattern of the Eötvös Prize, except that it is given in the spring while students are still in high school. The questions used are of striking originality, and have always been a definite challenge to the students; only a handful of entrants produce

solutions worth examining in detail, and from these, the top dozen can be selected as winners. The test questions, accompanied by a discussion of possible solutions by Professor Pólya, have been published each year [8]. The number of entrants has risen slowly, reaching 836 in 1958. The following sample question might be considered somewhat typical.

In any triangle, the sum of the . . . is greater than the semi perimeter. Replace the dots . . . successively by (I) altitudes (II) medians (III) angle bisectors. You obtain three different assertions. Examine each: is it true or false? Prove your answer.

6. The Wisconsin section M.A.A. contest. The following description applies to the 3rd annual contest (1958). The competitive examination was separated into two stages, a Preliminary Contest held in February, and a Final Contest held in April. The former consisted of a multiple choice test, conducted in the participating high schools, and proctored and scored by the teachers. There were 18 questions of varying difficulty, based on three semesters' work. 9516 contestants from 237 schools participated. Certificates were awarded to the top scorer at each school. The complete cost of this stage was more than covered by the charge of \$1.00 per school, and \$.05 per contestant. The score distributions were reported to the central committee who determined a recommended entrance score for admission to the Final Contest; this figure was arbitrarily set to admit about 1000 to the Final. Contestants reported to 27 centers scattered over the state where they took the second stage test; this was a nonobjective perceptive type test, modeled on that in the Eötvös Prize Contest. It was proctored by college personnel, and the papers sent to Madison where they were scored by a team of graduate students, and checked by members of the Committee; 916 students entered the Final, from which 38 were selected as first, second or third prize winners, and 100 as honorable mention. The cost of the contest and prizes (\$20 for first place, together with an Association pin, and a book set for the winner's high school, lesser cash awards for second and third prize winners) was covered completely by the fee of \$1.00 charged each entrant in the Final. We quote several sample questions.

Preliminary

1. Walk one mile East, two miles North, three miles West, four miles South, five miles East, and six miles North. How far are you from your starting point?

5 mi. 7 mi. 13 mi. 21 mi.

2. Let x, y and z be three different positive numbers, with x smaller than y, and z larger than y. Which of the following is always the largest?

y/x z/x z/y none

Final

1. Four points A, B, C, and D are given on a line. Show how to construct a pair of parallel lines through A and B, and another pair of parallel lines through C and D, so that both pairs intersect to form a square.

2. Find a pair of whole numbers x and y with $11x-13y=1$, and with $x+y$ larger than 50.

7. The national M.A.A. contest, 1958. I shall assume that most readers of the MONTHLY have had an opportunity to examine the test which was used in 1958, and have some acquaintance with the history of its development. An interesting summary of this, together with an evaluation of the results of the contest, may be found in a recent article by Professors Fagerstrom and Lloyd [2]. A few comments may be helpful. The test used emphasized both speed and power; it is too long to be completed by an average contestant in the allotted time. A definite advantage is enjoyed by those with more training, but occasionally winners emerge from the lower semester students. The maximum score was 150. One student did in fact achieve a score of 146, but the median score was 31, and only 70 students of the entire 80,000 succeeded in making better than 90. A high score can be achieved only by a student who is fantastically fast, or who is able to bring his intuition to bear and consistently guess the most likely answer from those presented. Every participating school received a pin to award to its top entrant; in addition, schools received certificates if their team (top three) placed in the best 10% in their national area. Although entering schools were charged $.20 per test, for the first twenty, and $.10 for each additional test, the contest was not self-supporting. In addition to the awards mentioned above, local contest committees were able in some cases to secure contributions from local industries. It has been felt that it would be inadvisable to make large awards to the winners in a high school contest which is given in the high schools and proctored by the mathematics teachers; rather than reflecting an uncharitable attitude toward teachers, this comment has in fact the support of members of the N.C.T.M. who feel that this responsibility should be placed elsewhere.

8. Other local contests. It is not feasible in this report to give a detailed description of each of the multitude of high school mathematics contests held each year. In a survey made several years ago, Professor Lloyd reported on almost sixty contests; this report contains much useful comparative information [7]. Some contests are similar to the former Metropolitan New York M.A.A. Contest; among these, one might mention the contest sponsored by the Maryland-District of Columbia-Virginia section. Others seem to reflect the viewpoint of the Wisconsin Section; in this category belongs the Central Valleys Mathematics Quiz of California, sponsored jointly by a newspaper chain and a group of state colleges, and the Michigan Mathematics Prize Competition. Such variation is indeed welcome; I hope that there will be continued experimentation, until each Section of the Association is actively sponsoring a contest program best suited to its local objectives and circumstances.

9. A proposed expansion of the national M.A.A. contest. The administration of the program would rest with a Central Contest Committee, assisted by Local Contest Committees within each section of the M.A.A. Their duties and separate responsibilities are specified below.

The Contest would be a two-stage tournament. The test would be prepared by the Central Committee. The first round would be held in early March or late February.

This would be a multiple-choice test, with subject matter confined to the first two years of high school mathematics. As at present, it would be given in the high schools and proctored by teachers. There might be 25–30 questions to be answered in 80–90 minutes.

The answer sheets from the Preliminary would be sent to the Local Committee which had made the initial arrangements, and would be machine-graded, under their direction. State-wide winners, and winners within each school, then would be announced. Appropriate prizes such as pins and certificates should be awarded. The method of selection of winners and the administration of awards is left to the Local Committee; perhaps the top ten per cent of the contestants should receive recognition.

Each Local Committee sends the score distribution from its Preliminary to the Central Committee. There they are collated and a *recommended admission score for the Final* is determined. This figure is set to admit approximately four per cent of the Preliminary contestants to the Final; the limiting factor here is the number of papers that can actually be dealt with.

Using the recommended score as a guide, suitably modified to meet local conditions, the Local Committees invite schools to select contestants to enter the Final round of the tournament. The Final test should consist of *five* questions to be worked during $1\frac{1}{2}$–2 hours. They should be nonroutine perceptive-type problems that call for unusual insight. The Final contest should be given at a number of centers within each local region, usually at colleges or universities. It is hoped that this could be done in conjunction with lectures, displays, expositions, *etc.*

The responsibility for scoring the Final rests with the Local Committee. The Central Committee will have supplied suggested solutions and scorings, but the judges must be on the alert for an unorthodox approach. All persons involved in the judging should be paid.

The Local Committees send their score distributions and the top three per cent of their papers to the Central Committee. There they are collated, re-examined for uniformity of grading, and winners selected. There might be 100–200 National Winners; the number would be determined by the number of prizes available. These are announced both by the Central Committee and by the appropriate Local Committees; there should also be some process for honoring the school and teacher responsible for helping produce a National Winner.

The cost of the contest should be carried by a *small* fee charged each school, together with a *small* per-student fee. Money for prizes would come from industry or government sponsorship.

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CONFERENCE ON THE COMMITTEE ON THE UNDERGRADUATE PROGRAM

A conference on the work of the Association's Committee on the Undergraduate Program in Mathematics was held at the Burlington Hotel, Washington, D. C., on November 15 and 16, 1958.

The following forty-seven persons attended the conference:

Max Beberman	UICSM Mathematics Project, University of Illinois
E. G. Begle	Yale University, School Mathematics Study Group
R. H. Bing	University of Wisconsin
R. C. Buck	Stanford University
R. R. Bush	University of Pennsylvania. (American Psychological Association)
E. A. Cameron	University of North Carolina
J. W. Cell	North Carolina State College. (American Society for Engineering Education)
L. W. Cohen	University of Maryland
R. P. Dilworth	California Institute of Technology
W. L. Duren, Jr.	University of Virginia
F. A. Ficken	University of Tennessee
R. C. Fisher	Ohio State University
G. E. Forsythe	Stanford University
H. M. Gehman	University of Buffalo
A. M. Gleason	Harvard University
W. T. Guy, Jr.	University of Texas
P. G. Hoel	University of California, Los Angeles
R. D. James	University of British Columbia
J. L. Kelley	University of California, Berkeley
J. G. Kemeny	Dartmouth College
C. B. Lindquist	U. S. Office of Education
W. G. Madow	Stanford Research Institute
J. R. Mayor	American Association for the Advancement of Science
E. J. McShane	University of Virginia
L. J. Montzingo	University of Buffalo
Ivan Niven	University of Oregon
E. P. Northrop	University of Chicago
F. G. O'Brien	National Science Foundation
R. E. Paulson	National Science Foundation
G. B. Price	University of Kansas
A. L. Putnam	University of Chicago
Mina Rees	Hunter College
P. C. Rosenbloom	University of Minnesota
A. E. Ross	University of Notre Dame
R. E. K. Rourke	Commission on Mathematics
Patrick Suppes	Stanford University
H. W. Syer	Kent School
G. B. Thomas, Jr.	Massachusetts Institute of Technology
D. L. Thomsen, Jr.	Watson Scientific Laboratory, Columbia University. (Society for Industrial and Applied Mathematics)
A. W. Tucker	Princeton University
F. E. Ulrich	Rice Institute
R. J. Walker	Cornell University

A. D. Wallace	Tulane University
S. S. Wilks	Princeton University
C. R. Wylie, Jr.	University of Utah
J. W. T. Youngs	Indiana University
Mark Zemansky	City College of New York. (American Institute of Physics, and American Association of Physics Teachers)

Speakers were invited to discuss the various topics on the program, and in most cases outlines were prepared and distributed to the participants in advance of the Conference. Summaries of the prepared talks and of some of the discussion are given herewith.

Price: Purpose of the Conference. The purpose of the Conference was to assist in formulating plans and policies for the future work of the Committee on the Undergraduate Program. This Committee, formed early in 1953, proposed in 1958 that the time had come to reorganize and expand its efforts. Accordingly, at the Committee's request, it was discharged at the end of August, 1958. This Conference was called to re-examine the assignment of the Committee on the Undergraduate Program and to take steps to establish a new Committee with adequate funds, personnel, and program.

Duren: Report from the original Committee on the Undergraduate Program. The speaker reviewed the goals and activities of the original CUP since its appointment in January 1953. The primary purpose of the CUP was to act as a bridge between research and curriculum. The effort of the CUP was focused on the first year of college. No attempt was made to write text books, but source books were written. Among the primary goals of the CUP were:

1. To interest young mathematicians in teaching problems, and to encourage senior mathematicians to become more aware of the needs of teaching; to point out the importance of finding a balance between critical mathematics, and the desires and motivations of 18-year olds;
2. To contribute to institute and lecture programs;
3. To establish contact with other groups with purposes similar to those of the CUP.

No specific curriculum suggestions were made until it was known that the original CUP was to be discharged.*

The speaker suggested that in the reorganization of the CUP some thought be given to reorganizing into groups, or one large group subdivided into subgroups by locality.

Rourke: Report from the Commission on Mathematics. The speaker drew attention to the books and pamphlets that have been written and distributed by the Commission, and to the conferences that have been arranged to discuss the work of the Commission. Out of these conferences there has emerged a clarification of the proposals of the Commission, especially on the following particular points:

(1) The role of set theory in the high school curriculum;
(2) the continuing role of skills;
(3) the increasing emphasis on structure as contrasted with manipulation in algebra;
(4) the use of the word "modern."

The speaker outlined the following nine point program of the Commission for the class of college-capable students in the secondary schools:

1. Strong preparation both in concepts *and* in skills, for college mathematics at the level of calculus and analytic geometry.

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* Anyone interested in the reports issued by this committee may obtain the following by writing to the Buffalo office of the Association: (1) Collected Reports of the CUP (1957) and (2) Outline of Recommended Courses (1958).

2. Understanding of the nature and role of deductive reasoning—in algebra, as well as in geometry.
3. Appreciation of mathematical structure ("patterns")—for example, properties of natural, rational, real, and complex numbers.
4. Judicious use of unifying ideas—set, variable, function, and relation.
5. Treatment of inequalities along with equations.
6. Incorporation with plane geometry of some coordinate geometry, and also essentials of solid geometry and space perception.
7. Introduction in grade eleven of fundamental trigonometry—centered on coordinates, vectors, and complex numbers.
8. Emphasis in grade twelve on elementary functions (polynomial, exponential, circular).
9. Recommendation of additional alternative units for grade twelve: *either* introductory probability with statistical applications *or* an introduction to modern algebra.

Finally, the speaker emphasized the need of appropriate teacher training programs, for prospective teachers as well as those in service. He cited cases of teachers attending university summer sessions in the hope of building up their knowledge of newer viewpoints in mathematics, only to get traditional courses of a sort that have almost disappeared from many university programs.

Discussion: The question was raised as to the desirability of teaching polynomial calculus in secondary school, and Mr. Rourke pointed to the critical shortage of qualified teachers. The problem of the certification of teachers was discussed; it was pointed out that the Commission has no authority here, but can only make recommendations. There was considerable feeling that the Mathematical Association of America should likewise formulate a set of strong recommendations.

Beberman: Report on the University of Illinois Committee on School Mathematics. The speaker stated that the aim of the UICSM is to develop a four-year program of high school mathematics that develops understanding as well as manipulative skills. The program is aimed primarily at the college-capable student. At present there are about 55 schools in 20 states using the UICSM materials as pilot schools.

The units and grade level which comprise the UICSM program are:

Unit	*Grade Level*	*Descriptive Title*
1	9th Grade	Arithmetic of real numbers.
2	9th Grade	Pronumerals, generalizations, manipulation.
3	9th Grade	Equations, inequations, applications.
4	9th Grade	Ordered pairs, graphs.
5	10th Grade	Relations, functions.
6	10th Grade	Geometry.
7	11th Grade	Real number system, induction.
8	11th Grade	Exponents, logarithms.
9	11th Grade	Complex numbers, systems of quadratics.
10	11th Grade	Polynomial functions, theory of equations.
11	12th Grade	Circular functions, trigonometry.
12	12th Grade	Postulational systems.
13	12th Grade	Analytic Geometry.

The discussion that followed centered around the question of finding a middle ground between a whole deductive system and none, "rigor" versus "intuitively obvious."

Begle: Report from the School Mathematics Study Group. The aim of the School Mathematics Study Group (SMSG) is to improve the amount and quality of mathematical

training in the secondary and elementary schools. It is attempting to make mathematics courses in the schools more interesting so that more students are attracted, to improve the curriculum by working for better mathematics, and to help teachers prepare themselves to instruct these improved courses. SMSG is financed by the National Science Foundation. It hopes to work toward its goals by a joint effort of representatives from all parts of the mathematical profession.

There are three projects in progress. The first is the production of sample textbooks for grades 9 through 12. The second is the production and testing of several experimental units of instruction for the 7th and 8th grades; these units are being tested in a large number of classrooms in all parts of the country. Both these projects will involve writing sessions in the summer of 1959. The third project under way is the production of a series of monographs designed for the better students in high schools. These monographs are intended to supplement the high school program by showing something of the scope and interest of mathematics in our culture.

There are two projects being organized, one on films and television as teaching aids, and another on the production of teacher training materials. In addition there are various projects under consideration, such as a study of the mathematics of elementary schools, and a study of such topics as concept formation and attitudes towards mathematics. This latter study would enlist the aid of psychologists and other social scientists.

Kemeny: Courses for Teacher Training. The speaker outlined a proposed mathematics requirement for high school mathematics teacher training programs that consisted of 36 semester hours of undergraduate mathematics courses. During the first two years the student would carry one course each semester and complete the work that is roughly equivalent to Universal Mathematics I and II, and a course in the calculus of $y=f(x)$. During the junior and senior years the student would carry two courses each semester in order to complete four one-year units specially designed for teachers. These four units would consist of:

(a) One unit in modern algebra: half of it in groups, *etc.*; and half in linear algebra.
(b) One unit in geometry: substantial work in analytic geometry with introductions to projective and non-Euclidean geometrics.
(c) One unit in probability and statistics: half of it a course in probability theory, half of it a course in statistical inference.
(d) One unit in the history of mathematics: half of it to cover the period from Euclid to non-Euclidean geometry, half to cover the last 120 years, with special emphasis on topics not covered in other courses.

The speaker went on record as being in favor of a statement by the MAA concerning minimum requirements for training of high school mathematics teachers.

The discussion that followed centered about three questions:

1. Are the standards set forth by the speaker realistic?
2. Can we give students better training in mathematics and expect them to remain in the high school teaching profession?
3. Should we not be more vitally concerned with the teachers in the high school teacher training programs?

Zemansky: The Relation of Mathematics to Physics Instruction. The speaker felt that it would be of considerable value to get a short article into one of the physics journals on the trends in mathematical instruction which had been presented by previous speakers. He urged that calculus be taught early in the college program so that students would have the subject available for use in their study of physics. The speaker questioned the emphasis on understanding and insight, not for the physics students, but for the pre-engineering students, on the grounds that there was a great deal of manipulative technique that the students needed.

Discussion: Several persons present responded to this last point of the speaker by giving evidence of the need for understanding in mathematics by engineering students.

Madow: The Mathematical Training of Social Scientists. The speaker reported an increased interest among social scientists concerning the mathematical training of their students, but that many departments in the social science areas could increase the use of mathematics in their own courses. It was pointed out that the ordinary use of mathematics in the social sciences is to construct theories rather than to work out problems. The speaker also noted that the major use of mathematics is the use of probability and statistics and hence it is not so important that the average social science student get his calculus early. Since it is important that physical scientists get calculus early, the speaker asked that the CUP reconsider the proposal of the original CUP concerning a common first year mathematics course for all students.

Hoel: Undergraduate Statistics in a Mathematics Department. The speaker addressed his remarks to those situations where a college or university has neither a department of statistics nor a collection of statisticians in a mathematics department. First, an elementary service course in statistics should be given in a mathematics department only if the department has someone trained in statistics and the department is much better equipped than other departments to give it. A year course in statistics without mathematical prerequisites is more useful than a one semester course requiring a semester of mathematics first.

Next, in the case of a statistics course with a calculus prerequisite, a knowledge of integration should be insisted upon since it is inefficient to teach such a course with lower prerequisites. Integral calculus is the major tool in beginning statistical theory.

The speaker also commented on universal mathematics courses, and urged that statistics be not included as a topic in such programs. The reasons for this are that it would be a duplication for those who will be taking a statistics course, and for the others the time available is inadequate to do justice to a good explanation of statistical theory.

Regarding the undergraduate program in algebra in its relation to statistics, the speaker urged the replacement of the traditional theory of equations course by an elementary course in linear algebra with a strong orientation towards geometry.

Forsythe: The Role of Numerical Analysis in an Undergraduate Program. The speaker pointed out that most of our mathematics students major in other fields and are primarily interested in the applications of mathematics, and not in its structure. He also pointed out that about half of those who attain a Ph.D. in mathematics go into industry and apply mathematics in various areas, and that almost all of them will be connected in some degree with automatic computation. The speaker noted that there is now a strong demand for the A.B. mathematician, but that his training has not made him a good practitioner of mathematics. The speaker suggested that the aims of an undergraduate mathematics education are the following:

a. To learn as much as possible about the structure of mathematics.
b. To learn to read independently the mathematical literature at his level.
c. To know the tools of mathematics, books and machines, and how and where to find them.
d. To cultivate and practice solution of mathematical problems new to him.
e. To go fairly deeply into some other field of knowledge where mathematics is used.
f. To learn to enjoy mathematical study.

The speaker criticized much of mathematical education as not intuitive enough nor well enough illustrated, and stated his agreement with the words of Felix Klein who said, "The living thing in mathematics, its most important stimulus, its effectiveness in all directions, depends entirely upon the applications."

The speaker stated his belief that an increased role must be given to numerical

analysis in the undergraduate program, and that for the most part this work should be mixed into undergraduate courses rather than to have a great many separate courses in numerical analysis. The speaker also favored a special coding course for all students, to be taken early in their studies.

The discussion that followed brought out the need to distinguish carefully between a vocational point of view and the point of view of illustrating the fundamental ideas in mathematics by computing techniques. It was noted that even a coding course gives training in precise thought since it forces a student to think in terms of exactly what he is doing, since he must write a complete set of directions.

Cohen: Report on Films and Television for Mathematical Instruction. The speaker pointed out that films and television as a means of mathematical instruction have some, but not all, of the properties of books and teachers; consequently their use is not to be undertaken with the object of replacing either of the older aids to learning. At the October 18 meeting of the MAA's Committee on Production of Films, it was decided

1. To produce motion picture films of three hour lectures.
2. To aim the instruction at highly competent students.
3. To make one film spanning the competence of high school seniors and college freshmen.

The topics and level under consideration are:

1. Mathematical Induction (H. S. Senior—College Freshmen)
2. Theory of Limits or Integration (Sophomore or Junior Undergraduates)
3. Topology (Undergraduate Math. Club)

Correspondence is under way to determine a lecturer for each topic. Questions relating to distribution, royalties and property rights in the films to be produced were recognized as requiring answers. These answers will be sought in due course. The cost of the program will be met through a grant which the Association has obtained from the National Science Foundation.

1. The preliminary script should be a draft of the lecture prepared by the lecturer.
2. The director and lecturer then collaborate in preparation of the script based on the draft lecture.
3. The director, lecturer and committee meet to review the script and production before the making of the film.
4. The committee should be present at the making of the first film.
5. There should be a mathematician present during the making of each film to catch slips overlooked by the lecturer and director.
6. Short, inexpensive tests for image and voice should be made as an aid to the choice of a lecturer.
7. Animation may be introduced into the films as an aid to clarity in blackboard writing and diagramming.

In the long discussions that followed the report many representatives reported experimental use of television and of the large lecture system. There seemed to be general agreement that there is student acceptance of the large lecture system, and it is preferred at this time to pure television courses. A question was raised regarding the merits of such instruction for the gifted student. It was generally agreed that special instruction should be provided for the "top 1%," but there was not general agreement as to whether or not the new CUP should concern itself with such special instructional programs.

Begle: Discussion of writing sessions. The speaker described the different kinds of writing programs which had been operated: small writing groups as contrasted with large ones, and both of these compared with scattered efforts by individuals.

Kemeny: Financial Arrangements for Authors. The speaker cited the experience of the Dartmouth writing group, which did not receive royalties from its published works. It was felt that in the future such limitations should be removed.

Discussion: There was a considerable response to this matter of financial arrangements. The basic problem is the granting of royalty rights to writers who have been paid for their writing efforts. It was felt that if an author was paid to write a book and in addition received full royalties, there might be criticism from the free-lance mathematical writers. This criticism could be met by subtracting the initial payment to the writer from subsequent royalty payments. Many preferred to avoid any such subtraction, in order to provide adequate economic motivation for high calibre persons in writing projects.

Discussion: The Nature of Books to be Written. The question discussed was the desirability of source-books vs. textbooks. Since source-books are written too compactly for classroom use, their role is to serve as materials for use by writers of texts. The experience of some writers of source-books is to say "never again!," but it was urged that some compromise stage between source-books and textbooks be sought.

RESOLUTIONS ADOPTED BY THE CUP CONFERENCE

Since the Committee on the Undergraduate Program is charged with responsibility for advising the Mathematical Association of America on all matters connected with the undergraduate program in mathematics, this Conference hereby adopts the following recommendations concerning certain special aspects of the work of the Committee:

1. This Conference recommends that the Committee on the Undergraduate Program in Mathematics consist of from ten to fifteen persons and that the Committee have power to delegate its various activities to subcommittees whose members need not be members of the Committee.

2. This Conference recommends that the Committee on the Undergraduate Program be provided with adequate facilities and staff, free of other responsibilities, sufficient to carry out the tasks assigned to it.

3. This Conference recommends that the Committee on the Undergraduate Program continue to be concerned with the development of undergraduate courses in mathematics and with the production of adequate materials and methods for these courses, keeping in mind the varying needs of students destined for the many careers which make use of mathematics, and particularly the needs of prospective high school mathematics teachers.

4. This Conference recommends that the Committee on the Undergraduate Program continue to plan its activities with due regard for other groups concerned with similar problems.

5. Since the encouragement of writing of adequate mathematical textbooks is a primary problem facing the Committee on the Undergraduate Program, this Conference recommends to the Board of Governors of the Association and to the Committee that the Committee develop a program to subsidize the preparation of textbooks and monographs without prejudice to the royalty rights of the authors.

6. This Conference recommends that, upon the recommendation of the Committee on the Undergraduate Program, the Mathematical Association of America publish a statement of minimal standards for teachers of mathematics in high schools, junior colleges, and colleges. The review and revision of such standards should be a continuing responsibility of the Committee and the Committee should recommend to the Association future changes in such standards. The existence of such statements of minimal

standards should be given wide publicity through the Sections of the Association to all interested state and local groups.

7. The Conference recommends that the Committee on the Undergraduate Program make recommendations to the Association regarding the desirable mathematical preparation of students expecting to pursue graduate work in mathematics or expecting to pursue mathematical careers in industry.

8. Since the vastly increased demands upon mathematicians and upon teachers of mathematics in the undergraduate colleges of this country have put upon the Mathematical Association of America the obligation of providing professional leadership in the design of appropriate courses to satisfy these demands, and since the obligations of the Association in this field have been delegated to the Committee on the Undergraduate Program, and since this delegation implies a vastly expanded program of activity as described in these resolutions, therefore this Conference recommends that the officers and the Board of Governors of the Association communicate to the appropriate agencies the compelling need for grants of funds commensurate with the magnitude of the task and with its importance for the development of mathematics and science in this country.

Concluding note. A Committee on the Undergraduate Program in Mathematics is now being appointed. Members of this committee met in New York City on December 29 and 30, 1958 to plan its future activities in view of the recommendations of the Conference.

R. C. FISHER AND IVAN NIVEN, *Recorders*
HARRY M. GEHMAN, *Secretary-Treasurer*

MATHEMATICAL NOTES

EDITED BY ROY DUBISCH, Fresno State College

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NOTE ON HYPERGEOMETRIC POLYNOMIALS

S. K. CHATTERJEA, Bangabasi College, Calcutta, India

1. In a recent paper [1], V. R. Thiruvenkatachar and T. S. Nanjundiah have proved the following nonlinear recurrence relation for ultraspherical polynomials:

$$(1.1)\quad (1-x^2)D_n^\lambda(x) = n(n+2\lambda)[P_n^\lambda(x)]^2 - (n+1)(n+2\lambda-1)P_{n-1}^\lambda(x)P_{n+1}^\lambda(x),$$

where

$$D_n^\lambda(x) \equiv [DP_n^\lambda(x)]^2 - [DP_{n-1}^\lambda(x)][DP_{n+1}^\lambda(x)].$$

M. S. Webster [2] has pointed out that (1.1) characterizes the ultraspherical polynomials. We show in this section that a very simple proof of the formula (1.1) may be given merely by considering the following two differentiation formulas ([3], vol. 2, p. 176):

$$xDP_n^{\lambda}(x) = DP_{n-1}^{\lambda}(x) = nP_n^{\lambda}(x), \tag{1.2}$$

$$DP_{n+1}^{\lambda}(x) - xDP_n^{\lambda}(x) = (n + 2\lambda)P_n^{\lambda}(x). \tag{1.3}$$

Changing n into $n+1$ in (1.2) and $n+1$ into n in (1.3) we obtain

$$xDP_{n+1}^{\lambda}(x) - DP_n^{\lambda}(x) = (n + 1)P_{n+1}^{\lambda}(x) \tag{1.4}$$

and

$$DP_n^{\lambda}(x) - xDP_{n-1}^{\lambda}(x) = (n + 2\lambda - 1)P_{n-1}^{\lambda}(x). \tag{1.5}$$

Next from (1.2) and (1.3) we have on multiplication

$$\begin{aligned} -x^2(DP_n^{\lambda})^2 - DP_{n-1}^{\lambda}\cdot DP_{n+1}^{\lambda} + x[DP_n^{\lambda}DP_{n+1}^{\lambda} + DP_{n-1}^{\lambda}\cdot DP_n^{\lambda}]& \\ = n(n + 2\lambda)\cdot(P_n^{\lambda})^2&. \end{aligned} \tag{1.6}$$

Similarly from (1.4) and (1.5) we get

$$\begin{aligned} (DP_n^{\lambda})^2 + x^2DP_{n-1}^{\lambda}\cdot DP_{n+1}^{\lambda} - x[DP_n^{\lambda}\cdot DP_{n+1}^{\lambda} + DP_{n-1}^{\lambda}\cdot DP_n^{\lambda}]& \\ = -(n + 1)(n + 2\lambda - 1)P_{n-1}^{\lambda}\cdot P_{n+1}^{\lambda}&. \end{aligned} \tag{1.7}$$

Finally from (1.6) and (1.7) we get on addition

$$\begin{aligned} (1 - x^2)\cdot[(DP_n^{\lambda})^2 - (DP_{n-1}^{\lambda})\cdot(DP_{n+1}^{\lambda})]& \\ = n(n + 2\lambda)(P_n^{\lambda})^2 - (n + 1)(n + 2\lambda - 1)P_{n-1}^{\lambda}P_{n+1}^{\lambda}&. \end{aligned}$$

We would like to point out that we succeeded in extending this result (1.1) as well as the characterization [4] by using the Jacobi polynomials in place of the ultraspherical polynomials.

2. The following formula is due to the German mathematician Alexander Dinghas [5]: If $P(x)$ denotes a linear polynomial in x of the form $Ax+B$, which remains positive throughout the closed interval $[a, b]$ and if α, β are two real positive numbers, we have

$$\frac{1}{P^{\alpha}(a)P^{\beta}(b)} = \frac{(b - a)^{1-\alpha-\beta}}{B(\alpha, \beta)}\int_a^b \frac{(b - x)^{\alpha-1}(x - a)^{\beta-1}}{P^{\alpha+\beta}(x)}\, dx, \tag{2.1}$$

where $P^{\alpha}(a) = (Aa+B)^{\alpha}$ and $B(\alpha, \beta)$ is Euler's integral of the first kind. We wish to give in this section several interesting results, *viz.*, (2.2), (2.5), (2.6), (2.7), (2.8), (2.9), all derived from (2.1).

Substituting $x=a\sin^2\theta+b\cos^2\theta$, we at once write (2.1) as follows:

$$\frac{1}{P^\alpha(a)P^\beta(b)}=\frac{2}{B(\alpha,\beta)}\int_0^{\pi/2}\frac{(\sin\theta)^{2\alpha-1}(\cos\theta)^{2\beta-1}}{\{P(a\sin^2\theta+b\cos^2\theta)\}^{\alpha+\beta}}d\theta. \tag{2.2}$$

From (2.2) it is easy to verify the well-known definite integral ([**3**], vol. I, p. 11)

$$\int_0^{\pi/2}\frac{(\sin\theta)^{2\alpha-1}\cdot(\cos\theta)^{2\beta-1}}{(a\sin^2\theta+\cos^2\theta)^{\alpha+\beta}}d\theta=\frac{1}{2}a^{-\alpha}B(\alpha,\beta),\qquad 0<a<1. \tag{2.3}$$

Next putting $P(x)=1-2xh+h^2$ in (2.2) we get for small h

$$\begin{aligned}&(1-2ah+h^2)^{-\alpha}(1-2bh+h^2)^{-\beta}\\ &\quad=\frac{2}{B(\alpha,\beta)}\int_0^{\pi/2}\frac{(\sin\theta)^{2\alpha-1}(\cos\theta)^{2\beta-1}}{\{1-2(a\sin^2\theta+b\cos^2\theta)h+h^2\}^{\alpha+\beta}}d\theta.\end{aligned} \tag{2.4}$$

Now we remember as usual the ultraspherical polynomial $P_n^\lambda(x)$, $(\lambda\neq 0)$, by the expansion

$$(1-2hx+h^2)^{-\lambda}=\sum_{n=0}^{\infty}P_n^\lambda(x)h^n.$$

Thus equating coefficients of h^n from both members of (2.4) we get

$$\begin{aligned}&\sum_{r=0}^{n}P_r^\alpha(a)P_{n-r}^\beta(b)\\ &\quad=\frac{2}{B(\alpha,\beta)}\int_0^{\pi/2}P_n^{\alpha+\beta}(a\sin^2\theta+b\cos^2\theta)(\sin\theta)^{2\alpha-1}(\cos\theta)^{2\beta-1}d\theta.\end{aligned} \tag{2.5}$$

In particular when $\alpha=\beta=\frac{1}{2}$ we get from (2.5)

$$\sum_{r=0}^{n}P_r(a)P_{n-r}(b)=\frac{2}{\pi}\int_0^{\pi/2}U_n(a\sin^2\theta+b\cos^2\theta)d\theta, \tag{2.6}$$

where $P_n(x)$ and $U_n(x)$ denote as usual the Legendre and Chebyshev (second kind) polynomials respectively.

Again putting $\alpha=\frac{1}{2}$, $\beta=1$ in (2.5) we obtain a more interesting result

$$\sum_{r=0}^{n}P_r(a)U_{n-r}(b)=4\cdot\int_0^{\pi/2}P'_{n+1}(a\sin^2\theta+b\cos^2\theta)\cos\theta d\theta, \tag{2.7}$$

where $P'_n(x)\equiv(d/dx)\{P_n(x)\}$.

Or we can write (2.7) as follows:

$$\sum_{r=0}^{n}U_r(a)P_{n-r}(b)=4\cdot\int_0^{\pi/2}P'_{n+1}(a\sin^2\theta+b\cos^2\theta)\sin\theta\, d\theta. \tag{2.8}$$

Lastly putting $a=-b$ in (2.6) and then changing b into x we get

$$\sum_{r=0}^{n} (-1)^r P_r(x) P_{n-r}(x) = \frac{2}{\pi} \int_0^{\pi/2} U_n(x \cos 2\theta) d\theta. \tag{2.9}$$

I am indebted to Dr. H. M. Sengupta (Department of Pure Mathematics, Calcutta University) for his kind help in the preparation of this note.

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AN INEQUALITY FOR THE FOURIER COEFFICIENTS OF A NONNEGATIVE FUNCTION

J. D. WESTON, King's College, University of Durham, England

In the analysis of an electrical network consisting of linear elements combined with a periodically varying conductance, one is led to represent the conductance function $g(\theta)$ by its Fourier series

$$\sum_{n=-\infty}^{\infty} c_n e^{in\theta},$$

where, for each n,

$$c_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} g(\theta) e^{-in\theta} d\theta.$$

Under certain assumptions, it can be shown that the system responds linearly to small perturbations, and is stable provided that

$$c_0^2 + c_0 \left| c_{2n} \right| \geqq 2 \left| c_n \right|^2. \tag{1}$$

On the other hand, energy considerations suggest the system will be stable if $g(\theta)$ is never negative. One is thus invited to prove that (1) holds for any nonnegative function $g(\theta)$ which has Fourier coefficients c_n. This can be done in the following way.

For a given value of n, we can choose a real number α so that $e^{in\alpha}c_n$ is real. Then

$$e^{in\alpha}c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)e^{-in(\theta-\alpha)}d\theta$$
$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)\cos n(\theta-\alpha)d\theta,$$

since $g(\theta)$ is real. Also,

$$|c_{2n}| \geqq \Re(e^{2in\alpha}c_{2n}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(\theta)\cos 2n(\theta-\alpha)d\theta.$$

Now suppose that $g(\theta) \geqq 0$ for all θ. Then $c_0 \geqq 0$, so that (1) is established if we can show that

$$c_0^2 + c_0\Re(e^{2in\alpha}c_{2n}) - 2|c_n|^2 \geqq 0. \tag{2}$$

Denoting the left-hand side of (2) by L, we have

$$4\pi^2 L = \left\{\int_{-\pi}^{\pi} g(\theta)d\theta\right\}^2 + \int_{-\pi}^{\pi} g(\theta)d\theta \int_{-\pi}^{\pi} g(\theta)\cos 2n(\theta-\alpha)d\theta$$
$$- 2\left\{\int_{-\pi}^{\pi} g(\theta)\cos n(\theta-\alpha)d\theta\right\}^2$$
$$= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} g(\theta)g(\phi)\{1 + \cos 2n(\phi-\alpha) - 2\cos n(\theta-\alpha)\cos n(\phi-\alpha)\}d\theta d\phi.$$

Thus

$$2\pi^2 L = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} g(\theta)g(\phi)\cos n(\phi-\alpha)\{\cos n(\phi-\alpha) - \cos n(\theta-\alpha)\}d\theta d\phi.$$

We may interchange θ and ϕ, so that

$$2\pi^2 L = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} g(\theta)g(\phi)\cos n(\theta-\alpha)\{\cos n(\theta-\alpha) - \cos n(\phi-\alpha)\}d\theta d\phi.$$

Adding these two expressions for $2\pi^2 L$, we see that

$$4\pi^2 L = \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} g(\theta)g(\phi)\{\cos n(\theta-\alpha) - \cos n(\phi-\alpha)\}^2 d\theta d\phi.$$

It follows that $L \geqq 0$, equality occurring only if $g(\theta) \equiv 0$.

The inequality (1) is fairly delicate, for although $c_0 \geqq |c_n|$ if $g(\theta) \geqq 0$ for all θ, we can also have $c_0 + |c_{2n}| < 2|c_n|$: for example, if $g(\theta) = 1$ when $\cos\theta \geqq 0$ and $g(\theta) = 0$ when $\cos\theta < 0$, then $c_0 = \frac{1}{2}$, $c_1 = 1/\pi$, and $c_2 = 0$.

CLASSROOM NOTES

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LAGRANGE'S MULTIPLIERS

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In most expositions of the Lagrange multiplier method, no motivation for the introduction of the multipliers is given. The discussion below presents a geometric interpretation.

A stationary point of a function $\phi(x_1 \cdots x_n)$ is, by definition, a point at which the gradient $(\phi_1, \cdots, \phi_n)$, where $\phi_k = \partial\phi/\partial x_k$ $(k=1, \cdots, n)$, is zero. If we regard the vector $(dx_1 \cdots dx_n)$ as a permissible displacement the condition that the gradient be zero may be replaced by the condition that the gradient be orthogonal to all permissible displacements which, in the unconstrained case, is any arbitrary displacement. If the function is subjected to m independent constraints

$$f_k(x_1, \cdots, x_n) = 0 \qquad (k = 1, \cdots, m),$$

then the permissible displacements are precisely those which are orthogonal to the gradients of these functions since

$$df_k = f_{k1}dx_1 + \cdots + f_{kn}dx_n = 0 \qquad (k = 1, \cdots, m),$$

where $f_{kr} = \partial f_k/\partial x_r$. The condition that the gradient of the function, ϕ, be orthogonal to all permissible displacements is satisfied if and only if the gradient of ϕ lies in the space spanned by the gradients of the constraining functions, *i.e.*, the gradient of ϕ must be a linear combination of the gradients of the constraining functions. Thus, if a new function is formed by adding a linear combination of the constraining functions to ϕ, and the gradient of the new function is equated to zero, the last condition is satisfied.

The above discussion is applicable also to nonholonomic constraints for which the gradient of the constraints is replaced by the vector whose components are the coefficients of the differentials in the form which defines the constraint.

A NOTE ON THE CONICS

J. GALLEGO-DIAZ, University of Puerto Rico

The purpose of this paper is to study one application of the transformation $\Omega = z^2$, where Ω and z are complex. We consider a given central conic in the Ω-plane with a focus at the pole and its major axis coinciding with the polar axis.*

* I have not seen such an application studied in any of the books that I have consulted, nor in the excellent *Dictionary of Conformal Representations*, by H. Kober, New York, 1952.

We suppose that the points of the Ω-plane are given either by cartesian coordinates (u, v) or by polar coordinates (ρ_1, ω_1). Similarly, points in the z-plane are given either by cartesian coordinates (x, y) or by polar coordinates (ρ, ω).

The polar-coordinate equation of a conic with a focus at $(0, 0)$ and its major axis on the polar axis is

$$\rho_1 = \frac{p_1}{1 - e_1 \cos \omega_1}. \tag{1}$$

The transformation we are considering may be written either as

$$u = x^2 - y^2, \qquad v = 2xy \tag{I}$$

or as

$$\rho_1 = \rho^2, \qquad \omega_1 = 2\omega. \tag{II}$$

By virtue of these, (1) becomes

$$\rho^2 = \frac{p}{1 - e_1 \cos 2\omega} \tag{2}$$

or

$$\frac{x^2}{p_1/(1 - e_1)} + \frac{y^2}{p_1/(1 + e_1)} = 1, \tag{3}$$

which means that the transformed curve in the z-plane is another conic whose center is the origin and whose axes coincide with the coordinate axes.

Letting a and b denote the semiaxes of the conic (3) and a_1 and b_1, those of the conic (1), we easily obtain

$$b^2 = a_1 - c_1, \qquad a^2 = a_1 + c_1; \tag{4}$$

$$a^2 + b^2 = 2a_1, \qquad ab = \pm b_1; \tag{5}$$

$$c_1 = c^2/2, \qquad e_1 = e^2/(2 - e^2), \tag{6}$$

where e_1 and e are the respective eccentricities.

It is seen without difficulty that if $e_1 < 1$ then $e < 1$ and that if $e_1 > 1$ then $e > 1$. In other words, *the transformation preserves the nature of the conic.*

Keeping in mind formulas (4), (5), and (6), we are able to prove easily and quickly a large number of theorems and to solve numerous problems concerning loci, envelopes, maxima and minima, orthogonal trajectories, graphical constructions, and so on. In some cases it is enough to remember that a straight line in the Ω-plane is transformed into a rectangular hyperbola with center at the origin in the z-plane and that a straight line in the z-plane is transformed into a parabola with focus at the origin in the Ω-plane. These results follow from (I) and (II) since $Mu + Nv + P = 0 \rightarrow M(x^2 - y^2) + 2Nxy + P = 0$ and

$$\frac{1}{\rho} = A \cos \omega + B \sin \omega \rightarrow \frac{1}{\sqrt{\rho_1}} = A \cos \tfrac{1}{2}\omega_1 + B \sin \tfrac{1}{2}\omega_1$$

$$\rightarrow \frac{2}{\rho_1} = A^2 + B^2 + (A^2 - B^2) \cos \omega_1 + 2AB \sin \omega_1.$$

The transformation is conformal and its critical points are $z=0$ and $z=\infty$.

Other important and obvious properties of this transformation are:

(a) To diametrically opposite points on the conic in the Ω-plane correspond the ends of semiconjugate diameters of the transformed conic in the z-plane. ($P\rightarrow M$, $Q\rightarrow L$ in Fig. 1.)

(b) The two foci F and F' of the conic in the z-plane correspond to the other focus O' of the conic in the Ω-plane (Fig. 1).

(c) The ends of two orthogonal semidiameters of the conic in the z-plane correspond to the ends of a focal chord of the conic in the Ω-plane.

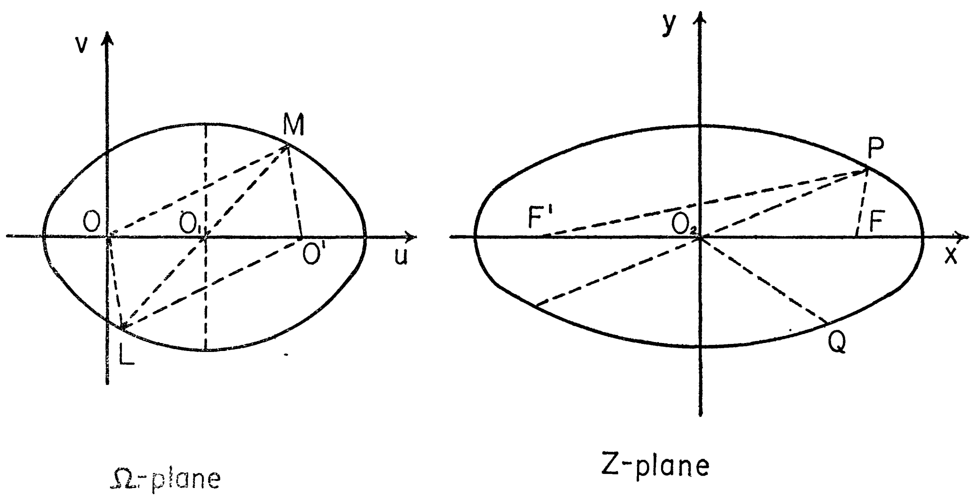

FIG. 1

We give two examples to illustrate simple proofs.

Example 1. Let O_2P and O_2Q be two semiconjugate diameters of the given ellipse in the z-plane, whose foci are F and F' corresponding to the focus O' of the given ellipse in the Ω-plane (Fig. 1). Since $MO'=OL$, it follows that $PF\cdot PF' = \overline{OQ}^2$. In other words, *the product of the radius vectors of an ellipse which begin at the point P is equal to the square of the semiconjugate diameter of O_2P.*

Example 2. Since $OM+OL=2a_1$, it follows that $\overline{O_2P}^2+\overline{O_2Q}^2=a^2+b^2$. (Theorem of Appolonius.)

The following are further examples of results that may easily be obtained:

1. The envelope of a family of coaxial ellipses which have equal area consists of two rectangular hyperbolas.

2. Let H_1 and H_2 be two rectangular hyperbolas whose common center is A and which are tangent at M and N, respectively, to a given ellipse whose center is also A. If H_1 and H_2 intersect at P, prove that AP bisects the angle MAN.

3. Let OA and OB be two orthogonal semidiameters of a given ellipse. Prove that $1/\overline{OA}^2+1/\overline{OB}^2=1/a^2+1/b^2$.

4. The orthogonal trajectories of the family of curves $(x^2+y^2)^2=k^2(x^2-y^2)$ are the family of curves $(x^2+y^2)^2=2c^2xy$.

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN A. BROWN, University of Delaware AND JOHN R. MAYOR, AAAS and University of Maryland

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THE BALL STATE EXPERIMENTAL PROGRAM IN GEOMETRY AND ALGEBRA

C. F. BRUMFIEL, Ball State Teachers College, Muncie, Indiana

A tenth grade plane geometry course, based upon a modified version of the Hilbert Postulates, and a ninth grade algebra that is a mild postulational development have been developed in the laboratory school of Ball State Teachers College.

Text material for these courses has been prepared by Professors Merrill Shanks of Purdue, Charles Brumfiel of Ball State and Robert Eicholz of the Ball State laboratory school. Under National Science Foundation grants for in-service institutes for teachers of mathematics a large group of Indiana teachers has received some training in the mathematical ideas underlying these experimental programs. The geometry is being taught experimentally in seven schools and the algebra in six. A classroom visitation and supervisory program is in effect for the teachers of this material.

Both the geometry and algebra are clear-cut attempts to introduce students to the axiomatic structure of mathematics. The geometry is at a high level of rigor. The use of axiomatics in algebra, however, is primarily for clarity rather than for proof. The point of view taken is that beginning algebra should be primarily a study of the algebra of the rational number system with an introduction to real numbers. This is the fourth year for the geometry. It has been through three revisions and is essentially in final form. A complete teachers' manual has been prepared. The algebra is a second edition and will undergo considerable revision during the next year.

After this introduction the usual theorems of plane geometry are considered and proofs are much the same as in conventional classes. Experience has shown that students who transfer out of the Ball State geometry to other schools during the school year can adjust without difficulty to conventional courses. However, it is not easy for students with a conventional geometry background to step into the Ball State geometry course after the first few weeks. The geometry engenders a considerable amount of enthusiasm among good students and teachers. A testing program last year which compared 250 students in the experimental classes to about the same number of students in conventional classes produced results favorable to the experimental group.

Throughout the algebra the axiomatic structure is held somewhat in the background. Much is postulated that is demonstrable. In order to emphasize the structure of the algebra of the rational number system, the postulates for the set consisting of zero and the natural numbers (existence and uniqueness, commutativity, associativity, distributivity, cancellation laws, and properties of 0 and 1) are repeated for the set of integers and then reiterated for the set of rational numbers. Students are encouraged to proceed intuitively and are rarely expected to present formal proofs. However, the teacher guides the class through several correct proofs.

Some systematic study of logic is made. Concepts of set theory are utilized and the language of quantification is used. In early work equations and inequalities are treated together. In many examples the variables are restricted to finite sets for simplicity.

When the negatives of the natural numbers are introduced it is at once postulated that the old familiar laws for the counting numbers hold for these new numbers. Now the usual calculation rules, *e.g.*, $a(-b) = -(ab)$, are proved as theorems.

The conventional topics of elementary algebra, linear equations, work with formulas, computation with polynomials, factoring, graphing, ratio and proportion, fractional equations, *etc.* are treated. But, emphasis is always placed upon the kinds of numbers that are being used. For example, computations that involve fractional equations are treated as computations with rational numbers.

In the Ball State geometry classes this year are some students who have had one year of the experimental algebra. These students adjust more easily to the abstract concepts of geometry than do the students who have come through traditional algebra classes.

A GEOMETRY COURSE FOR JUNIORS

CHARLES C. BUCK, University of Alabama

The geometry course described below is intended to be a course which grows naturally out of the student's background in high school geometry. It has been taught for several years to upperclassmen at the University of Alabama. Most of the students have been mathematics majors in the College of

Arts and Sciences or in the College of Education, but qualified liberal arts students, such as majors in philosophy, are welcome. A variant of the course is regularly taught in the summer school to high school geometry teachers working on advanced degrees.

The student at the post-calculus level needs some mathematics courses which bring him back to the realization that mathematics is deductive argument. In this connection an important committee of mathematicians has said, "high school geometry is usually the only course in high school or the early years of college which is taught as a mathematical subject" [1]. It is true that there is now an increasing emphasis on deduction in the elementary courses and that computation is itself a form of deduction, but in the mind of the student this feature is obscured by the need for knowing that $D_x \ln x = 1/x$ and that $\sin 2A = 2 \sin A \cos A$. The mathematics majors, who may have hungered for logical argument through years in which the emphasis was on the formal manipulation of symbols, deserve to be told that mathematics is proof and that logical proof is a vehicle in which new territory can be discovered and explored. It is a major purpose of this course to demonstrate this.

This course starts with a study of Book One of Euclid's *Elements* [2]. We note the obvious flaws and indicate how these may be patched up. (In general, I take a positive attitude and encourage the students to pick the flaws.) We also note the branch points into the two major non-Euclidean geometries. (The clean separation that Euclid made between those theorems dependent on his parallel postulate and those which are not, led Coxeter to refer to him [3] as "the first non-Euclidean geometer.")

Having used the *Elements* to establish a frame of reference (and as a review of the relevant parts of high school geometry), we explore a little way down the path opened up by Bolyai and Lobatschewsky—far enough to cover one topic in some detail. For the foundations of hyperbolic geometry, we read the first part of Lobatschewsky's *Theory of Parallels* (bound in [4]). For the past year or so we have studied, as the additional topic, the theory of plane area, presented by means of lectures based on the article *Area in non-Euclidean geometry* [5] by Kenneth Leisenring. In other years we have studied the circles and spheres of hyperbolic space and proved the theorem that the geometry on a horosphere is Euclidean.

The point at which the path to elliptic geometry branches off from Euclid is somewhat obscured, but the students see it easily when it is pointed out to them. We explore elliptic geometry at least far enough to see that every triangle has an angular excess and that there is a natural unit of length: the polar distance. If we have studied area in the hyperbolic plane, I ask for papers on area in the elliptic plane. If we have studied circles and spheres in hyperbolic geometry (which is a larger topic than area), there is no corresponding topic of equal interest and accessibility in elliptic geometry.

By the time we have built this much of a superstructure on our patched up

Euclid, the student has quite a stake in the validity of his reasoning processes. It may therefore come as a surprise to him that the propositions "All triangles are isosceles," and "A right angle is equal to an angle which is greater than a right angle," are theorems in the system he has been using so confidently [6].

When it is realized that these paradoxes are provable because we have ignored one-dimensional geometry (linear order) and that a whole new set of axioms are needed for this, it is seen that Euclid needs more than patching up and that we must, moreover, have some assurance that no more paradoxes can crop up.

With this in mind we study the development of the axiom system in Hilbert's *Foundations of Geometry* [7]. There are theorems of various degrees of difficulty for the students to prove. In these they can get experience with the technique of rigorous proof as demanded by Hilbert's material and by their own new understanding of the meaning of proof. The way in which the flaws in Euclid are corrected in Hilbert is easily seen. With the proof of the consistency of Hilbert's axioms, the course ends.

The students are required to buy three books [2], [4], and [7] for this course. The total cost is about five dollars. The parts of these books which are not directly used in the course form a collection of valuable primary reference material.

Aside from the major purposes of orienting the student correctly in mathematics as a whole and of providing an introduction on a mature level to one branch of mathematics, this course has some worthwhile by-products. Three of these are: (1) the texts that we use are not textbooks but beautifully constructed mathematical treatises; (2) the nature of the course encourages the use of the library and helps the student to get acquainted with some of the literature of mathematics; (3) it introduces a little of the history of mathematics in a natural way and portrays the subject as continually growing and not as a static thing completed in the seventeenth century.

References

1. The 1954 Summer Writing Group of the Department of Mathematics, University of Kansas, Universal Mathematics, Part 1. Student Union Book Store, Lawrence, Kansas, p. 7. (A rearranged edition entitled *Universal Mathematics* may be obtained free of charge by writing to Professor H. M. Gehman, Mathematical Association of America, University of Buffalo, Buffalo 14, New York.)

2. T. L. Heath, The Thirteen Books of Euclid's Elements. 2nd ed., vol. I, Introduction and Book I, II, New York, 1956.

3. H. S. M. Coxeter, Non-Euclidean Geometry, University of Toronto Press, 1942, p. 2.

4. Roberto Bonola, Non-Euclidean Geometry with a supplement containing the Dr. George Bruce Halsted translations of The Science Absolute of Space by Johann Bolyai and The Theory of Parallels by Nicholas Lobatschewsky, New York, 1955.

5. Kenneth Leisenring, Area in non-Euclidean geometry, this MONTHLY, vol. 58, 1951, pp. 315–322.

6. W. W. Rouse Ball, Mathematical Recreations and Essays, 10th ed., 1922, London, pp. 45, 48, 49.

7. David Hilbert, Foundations of Geometry, reprint ed., LaSalle, Illinois, pp. 1–30.

CURRICULUM STUDIES IN MATHEMATICS

A report from the School Mathematics Study Group, submitted at the Washington Conference on the Committee on the Undergraduate Program, appears on pp. 215-216 of this issue.

CURRENT ITEMS

Standards for Materials and Equipment for the Improvement of Instruction in Science, Mathematics, and Modern Foreign Languages is the title of a bulletin recently issued by the Council of Chief State School Officers. The bulletin reports a conference on this subject held by the Chief State School Officers at Michigan State University in November 1958. Mathematics consultants to the chief state school officers at the conference were: John A. Brown, University of Delaware; Harry M. Gehman, University of Buffalo; Donovan Johnson, University of Minnesota. John R. Mayor of AAAS served as director of the conference. Copies of the bulletin are available through state departments of education.

A Yardstick for Measuring your Science and Mathematics Programs is the title of a 4-page brochure recently issued by the President's Committee on Scientists and Engineers. The brochure describes programs for grades 7 through 12 in science and in mathematics.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1345 [1958, 775]. *Correction.*

Change the last subscript in line 4 of the statement of the problem to $2x+1$.

E 1356. *Proposed by F. Leuenberger, Zuoz, Switzerland*

Let I, O, r, R, K denote the incenter, circumcenter, inradius, circumradius, and area of a triangle T. Show that $(IO)^2+K=r^2+R^2$ if and only if T is a right triangle.

E 1357. *Proposed by Albert Wilansky, Lehigh University*

Prove that for every number a the equation $x=-a+\sqrt{2}\sin[(a-x)/\sqrt{2}]$ has a unique solution.

E 1358. *Proposed by N. R. Riesenberg, Brooklyn College*

Show that the roots of the cubic equation $64x^3-192x^2-60x-1=0$ are $\cos^3(2\pi/7)\sec(6\pi/7)$, $\cos^3(4\pi/7)\sec(2\pi/7)$, $\cos^3(6\pi/7)\sec(4\pi/7)$.

E 1359. *Proposed by Leo and William Moser, Universities of Alberta and Saskatchewan*

(1) Given eight positive integers $a_1<a_2< \cdots <a_8\leqq 16$. Prove that there exists a k such that $a_i-a_j=k$ has at least three solutions.

(2) Find a set $a_1, \cdots, a_8$ for which $a_i-a_j=k$ has at most three solutions for any k.

E 1360. *Proposed by J. L. Brenner, Stanford Research Institute*

Call two elements a, b of a group "k-commutative" (B. Friedman) if every product of k factors, each factor being a or b, commutes with every other such product. Show that for every two elements a, b of a group, the set of all k for which a, b are k-commutative is an ideal in the set of nonnegative integers.

SOLUTIONS

An Enumeration of Triangles

E 1326 [1958, 526]. *Proposed by P. L. Chessin, University of Maryland*

With n straight line segments of lengths $1, 2, 3, \cdots, n$, how many nondegenerate triangles can be constructed?

Solution by C. S. Ogilvy, Hamilton College. Each time n is increased by one to an even integer, the number of new triangles added is $(n-2)^2/4$; each time to an odd integer, $(n-1)(n-3)/4$. Applying well-known summation formulas, we find that if T is the total number of triangles constructible with n segments, then

$$T = n(n-2)(2n-5)/24, \quad n \text{ even},$$

$$T = (n-1)(n-3)(2n-1)/24, \quad n \text{ odd}.$$

Also solved by Eugene Albert, P. M. Anselone, J. H. Bailey, H. F. Bennett, W. J. Blundon, Julian Braun, D. A. Breault, Robert Burton, R. W. Estus, Susan L. Friedman, Emil Grosswald, B. A. Hausmann, S.J., H. K. Hilton, A. R. Hyde, C. H. King, Morton Kupperman, W. M. McKeeman, D. C. B. Marsh, Helen M. Marston, Leo Moser, Paul Payette, N. R. Riesenberg, Ricky Ritterman, Daniel Serebrakian, and the proposer. Late solutions by Merrill Barnebey, D. R. Brillinger, Oldrich Buchta, André Dupras, S. H. Greene, Joe Lipman, Frederick Luttmann, J. B. Muskat, Walter Penney, and Benjamin Sapolsky.

Editorial Note. If a distinction is to be made between the clockwise and the counterclockwise triangles with sides a, b, c, then the answers above should be doubled. The two formulas can variously be combined into one, as for example

$$T = n(n-2)(2n-5)/24 + \{-1+(-1)^n\}/16.$$

Blundon and Kupperman employed the fact that the number of triangles with longest side m is the number of lattice points interior to the triangle with vertices $(0, m)$, $(m. m)$, $(m/2, m/2)$.

Bailey and Braun showed that T is given by the sum of the numbers in the first $n-3$ rows of the arithmetical triangle.

1							
1	1						
1	2	1					
1	2	2	1				
1	2	3	2	1			
1	2	3	3	2	1		
1	2	3	4	3	2	1	
.	.	.	.	.	.	.	.

Trisection with Clocks

E 1327 [1958, 526]. *Proposed by Marlow Sholander, Carnegie Institute of Technology*

A confirmed angle-watcher marked the angle α formed by the hands of a clock. Some time later he noticed that the hands trisected α. In how short a time could this have happened? How soon after 3:00 could one start such an experiment?

Solution by Hyman Orlin, Coast and Geodetic Survey. We have $m_0 - h_0 = \alpha$, where m_0 and h_0 are the angular positions (measured clockwise from the 12 o'clock position) of the minute and hour hands at time t_0. We seek a solution where the time interval Δt leading to a trisection is less than one hour. In this time interval either

$$(1)\qquad [h_0 + (2\pi/12)\Delta t] - h_0 = \alpha/3, \quad (m_0 + 2\pi\Delta t) - m_0 = 2\pi - \alpha/3,$$

for which $\Delta t = 12/13$ hours and $\alpha = (6/13)\pi$, or

$$(2)\qquad [h_0 + (2\pi/12)\Delta t] - h_0 = 2\alpha/3, \quad (m_0 + 2\pi\Delta t) - m_0 = 2\pi - 2\alpha/3,$$

for which $\Delta t = 12/13$ hours and $\alpha = (3/13)\pi$. Hence the shortest time interval is 12/13 hours.

The hands will form the smaller angle $\alpha = (3/13)\pi$ at t hours after 3:00 when $(0 + 2\pi t) - (\pi/2 + 2\pi t/12) = (3/13)\pi$, or $t = 57/143$. Hence the earliest one could begin the experiment is 57/143 hours, or 23 131/143 minutes, after 3:00 o'clock.

Also solved by Eugene Albert, Leon Bankoff, Merrill Barnebey and F. D. Ceruti (jointly), H. F. Bennett, Julian Braun, E. W. Brown, P. F. Clemens, P. J. Diamandis, Michael Goldberg, S. T. Gormsen, Sidney Kravitz, J. W. Layman, J. L. Leonard, D. C. B. Marsh, Helen M. Marston, L. V. Mead, G. J. Michaelides, and the proposer. Late solutions by C. D. Anderson, J. W. Baldwin, André Dupras, Joe Lipman, J. B. Muskat, and C. F. Pinzka.

Editorial Note. Most of the solvers interpreted the second question of the problem to mean (as in the above solution): How soon after 3:00 can one start an experiment which will lead to trisection *in the shortest possible time*? Braun and Marston interpreted the second question of the problem to mean: How soon after 3:00 can one start an experiment which will ultimately lead to a trisection? Under this interpretation the answer is 1 37/143 minutes after 3:00 o'clock, but one will have to wait 10 2/13 hours for the trisection. A number of the submitted solutions gave entirely different answers.

"I Shall Arise the Same, though Changed"

E 1328 [1958, 526]. *Proposed by Winton Laubach, Colorado School of Mines*

A ray from the origin intersects the circle $\rho=1$ at C and the spiral $\rho=e^{\theta}$, $\theta>0$, at S. Tangents to the circle at C and to the spiral at S intersect at P. Identify the locus of P.

Solution by C. S. Ogilvy, Hamilton College. The arc length of the spiral from the point $(1, 0)$ to S is given by

$$\int_1^{OS} [1 + \rho^2(d\theta/d\rho)^2]^{1/2} d\rho = \sqrt{2}(OS - 1) = \sqrt{2}SC = SP,$$

because SPC is an isosceles right triangle. Thus P lies on that involute of the spiral which intersects it at $(1, 0)$.

Also solved by Eugene Albert, A. P. Boblétt, Stuart Friedman, Michael Goldberg, A. G. Grace, Jr., L. I. Lowell, D. C. B. Marsh, Helen M. Marston, Paul Payette, Ricky Ritterman, D. A. Robinson, E. M. Scheuer, David Zeitlin, and the proposer. Late solutions by C. D. Anderson, J. W. Baldwin, Oldrich Buchta, J. E. Darraugh, S. H. Greene, and Joe Lipman.

Editorial Note. The locus of P is another logarithmic spiral. Its polar equation is $\rho^2=1+(1-e^{\theta})^2$. For an allied problem see E 784 [1948, 317].

Spherical Analogue of the Pythagorean Theorem

E 1329 [1958, 526]. *Proposed by Jose Gallego-Diaz, Vanderbilt University*

A spherical square is a spherical quadrilateral whose four sides are equal and whose four angles are equal. If we let a, b, c denote the areas of the spherical squares constructed on the legs and hypotenuse of a right spherical triangle, show that

$$\tanh^{-1}(\sin c/4) = \tanh^{-1}(\sin b/4) + \tanh^{-1}(\sin a/4).$$

Solution by D. A. Robinson, University of Wisconsin. Let the sphere have radius 1 and let a_1 be the angular measure of the side of the right spherical triangle on which is constructed the spherical square of area a. Each of the four equal angles of this square is $a/4+\pi/2$. By considering the two isosceles spherical triangles which comprise this square and applying the cosine law for angles, one obtains upon solving for $\cos a_1$ and employing standard trigonometric identies

$$\cos a_1 = \frac{1 - \sin a/4}{1 + \sin a/4}.$$

If b_1 and c_1 are defined similarly, analogous expressions are obtained for $\cos b_1$ and $\cos c_1$. By Napier's Rules, $\cos c_1 = \cos a_1 \cos b_1$. Hence

$$\frac{1 - \sin c/4}{1 + \sin c/4} = \frac{1 - \sin a/4}{1 + \sin a/4} \cdot \frac{1 - \sin b/4}{1 + \sin b/4}.$$

Taking the natural logarithm of both sides, multiplying both sides by $-1/2$, and using the fact that

$$\tanh^{-1} x = \frac{1}{2} \ln \frac{1+x}{1-x},$$

the desired result is obtained.

Also solved by Eugene Albert, D. C. B. Marsh, Paul Payette, Ricky Ritterman, and the proposer. Late solutions by D. R. Brillinger, Joe Lipman, and Benjamin Sapolsky.

An Infinite Product Representing e

E 1330 [1958, 527]. *Proposed by J. L. Pietenpol, Columbia University*

Find the value of the infinite product $\prod_{n=1}^{\infty}(1+1/a_n)$, where $a_1=1$, $a_n=n(a_{n-1}+1)$.

Solution by N. J. Fine, Institute for Advanced Study. The nth partial product is

$$\begin{aligned} P_n &= [(a_1+1)/a_1][(a_2+1)/a_2]\cdots[(a_n+1)/a_n] \\ &= [(a_1+1)/a_2][(a_2+1)/a_3]\cdots[(a_{n-1}+1)/a_n](a_n+1) \\ &= (a_n+1)/n!. \end{aligned}$$

Now

$$\begin{aligned} P_n - P_{n-1} &= [(a_n+1)/n!] - [(a_{n-1}+1)/(n-1)!] \\ &= [(a_n+1)/n!] - [a_n/n!] = 1/n!. \end{aligned}$$

Hence

$$\begin{aligned} P_n &= P_1 + 1/2! + 1/3! + \cdots + 1/n! \\ &= 1 + 1/1! + 1/2! + 1/3! + \cdots + 1/n! \end{aligned}$$

and $\lim_{n\to\infty} P_n = e$.

Also solved by Eugene Albert, R. G. Albert, Merrill Barnebey, David Barr, Julian Braun, D. A. Breault, R. L. Causey, P. L. Chessin, R. J. Cormier, Underwood Dudley, E. L. Ellis, F. A. Ficken, Fred Galvin, Michael Goldberg, A. G. Grace, Jr., Emil Grosswald, J. H. Hodges, J. Hooley, E. H. Kanning III, R. P. Kelisky, D. A. Kearns, P. G. Kirmser, A. G. Konheim, M. I. Knopp and E. H. Scheuer (jointly), Morton Kupperman, Gerald Leibowitz, R. W. McChesney, D. C. B. Marsh, Clifford Marshall, G. I. Michaelides, Leo Moser, Joseph Muskat, C. S. Ogilvy, Paul Payette, Ricky Ritterman, D. A. Robinson, H. D. Ruderman, Jeff Scargle, R. E. Shafer, R. P. Tapscott, W. F. Trench, Chih-yi Wang, Kenneth Williams, David Zeitlin, and the proposer. Late solutions by H. F. Bechtell, D. R. Brillinger, Oldrich Buchta, J. E. Darraugh, D. A. Freedman, S. H. Greene, Walter Penney, C. F. Pinzka, and Benjamin Sapolsky.

Wang showed, more generally, that if $a_n = n(a_{n-1}+z^{n-1})$, where z is any complex number, then the value of the infinite product is e^z.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4834. *Proposed by Oystein Ore, Yale University*

It is well known that the number 30 is the largest integer such that the set of reduced residues (mod 30) includes no compositive numbers. Determine all integers n such that the $\phi(n)$ reduced residues (mod n) are powers of primes.

4835. *Proposed by F. H. Northover, Carleton University, Ottawa, Canada*

Prove

$$\sum_{\nu=0}^{\infty}\binom{n+\nu-1}{\nu}^2 x^{\nu} = (1-x)^{1-2n}\sum_{k=0}^{n-1}\binom{n-1}{k}^2 x^k.$$

4836. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Consider the equation $x^2-dy^2=-1$ where d is a nonsquare integer, and suppose that $x=a$ and $y=b$ is any solution in integers. Show that there are infinitely many solutions (x, y) in which x is a multiple of a and y is a multiple of b.

4837. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Consider the solutions (x, y) of the equation $x^2-dy^2=-1$. Show that every x is relatively prime to every y.

4838. *Proposed by R. L. Duncan, Pennsylvania State University*

Let $A=\{a_n\}$ and $B=\{b_n\}$ denote strictly monotone sequences of positive integers and define the product $AB=\{b_{a_n}\}$. Also, let $\delta(A)=\lim A(n)/n$ denote the natural density of A, where $A(n)$ is the number of elements of A not exceeding n, and let S denote the set of all sequences A for which $\delta(A)$ exists. It has been shown by Niven that if $\delta(A)$ and $\delta(B)$ exist then $\delta(AB)$ exists and $\delta(AB)=\delta(A)\delta(B)$. It is easy to see that S is a semigroup with a unit and that the mapping $A\rightarrow\delta(A)$ is a homomorphism of S onto the unit interval. Is this the only such homomorphism? If there are others can they be completely determined and can they be eliminated by requiring that two sequences have the same image whenever they differ in a finite number of places?

4839. *Proposed by R. G. Bushman, University of Wichita*

Let $d(n)$ denote the number of divisors of n, and $[x]$ denote the greatest integer $\leqq x$. Determine the validity of the identity

$$\int_1^x\left(\frac{1}{v}\sum_{n=1}^{[v]} d(n)\right)dv \equiv \int_1^x\left(\frac{[v]}{v}\left[\frac{x}{v}\right]\right)dv, \qquad x \geqq 1.$$

SOLUTIONS

Binomial Theorem in an Associative Algebra

4789 [1958, 370]. *Proposed by Kurt Mahler and P. M. Cohn, the University, Manchester, England*

Let A be an associative algebra over a field of characteristic zero. If $(x+y)^n = \sum_r \binom{n}{r} x^r y^{n-r}$ for some $n \geqq 2$ and for all x and y in A, is it true that $(x+y)^{n+1} = \sum_r \binom{n+1}{r} x^r y^{n+1-r}$ for all x and y in A?

Solution by R. C. Lyndon, University of Michigan. We prove the proposition also for fields of characteristic p (with the assumption $n \leqq p$).

Write $(x+y)^n = x^n + S_1 + \cdots + S_{n-1} + y^n$, a sum of homogeneous parts. The hypothesis gives

$$[nx^{n-1}y - S_1] + \left[\binom{n}{2} x^{n-2}y^2 - S_2\right] + \cdots + [nxy^{n-1} - S_{n-1}] = 0,$$

where each bracketed term is homogeneous. Replacing x in this equation successively by $1x, 2x, \cdots, (n-1)x$ yields a system of equations with nonvanishing Vandermonde determinant. It follows, in particular, that $nx^{n-1}y - S_1 = 0$. Now $S_1 = x^{n-1}y + S'x$, where $S' = \sum x^h y x^k$, $h+k = n-2$. Therefore $(n-1)x^{n-1}y = S'x$, and $(n-1)x^n y = xS'x$. Since $(x+y)^n = (y+x)^n$, the hypothesis is symmetric under interchange of left and right, and therefore by symmetry we have $(n-1)yx^n = xS'x$. It follows that $(n-1)x^n y = (n-1)yx^n$, and that $x^n y = yx^n$, for all x, y. In view of this, $(x+y)^{n+1} = (x+y)(x+y)^n = x(x+y)^n + y(x+y)^n = x(x+y)^n + (x+y)^n y$. Substituting for $(x+y)^n$ the expression given by the hypothesis and collecting coefficients now gives the desired conclusion.

Also solved by L. Carlitz, and the proposers.

A Sum of Legendre Symbols

4790 [1958, 370]. *Proposed by Leonard Carlitz, Duke University*

Let p be a fixed prime >2 and put $\psi(x) = (x/p)$, the Legendre symbol. Evaluate the sum

$$N_r(a) = \sum_{x_1, \cdots, x_r = 1}^{p-1} \psi\{x_1 \cdots x_r(a - x_1 - \cdots - x_r)\}.$$

Solution by Robert Breusch, Amherst College, Massachusetts. We first show that

$$N_1(a) = \begin{cases} -\psi(-1) & \text{if } a \not\equiv 0, \\ (p-1)\psi(-1) & \text{if } a \equiv 0, \end{cases} \tag{1}$$

where, as in all that follows, congruences are taken modulo p. Obviously $N_1(0) = \sum_{x=1}^{p-1} \psi(-x^2) = (p-1)\psi(-1)$. Now, let $a \not\equiv 0$, $b \not\equiv 0$, $a \not\equiv b$, and let s satisfy $b \equiv s \cdot a$. Then

$$N_1(a) = \sum_{x=1}^{p-1} \psi(x(a-x)) = \sum_{x=1}^{p-1} \psi(sx(sa-sx)) = \sum_{y=1}^{p-1} \psi(y(b-y)),$$

because $y \equiv sx$ again takes all the values from 1 to $p-1$. Thus $N_1(a) = N_1(b)$. But

$$\sum_{a=0}^{p-1} N_1(a) = \sum_{x=1}^{p-1} \psi(x) \sum_{a=0}^{p-1} \psi(a-x) = \sum_{x=1}^{p-1} \psi(x) \cdot 0 = 0.$$

Thus $\sum_{a=1}^{p-1} N_1(a) = -(p-1)\psi(-1)$, and therefore $N_1(a) = -\psi(-1)$ for $a \not\equiv 0$.

We wish to prove that, for $k \geqq 1$,

$$N_{2k}(a) = \psi(-1)^k \cdot \psi(a) \cdot p^k; \quad N_{2k-1}(a) = \begin{cases} -\psi(-1)^k \cdot p^{k-1} & \text{if } a \not\equiv 0, \\ (p-1)\psi(-1)^k \cdot p^{k-1} & \text{if } a \equiv 0. \end{cases} \tag{2}$$

The proof follows by induction on k. The definition implies the reduction formula

$$N_r(a) = \sum_{x_r=1}^{p-1} \psi(x_r) \cdot N_{r-1}(a - x_r).$$

The second formula of (2) is true for $k=1$ by (1). Assume it holds for $k=m$. Then

$$\begin{aligned} N_{2m}(a) &= \sum_{x_{2m}=1}^{p-1} \psi(x_{2m}) \cdot N_{2m-1}(a - x_{2m}) \\ &= \psi(a)\psi(-1)^m p^{m-1}(p-1) + \sum_{x_{2m} \not\equiv a} \psi(x_{2m})\psi(-1)^m(-p^{m-1}) \\ &= \psi(a)\psi(-1)^m p^{m-1}(p-1) + \psi(a)\psi(-1)^m p^{m-1} = \psi^{(a)}\psi(-1)^m p^m. \end{aligned}$$

This argument is valid for $a \equiv 0$ as well as for $a \not\equiv 0$. Furthermore

$$\begin{aligned} N_{2m+1}(a) &= \sum_{x_{2m+1}=1}^{p-1} \psi(x_{2m+1}) \cdot N_{2m}(a - x_{2m+1}) \\ &= \psi(-1)^m \cdot p^m \sum_{x=1}^{p-1} \psi(x(a-x)) \\ &= \psi(-1)^m p^m N_1(a) = \begin{cases} \psi(-1)^{m+1} \cdot p^m \cdot (-1) & \text{if } a \not\equiv 0, \\ \psi(-1)^{m+1} p^m (p-1) & \text{if } a \equiv 0. \end{cases} \end{aligned}$$

The proof is thus complete.

Also solved by N. J. Fine, Emil Grosswald, J. H. Hodges, Joe Lipman, and the proposer.

An Elementary Inequality

4791 [1958, 370]. *Proposed by R. C. Lyness, Blackpool, England*

If α, β, γ are real and $\alpha^3+\beta^3+\gamma^3=0$, prove

$$[\sum (\beta-\gamma)^2][\sum \alpha^4] \geqq [\sum \alpha^2]^3.$$

Solution by Leonard Carlitz, Duke University. Put $s_n=\sum \alpha^n$. Then

$$\begin{aligned}\Delta &= (\alpha-\beta)^2(\beta-\gamma)^2(\gamma-\alpha)^2\\ &= \begin{vmatrix} 1 & 1 & 1\\ \alpha & \beta & \gamma \\ \alpha^2 & \beta^2 & \gamma^2 \end{vmatrix}^2 = \begin{vmatrix} 3 & s_1 & s_2 \\ s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \end{vmatrix}\\ &= (3s_2-s_1^2)s_4 - s_3(3s_3-s_1s_2)+s_2(s_1s_3-s_2^2).\end{aligned}$$

Since $\sum (\alpha-\beta)^2=2s_2-2\sum \alpha\beta=3s_2-s_1^2$, it follows that

$$\sum (\alpha-\beta)^2\cdot \sum \alpha^4 - (\sum \alpha^2)^3 = (3s_2-s_1^2)s_4 - s_2^3 = \Delta + s_3(3s_3-2s_1s_3).$$

Therefore, when $s_3=0$, $\sum (\alpha-\beta)^2\cdot \sum \alpha^4-(\sum \alpha^2)^3=\Delta\geqq 0$.

Also solved by D. M. Brown, A. G. Clark, James Clunie, D. C. B. Marsh, Yoshio Matsuoka, D. S. Passman, and the proposer.

Reducible Linear Differential Operators

4792 [1958, 370]. *Proposed by D. J. Newman and M. S. Klamkin, AVCO Research, Wilmington, Mass.*

Show that the following operators are reducible:

(1) x^nD^{2n}, (2) $x^{2n}D^n$,

and thus solve the differential equations

(1a) $[x^nD^{2n}-\lambda]y=0$, (2a) $[x^{2n}D^n-\lambda]y=0$.

Solution by D. C. B. Marsh, Colorado School of Mines. x^nD^{2n} may be factored as $\{xD^2+(1-n)D\}^n$, as is most easily verified after setting $x=e^z$. This yields the set of reduced equations

$$\{xD^2+(1-n)D\}y=r_iy,$$

where the r_i range over the nth roots of λ. This is one form of the Bessel equation whose solution is familiar. The complete solution of (1a) is therefore

$$y = x^{n/2}\sum_{i=1}^{n}\{A_iJ_n(2r_i^{1/2}\sqrt{-x})+B_iY_n(2r_i^{1/2}\sqrt{-x})\},$$

where the r_i range as before, A_i and B_i are arbitrary constants. $x^{2n}D^n$ may be factored as $\{x^2D+(1-n)x\}^n$, yielding the set of reduced equations $\{x^2D+(1-n)x\}y=r_iy$, where the r_i range over the nth roots of λ. This equation is

separable, giving the complete solution of (2a) as

$$y = \sum_{i=1}^{n} C_i x^{n-1} e^{-r_i/x}.$$

Also solved by H. E. Stelson and by the proposers.

Editorial Note. The proposers remark that both equations have been solved previously but not so simply. (1a) is due to Lommel, see Watson, *Bessel Functions*, p. 106. (2a) is given in Kamke, *Differential Gleichungen*, p. 540, with references to Steen and Krug.

An Infinite Sequence of Inscribed Polygons

4793 [1958, 451]. *Proposed by M. H. Lietzke and C. W. Nestor, Jr., Oak Ridge National Laboratory*

In *Mathematics and the Imagination*, Kasner and Newman present the following problem. An equilateral triangle is inscribed in a circle of unit radius, another circle is inscribed in the triangle, a square in this circle, *etc.* Continue the procedure, increasing the number of sides of the regular polygon each time by one. As the number of sides of the inscribed polygon increases, the radii of the shrinking circles converge to a definite limiting value. Find this value. The proposed answer (approximately 1/12) seems to be considerably in error. What should it be?

Solution by Julian Braun, Las Vegas, Nevada. Let L be the required limit. Since the ratio of the apothem to the radius of a regular polygon of k sides is $\cos(\pi/k)$ it follows that

$$L = \lim_{n\to\infty} \prod_{k=3}^{n} \cos(\pi/k).$$

Let $\log L = \sum_{k=3}^{n} \log\cos(\pi/k) + R_n$. From graphical considerations it is seen that

$$\left| \int_{n+1/2}^{\infty} \log\cos\frac{\pi}{k}\, dk \right| < |R_n| < \left| \int_{n}^{\infty} \log\cos\frac{\pi}{k}\, dk - \frac{1}{2}\log\cos\frac{\pi}{n} \right| \cdot$$

By setting $n=12$ (*i.e.*, summing ten terms with estimate of the remainder) calculation with the above formulas yields

$$0.114917 < L < 0.114998.$$

Setting $n=22$ yields $0.114938 < L < 0.114951$. One can carry the process further to obtain 0.11494 correct to five significant figures.

The problem is equivalent to one involving circumscribed polygons solved in *School Science and Mathematics*, 1953, pp. 575–6.

Also solved by E. H. Blum, A. P. Boblétt, A. G. Clark, Michael Goldberg, Emil Grosswald, Frank Herlihy, Edgar Karst, A. J. Kokar, T. M. Little, L. V. Mead, K. K. Norton, F. R. Urbanus, Alan Wayne, and the proposers.

Editorial Note. If $\log\cos(\pi/k)$ is replaced by its Taylor series expansion and the order of sum-

mation reversed, we can reduce the result to a form given by several solvers

$$-\log L = \sum_{n=1}^{\infty} \frac{2^{2n}-1}{n} \zeta(2n)[\zeta(2n) - 1 - 2^{-2n}],$$

where ζ is the Riemann zeta function. Using tabulated values (*e.g. Jahnke and Emde*, p. 273) the desired approximation to the limit is obtained.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

An Introduction to Fourier Analysis and Generalized Functions. By M. J. Lighthill. Cambridge University Press, London, 1958. 79 pp. $3.50.

The author's announced purpose is to provide a simple account of the principal and useful facts about Fourier analysis on the line, or what might more properly be called generalized Fourier analysis. Distributions or generalized functions, the term used in the book, play an important if not predominant role in the author's treatment of the subject.

A generalized function is defined in a rather simple way in terms of the sequences which approximate it, in much the same fashion as a real number is defined as an equivalence class of Cauchy sequences of rationals. The resulting objects are then shown to have derivatives of all orders, Fourier transforms, and appropriate formal properties.

One chapter is devoted to the study of those generalized functions which behave like powers of x or $|x|$, and products of these by logarithmic factors. The Fourier transforms of these particular generalized functions are obtained and are tabulated in convenient form. In this part, however, the author is forced to make some rather arbitrary definitions and the theory loses some of its naturalness.

In the next chapter a rather useful and general technique is developed for obtaining asymptotic expressions for the behavior at infinity of the Fourier transforms of well-behaved generalized functions. Trigonometric series are treated in the final chapter as periodic generalized functions on the line.

On the whole the book strikes the reviewer as being somewhat short of its goal of simplicity. Brevity has replaced readability in many instances, and a number of the proofs are too short to be easily grasped; nevertheless the determined reader will gain much from the book.

R. A. KUNZE
Massachusetts Institute of Technology

College Plane Geometry. By Edwin M. Hemmerling. Wiley, New York, 1958. 310 pp. $4.95.

This book is written for college students who have not studied geometry previously and who will probably not elect any further course in mathematics. It covers much more than the elementary concepts and mensuration formulas of plane geometry. Many historical facts are noted to inform the student of the contributions of mathematics to our culture. The process of deductive reasoning and the postulational method are stressed. Particular attention is paid to reasoning in nongeometric situations. Special exercises are included for the purpose of sharpening the students' critical powers.

The material is well selected and well arranged. Photographs which illustrate important geometric concepts are used effectively. An exceptionally large number of well-drawn figures accompanies both text and exercises. The figures are widely spaced on the pages. Many suitable exercises are included. Beginning teachers, or persons using the text for self-instruction, will find the collection of Summary Tests a valuable feature.

The title is too modest. The book contains chapters on inequalities and variation, trigonometry of right triangles and solid geometry. The latter chapter covers all essential definitions and postulates and the statements (without proofs) of basic theorems.

The reviewer noted several minor defects. Readability of the tables of square roots would be improved by the insertion of vertical rulings. Axioms 13 and 14 on inequalities hold only if the multiplier or divisor is positive. The undefined nature of *point* and *line* could well have been emphasized in the text proper rather than being relegated to a footnote.

These minor criticisms are not meant to detract from the overall excellence of the work. The author has written a fine text, and the publisher has prepared it in an attractive form. They are to be commended for preparing a text in plane geometry for college students. This stimulating approach may well serve as a means of retaining plane geometry as a valuable subject of instruction.

H. S. KALTENBORN
Memphis State University

Combinations of Observations. By W. M. Smart. Cambridge University Press, New York, 1958. 253 pp. $6.50.

This book deals with the mathematical formulation of laws of behavior one encounters in various branches of experimental science and is much concerned with the treatment of observations and measures which are subject to accidental errors. One will also find an introduction to the concept of curve fitting.

The content is divided into nine chapters and three appendices.

Chapter I gives a very good and clear discussion of the notion of frequency distributions and the statistical methods for their theoretical and numerical characterization. The normal distribution, its basis and certain integrals con-

nected with it are discussed quite adequately. Sheppards' corrections to moments are presented and it is most gratifying to note that their limitations are indicated.

The next five chapters discuss certain fundamentals of the theory of errors of observation and the principle of least squares. The basic concepts of probability theory and several derivations of the normal law are clearly presented. Measure of precision (reliability) and the relative importance of observations is carefully discussed. It is gratifying to find in this book a discussion of the representation of a frequency distribution by polynomials, by a trigonometric series, by the Gram-Charlier series and by the Pearson system of curves. This, in the reviewer's opinion, is very important. This latter is found in Chapter VII.

Chapter VI concerns itself with equations of condition in several unknowns including equations of different weights, the concept of residuals, and the formal solution of the normal equations. Several interesting examples are given.

Chapter VIII deals with the correction of statistics and the possible rejection of an observation.

Chapter IX gives an adequate introduction to the conception of correlation.

The book is clearly and well written and is highly recommended as a text for a one-semester course, having calculus as a prerequisite, for students in mathematics and the sciences.

FRANK M. WEIDA
The George Washington University

Analytic Geometry. By E. J. Purcell. Appleton-Century-Crofts, New York, 1958. 289 pp. $4.50.

This new analytic geometry text appears (to one who has not yet taught from it) to be a superior, albeit conventional, book. Among its merits are its good format, its clear printing (the theorems being clearly set apart from discussion), and its length. The book is designed to get a student to the calculus as smoothly as possible. It is short enough to give the instructor no trouble in finishing the material in one semester.

There is no attempt to use vectors, nor to give a simultaneous introduction to plane and solid analytic geometry. The chapter on curve sketching is better and more complete than that in many books; for instance, it uses the theorem from higher plane curves on horizontal and vertical asymptotes of an algebraic curve. However, in this chapter a section on cubic curves adds little. Material on curve fitting and on cycloids has been omitted. In the opinion of this reviewer it would have been better to have included these topics and to have omitted the chapter on tangents to curves.

The student should find the text clear, and he should be able to read it with understanding.

ALICE T. SCHAFER
Connecticut College

A Short Course in Differential Equations and *Elementary Differential Equations.* Both by E. D. Rainville. Macmillan, New York, 1958. x+259 pp. $4.50 and xii+449 pp. $5.50.

The first of these two books was designed for a semester course in ordinary differential equations. Its seventeen chapters cover the usual topics exceptionally well with an adequate number of exercises—approximately 1250. The author's aim in trying to exhibit both the techniques for obtaining solutions and the basic ideas and theory behind these techniques is fully realized.

The first seventeen chapters of the second book are identical to the above; nine additional chapters are included in order to make this text suitable for a two-semester course in differential equations. Of special note are the chapters on the Power Series Method and Solutions Near Regular Singular Points. Chapters on Equations of Hypergeometric Type—covering a short introduction to the gamma and factorial functions, the hypergeometric and confluent hypergeometric equations, Bessel's equations, and the polynomials of Legendre, Hermite, and Laguerre—and Orthogonal Sets contain material not included in the first edition. The text concludes with sections on Partial Differential Equations, Fourier Series, and Boundary Value Problems.

Both texts contain excellent treatments of the material considered, especially from the point of view of the student. Explanations are clear and examples are helpful. The difference of only one dollar in price between these two books leaves no doubt in the mind of the reviewer that the Elementary Differential Equations book is preferable under any circumstances.

D. TRIFAN
University of Arizona

Modern Mathematical Methods and Models, Volume 1. Multicomponent Methods. By the Dartmouth College Writing Group. Photolithoprinted, 1958. iii+327 pp. (Any interested person may get a free copy by writing to Professor H. M. Gehman, Mathematical Association of America, University of Buffalo, Buffalo 14, N. Y.)

"This is the first of two volumes of experimental-text materials written by the Dartmouth College Writing Group for the Committee on the Undergraduate Program (CUP) of the Mathematical Association of America. These two volumes form the basis of a CUP course for sophomores specially interested in the biological and social sciences. Volume 1 covers the first semester of this course."

Volume 1 consists of four units of approximately equal length. The first unit, on matrix and vector algebras, develops the necessary algebraic language and tools for both volumes. Topics discussed include systems of linear equations and inequalities, inverse and characteristic values of a matrix, and linear and affine transformations.

Unit two is on calculus and finite differences. Extensive use is made in this unit of *A Handbook for Calculus, Difference and Differential Equations* by E. J. Cogan and R. Z. Norman (New York, 1958). Various types of differential and difference equations are solved using the *Handbook*.

The third unit is on multivariable functions, with emphasis on linear and quadratic functions. Topics discussed include linear and quadratic approximations of functions, canonical quadratic forms, and convex sets and functions.

The final unit is on optimization problems. After a discussion of the usual extrema problems of the calculus, extrema problems of a function subject to constraint equations and inequalities are presented. This leads naturally into linear programming, the final topic of this volume.

This book is written in a pleasingly informal way, with many interesting examples from economics, biology, *etc.*, throughout. It is the opinion of the reviewer that the authors have succeeded admirably in performing their assigned task.

RICHARD JOHNSON
Smith College

Introduction to The Theory of Determinants and Matrices. By Edward Tankard Browne. The University of North Carolina Press, Chapel Hill, 1958. 270 pp. $7.50.

The style of this book is that of formally stated theorem followed by proof. The approach is that of the explicit writing out of the components of matrices, and numerical illustrations are frequent. Among the attractive features of the book are the clear exposition (with some exceptions as in the treatment of determinants), the extensive lists of problems and the generally clear printing and organization of material. Many teachers will welcome the quite explicit treatment (illustrated by worked numerical examples) of lambda matrices, elementary divisors and related topics. Some items in the book will be novel and of especial interest for numerical computations with matrices. Thus the signature of a quadratic form is used (p. 136) to determine the number of distinct pairs of conjugate roots of a polynomial and this is applied in the problems with good effect to the study of the cubic and quartic cases.

Although vector spaces are defined (p. 51), there is very little use of the concept and a matrix is rarely regarded as an operator on a space. The concept of dual space appears not to be used at all. For a course serving as a foundation for further study of algebra, this point of view makes the book unsatisfactory despite its positive virtues for the explicit numerical treatment of matrix problems. In a few places the basic concept is not brought out clearly. For example, the Kronecker reduction of a quadratic form is proved with some care, yet the basic idea that the process consists essentially in completing the square is not emphasized.

The material covered includes that common to most current books on matrix algebra and also considers the equivalence of pairs of bilinear forms. The last

two chapters contain an interesting and useful treatment of commutative matrices and systems of differential equations.

The book has enough novelty in content and treatment to deserve a place in every general mathematics library. Teachers of courses in applied matrix algebra may well consider it seriously as a possible text and will certainly want a copy as a source of examples of alternative and often excellent proofs of standard results and of supplementary material usually not included in a first course.

WALLACE GIVENS
Wayne State University

Statistics Manual With Examples Taken From Ordnance Development. By Edwin L. Crow, Frances A. Davis, Margaret W. Maxfield, U. S. Naval Ordnance Test Stations, China Lake, California, 1955. 279 pp. $6.00 (Copies of this manual may be obtained by the public from Technology Division, Office of Technical Services, Department of Commerce, Washington 25, D. C. Order number PB131483.)

This is a manual written internally in the U. S. Naval ordnance test station. It is a "cookbook" outlining techniques of analyzing some "standard" statistical problems. Though the examples are chosen from ordnance development, the manual should be useful to anyone, (including a social scientist, for example), who needs a summarizing handbook of techniques. The manual tends to be old fashioned, ignoring many recent developments of statistical theory which by now are quite accessible for the experimenter. In fact, it seems to be a shorter and more concise version of dozens of elementary statistics books that already exist. Its main emphasis is on "how to do it" rather than on theory. It should prove useful for a statistically sophisticated scientist who needs a compact collection of certain recipes.

Technical Comments. The main topics covered are standard analysis of variance, regression, and goodness-of-fit techniques, and quality control methods. Confidence set procedures are given but the very useful multiple confidence procedures are never mentioned. There is very little about nonparametric procedures. There is some discussion of power (operating characteristic), but there is little suggestion of the applicability of these ideas to the more interesting problems of the manual.

MEYER DWASS
Northwestern University

BRIEF MENTION

The Neutrino. By James S. Allen. Princeton University Press, 1958. vii+168 pp. $4.50.

A survey of the information currently available on the neutrino, including reports on the progress of investigations through March, 1958. Both pre-parity and the newer post-parity experiments are discussed.

Mathematics for Engineers (Part 2, 5th Ed.). By W. N. Rose. Chapman and Hall, London, 1958. xii+403 pp. 25s. (About $3.50).

Library of Mathematics Books. Edited by Walter Ledermann. The Free Press of Glencoe, Ill., 1958. $4.00 per set.

This series of books seems to be intended primarily for cookbook users of mathematics rather than mathematicians.

Differential Calculus. By P. J. Hilton. vii+56 pp. $1.25. *Sequences and Series.* By J. A. Green. viii+78 pp. $1.25. *Linear Equations.* By P. M. Cohn. viii+74 pp. $1.25. *Elementary Differential Equations and Operators.* By G. E. H. Reuter. viii+67 pp. $1.25. Routledge & Kegan Paul, London.

Some Problems in Chemical Kinetics and Reactivity, Volume I. By N. N. Semenov, Translated from the Russian by Michel Boudart. Princeton University Press, 1958. xii+239 pp. $4.50.

One of the Russian translations prepared under contract with the National Science Foundation, reproduced by photo-offset from typewritten copy.

CORRECTION

The book *Notes on Analog-Digital Conversion Techniques*, reviewed in this MONTHLY, vol. 66, 1959, p. 76, should have been listed as a joint publication of Wiley and The Technology Press and the publication date should have been 1958.

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

SYMPOSIUM ON BOUNDARY PROBLEMS IN DIFFERENTIAL EQUATIONS

A symposium on Boundary Problems in Differential Equations, with especial reference to recent developments in this field, will be held by the Mathematics Research Center, U. S. Army, at the University of Wisconsin, April 20–22, 1959. Invited speakers, about twenty in number, will each present a thirty-minute paper. Both ordinary and partial differential equations will be considered, the emphasis to be upon methods that are potentially adapted to computation.

Among the speakers will be L. Collatz, G. Fichera, L. Fox, W. T. Koiter, J. Schröder, I. N. Sneddon, R. Bellman, G. Birkhoff, H. Bueckner, R. Courant, J. B. Diaz, J. Douglas, K. Friedrichs, P. Garabedian, B. A. Troesch, R. Varga, C. Wilcox, D. Young.

Persons interested in attending the symposium may receive the program and other details by writing to R. E. Langer, Director, Mathematics Research Center, U. S. Army, 1118 West Johnson Street, Madison 6, Wisconsin.

COLLEGE TEACHING AS A CAREER

A pamphlet entitled "College Teaching as a Career," published by the American Council on Education, is available for free distribution to members of the Association and other interested persons. Requests for copies should be sent to: Harry M. Gehman, Secretary-Treasurer, Mathematical Association of America, University of Buffalo, Buffalo 14, New York.

PERSONAL ITEMS

University of Cincinnati: Professor A. J. Macintyre, University of Aberdeen, Scotland, is Visiting Research Professor of Mathematics for the academic year 1958–59; Professor I. A. Barnett, on a year's sabbatical leave, is at the Institute for Advanced Mathematics, Dublin, Ireland; Professor G. M. Merriman, head of the department, is on leave of absence the second semester as Visiting Professor of Mathematics, University of North Carolina; Professor Harry Miller is retiring at the end of the current academic year; Mr. Wesley Love is on leave, pursuing an advanced degree.

Mississippi State University: Mr. F. P. Williams, Rear-Admiral (Ret.) U.S.N., has been appointed Instructor; Mr. H. H. Hight has been appointed Temporary Instructor; Assistant Professor Monica Goen has been promoted to Associate Professor; Assistant Professor M. M. Temple has been promoted to Associate Professor; Miss Mary J. Koelz has been promoted to Assistant Professor.

North Texas State College: Associate Professor George Copp has been promoted to Professor; Associate Professor H. C. Parris has been promoted to Professor and Director of the Mathematics Department.

University of Wisconsin-Milwaukee: Dr. Nicolas Artemiadis, University of Paris, has been appointed Assistant Professor; Mr. M. C. Austin, Lecturer at the Imperial College, University of London, England, has been appointed Visiting Lecturer for one year; Mr. W. G. Collar, University of Wisconsin, has been appointed Instructor.

Mr. R. D. Aeder, formerly Applied Science Representative, International Business Machines Corporation, Los Angeles, California, has been appointed Manager, Special Projects Section.

Dr. W. A. Al-Salam, Duke University, has been appointed Assistant Professor at the University of Baghdad, Iraq.

Professor William Bender, University of South Dakota, has accepted the position of Consulting Physicist, Bendix Aviation Corporation, Ann Arbor, Michigan.

Dr. F. G. Brauer, University of Chicago, has been appointed Lecturer at the University of British Columbia.

Dr. R. W. Brown, Oregon State College, has accepted a position as Research Analyst with Boeing Airplane Company, Seattle, Washington.

Assistant Professor R. K. Butz, Colorado State University, has been appointed Associate Professor at Alabama Polytechnic Institute.

Mr. H. S. Christian, Jr., Sentinel Radio Corporation, Evanston, Illinois, has accepted the position of Chief Engineer, Diamond Power Specialty Corporation, Lancaster, Ohio.

Dr. D. R. Clutterham, Convair, Fort Worth, Texas, has accepted the position of Chief Engineer with the Martin Company, Orlando, Florida.

Mr. A. E. Dean, University of North Carolina, has been appointed Assistant Professor at Rollins College.

Dr. F. W. Donaldson, General Electric Company, Philadelphia, Pennsylvania, has accepted the position of Head, Programming and Operations Section, Ramo-Wooldridge, Inc., Fort Huachuca, Arizona.

Mr. J. F. Elliott, Dobyns Bennett High School, Kingsport, Tennessee, has been ap-

pointed Teacher, Main Township High School, Des Plaines, Illinois.

Dr. A. L. Fass, Queens College, has been promoted to Assistant Professor.

Associate Professor W. H. Fuchs, Cornell University, has been promoted to Professor.

Associate Professor Leonard Gillman, on leave from Purdue University 1958–59 on a Guggenheim Fellowship, is a Member at the Institute for Advanced Study.

Mrs. Mildred W. Going, Army Map Service, Washington, D. C., has accepted a position as Associate Mathematician with the Applied Physics Laboratory, Johns Hopkins University, Silver Spring, Maryland.

Miss Louisa S. Grinstein, Hunter College, has accepted a position as Systems Analysis Engineer with Republic Aviation Corporation, Mineola, New York.

Associate Professor John Gurland, Iowa State College of Agriculture and Mechanic Arts, has been promoted to Professor.

Mr. C. D. Gustafson, Abilene Christian College, has been appointed Instructor of Advanced Mathematics, Nevada Union High School, Grass Valley, California.

Mr. J. V. Hancock, University of Georgia, has been appointed Associate Professor, Wofford College.

Dr. P. W. Healy, Phillips Petroleum Company, Atomic Energy Division, Idaho Falls, Idaho, has accepted a position as Head of the Computing Section, Aerojet-General Nucleonics, San Ramon, California.

Mr. R. T. Heimer, Pennsylvania State University, has accepted a position as Head of the Mathematics Department, Lockhaven State Teachers College.

Dr. Rufus Isaacs, Hughes Aircraft Company, Culver City, California, has accepted a position as Mathematician, Institute of Defense Analysis, The Pentagon, Washington, D. C.

Dr. H. H. Johnson, Stanford University, has been appointed Instructor at Princeton University.

Dr. A. B. Lehman, Case Institute of Technology, has been promoted to Assistant Professor.

Mr. S. R. Lenihan, General Electric Company, Evendale, Ohio, has accepted a position as Mathematician with the University of California Radiation Laboratory, Livermore, California.

Dr. D. B. Lowdenslager, University of Illinois, has been appointed a Lecturer at Princeton University.

Mr. C. W. Marshall has accepted a position as Applied Mathematician with the Institute for Defense Analysis, Washington, D. C.

Associate Professor K. F. McLaughlin, Florida State University, has been appointed Specialist for Appraisal of the Individual, U. S. Office of Education, Washington, D. C.

Mr. Richard McQuillin, Brown University, has accepted a position as Physicist with Bolt, Beranek and Newman, Inc., Cambridge, Massachusetts.

Associate Professor G. W. Medlin, Wake Forest College, has been appointed Head of the Mathematics Department at Stetson University.

Dr. A. B. Novikoff, University of California, Berkeley, has accepted a position as Research Mathematician with the Stanford Research Institute, Menlo Park, California.

Dr. P. A. Penzo, Convair-Astronautics, San Diego, California, is now a Member of the Technical Staff at the Space Technology Laboratories, Los Angeles, California.

Miss Jean E. Sammet, Sperry Gyroscope Company, Great Neck, New York, has been appointed Supervisor, MOBIDIC Programming Section, Programming and Simulation Department of the Data Processing Laboratory, Sylvania Electric Products, Needham, Massachusetts.

Mr. E. D. Schell, Sperry Rand Corporation, has accepted a position as Advisory Mathematician with the International Business Machines Corporation, Yorktown Heights, New York.

Dr. S. H. Schot, University of Maryland, has been appointed Assistant Professor at the American University.

Mr. W. S. Soar, Ballistic Research Laboratory, Aberdeen Proving Ground, has accepted a position as Mathematician with the National Security Agency, Fort George G. Meade, Maryland.

Dr. O. E. Taulbee, Sperry Rand Corporation, Saint Paul, Minnesota, has accepted a position as Mathematician with Lockheed Aircraft Corporation, Marietta, Georgia.

Dr. D. E. Thoro, University of Florida, has been appointed Assistant Professor at San Jose State College.

Mr. F. S. Van Vleck, University of Nebraska, has been appointed Instructor at the Institute of Technology, University of Minnesota.

Assistant Professor D. W. Wall, on leave from the University of North Carolina from February 1, 1959 to January 31, 1960, will be in residence at the University of Michigan on a National Science Foundation Research Grant.

Dr. T. C. Wallace, Iowa State College of Agriculture and Mechanic Arts, has been appointed Staff Member at the Los Alamos Scientific Laboratory, Los Alamos, New Mexico.

Mrs. Elaine G. Lasecki, University of Massachusetts, has been appointed Assistant Professor at Edgewood College of the Sacred Heart.

Assistant Professor E. M. Zaustinsky, San Jose State College, has been appointed Assistant Professor at Santa Barbara College, University of California.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 199 persons have been elected to membership by the Board of Governors on applications duly certified.

ROBERT O. ABERNATHY, M.A. (California, Berkeley) Asst. Professor, Southern University.

OLIVER ABERTH, M.S. (M.I.T.) Grad. Student, University of Pennsylvania.

MURRAY ABRAMSON, M.A. (Syracuse) Mathematician-Programmer, I.B.M., Endicott, New York.

THOMAS W. ACHESON, M.D. (Manitoba) Physician and Surgeon, Vancouver, B.C.

MRS. JEANNE L. AGNEW, Ph.D. (Radcliffe) Asst. Professor, Oklahoma State University.

YOUSEF ALAVI, Ph.D. (Western Michigan) Asst. Professor, Western Michigan University.

C. DEAN ALDERS, M.A. (Colorado S.U.) Instr., Mankato State Teachers College.

JOSEPH ALTINGER, B.S. (Dayton) Teacher, Cathedral Latin School, Cleveland, Ohio.

BERNARD C. ANDERSON, M.A. (Wayne) Teacher, Redford High School, Detroit.

CHARLES A. ANDERSON, B.S. (Pennsylvania S.U.) Grad. Student, Pennsylvania State University.

GLEN D. ANDERSON, A.B. (Drury) Grad. Student, University of Michigan.

STEPHEN A. ANDREA, Student, Oberlin College.

JACK BARR, M.S. (S.U. of Iowa) Instr., Bradley University.

CLARENCE H. BARTHELMAN, M.A. (Harvard) Head of Dept., Worcester Academy.

MRS. FELICE D. BATEMAN, Ph.D. (Michigan) Acting Asst. Professor, University of Illinois.

GERALD A. BECKER, M.S. (S.U. of Iowa) Asst. Professor, San Diego State College.

COL. LAWRENCE D. BELL, B.E., C.E. (N. Carolina S.C.) Asst. Professor, Lamar State College.

CHRISTOPHER BILLINGS, Student, Oberlin College.

RICHARD A. BLADE, Student, University of Colorado.

MRS. KAY W. BLAIR, M.S. (Minnesota) Instr., Macalester College.

JULIUS R. BLUM, Ph.D. (California) Asst. Professor, Indiana University.

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ALBERT S. KAMP, Student, St. Ambrose College.
ARTHUR J. KARSON, Student, University of Tulsa.
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COL. PIERRE A. KLEFF, M.S. (Holy Cross) Deputy Commander, Rocky Mountain Arsenal.
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JOHN J. SACCOMAN, Student, Seton Hall University.
MRS. NAOMI R. SALAMAN, Student, Colorado College.
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GLYNN D. SEARL, Student, University of British Columbia.
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SISTER ELIZABETH LOUISE, M.A. (Boston) Instr., Emmanuel College.
SISTER HELEN CLARE FITZSIMMONS, M.S. (Notre Dame) Asso. Professor, Webster College.
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D. BRENDAN SMITH, M.A. (Oxford) Head Teacher, St. Louis Priory School, Creve Coeur, Missouri.
WALTER A. SPIVEY, Ph.D. (North Carolina) Asst. Professor, University of Michigan.
ROY O. SPRUIELL, Student, New Mexico College of A. & M.A.
CAPT. MICHAEL J. STEGER, B.S. (U.S.M.A.) U.S.A.F.; Grad. Student, Texas A. & M. College.
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DON H. TUCKER, Ph.D. (Texas) Asst. Professor, University of Utah.
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JOHN C. WHITE, M.A. (Texas) Asst. Professor, Pacific University.
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JAMES WIGMORE, M.S. (Purdue) Asso. Scientist, Avco Mfg. Corp.
RICHARD H. WINKLER, Student, Massachusetts Institute of Technology.
FRED A. WOMACK, JR., Student, Texas Christian University.
CRAIG C. WORK, B.A. (Wisconsin) Grad. Asst., Stanford Univeristy.
LLOYD E. YATES, Student, University of Colorado.
JAMES F. ZANE, Student, University of Colorado.
IRVING ZUCKER, Student, McGill University.

THE OCTOBER MEETING OF THE IOWA SECTION

The Iowa Section of the Mathematical Association of America held its second annual fall meeting on Friday, October 17, 1958 at the State University of Iowa, Iowa City. This was held in conjunction with the 28th Annual Conference for Teachers of Mathematics in Iowa. Approximately 185 teachers attended the conference, which included 23 members of the Association.

The theme of the conference was: *Providing for the superior student.* Professors M. F. Smiley and H. V. Price, State University of Iowa, presided at the morning and afternoon sessions respectively.

The program consisted of three invited lectures in the morning and a panel discussion in the afternoon. The morning addresses were: *What the High Schools are doing for the*

superior pupils—the Evanston picture, by Mr. F. P. May, Evanston Township High School, Evanston, Illinois; *A model seminar for superior junior high school pupils in science and mathematics*, by Professor R. A. Nielsen, Iowa State Teachers College, Cedar Falls; and *A model seminar for superior senior high school pupils in mathematics*, by Professor W. W. Gutzman, University of South Dakota.

The subject, *The collegiate honors program in mathematics*, was discussed by a panel consisting of Professor Rhodes Dunlap, State University of Iowa, Moderator, Professor L. C. Graue, Coe College, Professor W. W. Gutzman, and Professor H. T. Muhly, State University of Iowa.

E. N. OBERG, *Chairman*

THE OCTOBER MEETING OF THE MINNESOTA SECTION

The fall meeting of the Minnesota Section of the Mathematical Association of America was held on October 11, 1958 at the University of Minnesota, Duluth Branch, Duluth, Minnesota. Professor J. E. Hafstrom of the University of Minnesota, Duluth, presided at the morning session. The section chairman, The Rev. W. C. Kalinowski, O.S.B., of St. John's University, presided at the afternoon session. There were 53 persons in attendance, including 32 members of the Association.

At the business meeting Professor Robert Sloan of Carleton College reported on the activities of the Committee on High School-College Relations. As directed by a vote of the section at the previous meeting, this committee has informed the Minnesota State Department of Education of the availability and willingness of members of the Association to serve as invited consultants at high schools which are in the process of re-evaluating their mathematics programs. Several other projects of this committee were discussed but are in a formative stage. Professor G. K. Kalisch of the University of Minnesota reported on the 1958 high school mathematics contest sponsored by the section and described plans for the 1959 contest. It is planned to finance the 1959 contest through the newly formed Industry-Education Board of the Minnesota Academy of Science.

Professor K. W. Wegner of Carleton College, who was the section representative at the summer meeting of the Association, reported on that meeting.

The following papers were presented:

1. *Asymptotic solution of an implicit function*, by Professor G. A. Heuer, Concordia College.

Let an experiment with N equally likely outcomes be performed K times. The probability of a repeated outcome is $P(N, K)=1-N!/(N-K)!N^K$. Denoting this number by t, the problem of solving for K as a function of N and t is considered. THEOREM. *For* $0<t<1$, K *is asymptotic to* $\sqrt{-2\log(1-t)}\sqrt{N}$. Machine computation shows the approximation to be good for small N.

2. *On numerical evaluation of series*, by Professor O. E. Stanaitis, St. Olaf College.

The sum of the first k terms, say $S_k=\alpha$ is calculated. The remainder is evaluated by Euler's summation formula, say $\gamma<R_k<\beta$. Hence, the approximation $\gamma+\alpha<S<\beta+\alpha$ is obtained. It has been shown that for the series $S=\sum 1/n^2$ it would be necessary to calculate a hundred terms in order to secure accuracy to 2 decimal places. By the above technique the same accuracy is reached by adding only six terms of the series.

3. *Mathematics education in Italy*, by Dr. Amos Nannini, Visiting Lecturer at the University of Minnesota, Duluth. (By invitation of the Executive Committee.)

Education in Italy is very much the same all over the country as there is only one government, in Rome. Italian educational institutions follow the 5-3-5 system. Education is free and compulsory up to the age of fourteen. Youngsters at the age of 14 start receiving instruction in rational geometry and in algebra.

In about 1900, a group of eminent mathematicians, led by Professor Federigo Enriques, intro-

duced more rigor into geometry as it is taught, and discussed some topics of mathematical interest in the elementary field.

Italian college students have a fairly good background in mathematics, and college standards are high.

4. *A program for advanced standing in mathematics for high school students*, by Professor W. B. Stenberg, University of Minnesota, introduced by Professor H. L. Tumittin.

The program at the University of Minnesota was described.

5. *The Minnesota program of educational experimentation in mathematics*, by Professor P. C. Rosenbloom, University of Minnesota.

The School Mathematics Study Group and the Minnesota National Laboratory for the Improvement of Secondary School Mathematics were described. The materials produced at the writing conference at Yale last summer were outlined. The designs of the 7–8th grade and the 9th grade experiments to be performed this year, were also described. The cooperation of college mathematicians was requested to provide appropriate courses for prospective teachers and teachers in service in order to increase the number of teachers qualified to participate.

F. L. WOLF, *Secretary*

OCTOBER MEETING OF THE OKLAHOMA SECTION

The fall meeting of the Oklahoma Section of the Mathematical Association of America was held at Oklahoma City University on October 24, 1958. The fall meeting of the Oklahoma Section is held in conjunction with the Oklahoma Education Association, and is devoted to papers of particular interest to high school teachers. Research papers are presented in the spring meeting.

In the absence of the Chairman and the Vice-chairman, Professor R. V. Andree, the Secretary, presided. There were 299 persons in attendance, including 72 members of the Association. The following officers were elected for one-year terms: Chairman, Professor D. P. Richardson, University of Arkansas; Vice-chairman, Professor Kathrine Mires, Northwestern State College; Secretary-treasurer, Professor R. V. Andree, University of Oklahoma.

The Association's Representative to the Junior Academy of Science, Professor R. B. Deal, reported that 50 papers had been submitted by high school students, and from this group 25 papers were selected for the program.

The Traveling Lecturer Committee, under the chairmanship of Professor Kathrine Mires, reported that progress was being made in establishing a lecture bureau for high schools in the state of Oklahoma. Professor D. P. Richardson, liaison officer for the Oklahoma Section of the Association with the State Academies of Science, reported that the Academies of Science of the states of Arkansas and Oklahoma and the Oklahoma Section of the Association would soon submit a proposal to the National Science Foundation for the sponsorship of a visiting lecturer program in the two states. Professor Carter, chairman of the High School Contest Committee, reported that plans were underway for Oklahoma and Arkansas to participate even more actively during the coming year in the Mathematical Association of America's Contest than they did last year.

The Secretary reported upon the national meeting of sectional officers.

The following program was presented as being of particular interest to high school teachers. The enthusiasm with which it was received seems to indicate that the topics were timely.

The concept of number, Professor Gene Levy, University of Oklahoma.

Functions, Professor R. B. Deal, Oklahoma State University.

Operating with sets, Mr. A. S. Davis, University of Oklahoma.

Point-Set topology, Professor O. H. Hamilton, Oklahoma State University.

The scientist—man of destiny, (luncheon program), Mr. H. M. Smith, Regional Director, U. S. Bureau of Mines, introduced by the Secretary.

The hypercircle in modern physics, Professor C. E. Springer, University of Oklahoma.

Statistics and data handling for high schools, Professor J. C. Brixey, University of Oklahoma.

Limits in geometry and algebra, Professor Arthur Bernhart, University of Oklahoma.

R. V. ANDREE, *Secretary*

THE NOVEMBER MEETING OF THE NORTHEASTERN SECTION

The annual meeting of the Northeastern Section of the Mathematical Association of America was held at the College of the Holy Cross, Worcester, Massachusetts on November 29, 1958. Professor D. E. Richmond, Chairman of the Section, presided. There were 115 persons registered for the meeting, including 90 members of the Association.

Officers chosen for 1958–59 were: N. H. McCoy, Smith College, Chairman; J. G. Kemeny, Dartmouth College, Vice-Chairman; V. O. McBrien, College of the Holy Cross, Secretary-Treasurer. The Section voted to sponsor the National High School Mathematics Contest and Rev. S. J. Bezuszka, S. J., Boston College, was appointed chairman of the Section Contest Committee.

The following members of the Association were appointed to act as liaison officers to deal with mathematical matters within their respective states or provinces: C. T. Holmes, Bowdoin College, Maine; J. B. Adkins, Phillips Exeter Academy, New Hampshire; D. H. Ballou, Middlebury College, Vermont; A. A. Bennett, Brown University, Rhode Island; J. W. Bower, Connecticut College, Connecticut; W. J. Blundon, Memorial University, Newfoundland; Rev. E. J. Roche, St. Dunstan's University, Prince Edward Island; W. S. H. Crawford, Mt. Allison University, New Brunswick; J. J. Adshead, Dalhousie University, Nova Scotia.

By invitation of the Section, the following papers were presented:

1. *The report of the Commission on Mathematics*, by Professor A. W. Tucker, Princeton University.

2. *The School Mathematics Study Group*, by Professor D. E. Richmond, Williams College. (Professor Richmond spoke in place of Professor E. G. Begle of Yale University who was unable to attend the meeting.)

3. *The concept of a "truth set,"* by Professor J. G. Kemeny, Dartmouth College.

The concept of a "truth set" may be used as a unifying concept in the presentation of many branches of mathematics. Given a set of logical possibilities, we may associate with any statement concerning these possibilities a subset of the given set. The truth set of the statement p is the set P of all possibilities in which the statement is true. This establishes a natural isomorphism between logic and set theory, and many applications may be given. Applications to linear equations and inequalities, to systems of functional equations, and to probability theory are discussed.

4. *Incidence geometry—a unifying concept*, by Professor Walter Prenowitz, Brooklyn College.

The notion of incidence geometry is suggested by Hilbert's set of Postulates of Incidence in his *Foundations of Geometry*. The concept and name are due to Gorn, Bull. Amer. Math. Soc. vol. 46, 1940, pp. 158–167. In the present paper an incidence geometry is characterized as a system (S_1, S_2, S_3) composed of three sets whose elements are called respectively *points*, *lines*, *planes* which satisfy a set of postulates essentially equivalent to that of Hilbert. A few elementary theorems are stated, and a list of ten incidence geometries is given. How to characterize most familiar types of geometry as incidence geometries is indicated.

5. *Recursive functions and set theory*, by Professor Hartley Rogers, Massachusetts Institute of Technology.

An expository discussion is given, presupposing a minimum of special background, of recursive function theoretic constructions on the integers that provide countable analogues to various transfinite structures of classical set theory. References include work of Dekker and Myhill on constructive versions of cardinal number theory, and work of Church, Kleene, Markwald, Spector, Kreider and the author on constructive versions of ordinal number theory.

V. O. McBRIEN, *Secretary*

CORRECTION

In the report of the March 1958 meeting of the Michigan Section and in the index (this MONTHLY, vol. 65, 1958, p. 551 and p. 810, respectively) the name P. J. Nikolai was misspelled "Nakolai."

CALENDAR OF FUTURE MEETINGS

Fortieth Summer Meeting, University of Utah, Salt Lake City, Utah, August 31–September 3, 1959.

Forty-third Annual Meeting, Conrad Hilton Hotel, Chicago, Illinois, January 28–30, 1960.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, University of Pittsburgh, May 2, 1959.
ILLINOIS, Millikin University, Decatur, May 8–9, 1959.
INDIANA, Valparaiso University, May 2, 1959.
IOWA, Iowa Wesleyan University, Mount Pleasant, April 17, 1959.
KANSAS, Marymount College, Salina, April 11, 1959.
KENTUCKY, Centre College of Kentucky, Danville, April 25, 1959.
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Goucher College, Towson, Maryland, May 2, 1959.
METROPOLITAN NEW YORK, Polytechnic Institute of Brooklyn, April 18, 1959.
MICHIGAN, Michigan State University, East Lansing, March 28, 1959.
MINNESOTA, University of Minnesota, Minneapolis, April 25, 1959.
MISSOURI, Lindenwood College, St. Charles, April 25, 1959.
NEBRASKA, University of Nebraska, Lincoln, April 18, 1959.
NEW JERSEY, Princeton University, November 7, 1959.
NORTHEASTERN
NORTHERN CALIFORNIA
OHIO, Miami University, Oxford, May 9, 1959.
OKLAHOMA, Tulsa University, April 10–11, 1959.
PACIFIC NORTHWEST, University of Oregon, Eugene, June 19, 1959.
PHILADELPHIA, University of Delaware, Newark, November 28, 1959.
ROCKY MOUNTAIN, Utah State University, Logan, May 8–9, 1959.
SOUTHEASTERN, East Tennessee State College, Johnson City, March 20–21, 1959.
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SOME USES OF LINEAR SPACES IN ANALYSIS*

F. A. FICKEN, University of Tennessee

During recent decades much progress in analysis has resulted from the application of algebraic and topological ideas to classes of functions. How does it happen that these ideas, which arise so naturally in Euclidean space of dimension two or three, can be extended so usefully to classes of functions? We shall try to throw some light on this question by describing and illustrating a few of the conceptual relationships involved.

Much of the usefulness of linear spaces arises from very simple algebraic facts. Let D denote any set and let R denote the class of all real-valued functions defined on D, with the understanding that two functions are equal if their values are equal at every point of D. For example, if D is the set of positive integers, then R is the class of all real sequences. If D is the real line, or a half line, or an interval, we have the class of all real-valued functions of a real variable with domain D.

There is an obvious way of adding elements of R. If $u \in R$ and $v \in R$, and the function $w = u + v$ is defined by agreeing that, for every $P \in D$

$$w(P) = (u + v)(P) = u(P) + v(P),$$

where the last expression is the numerical sum of the numbers $u(P)$ and $v(P)$, then clearly $w \in R$. It is evident that R with this operation of addition is an abelian group; the identity is the zero function, which vanishes identically on D.

Elements of R can also be multiplied by real numbers in the same pointwise manner. If $u \in R$ and a is a real number, and the function $v = au$ is defined by agreeing that, for every $P \in D$

$$v(P) = (au)(P) = au(P),$$

where the last expression is the numerical product of the number a and the number $u(P)$, then clearly $v \in R$. One verifies readily that the abelian group just mentioned "admits" the operation of multiplication by real numbers in the sense that the following equalities hold for all elements u and v of R and all real numbers a and b:

$$a(u + v) = au + av, \qquad (a + b)u = au + bu,$$
$$a(bu) = (ab)u, \qquad 0u = 0, \qquad 1u = u.$$

Under these circumstances R is said to be a real linear or vector space. ([1], p. 162; [2], Sec. 2; [3], Sec. 1.4; [4], p. 6; [5], Sec. 1.2. These and following similar indications refer to page or section of the numbered item in the bibliography.) The elements of R (the functions u) are the vectors and the real numbers are the scalars. When reference is made to a space of functions with a common

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domain, it will always be understood that this natural algebraic structure is being used. The functions will always be real-valued; thus we deal exclusively with real linear spaces of functions. It may also be noted that we are thinking in terms of the affine rather than the projective model of a linear space.

We first review Picard's theorem for the differential equation $y'=f(x, y)$ in such a way as to arrive at the notion of a complete metric linear space. Next we suggest considerations, first algebraic and then topological, underlying the idea of a Banach space. Then the use of Banach spaces is illustrated and, finally, some hint is given of the rich properties of Hilbert space.

Our aim is strictly expository, and proofs will be found in the references. It should also be emphasized that we can mention only a few fragments, chosen somewhat capriciously from a vast area of which many parts are still being studied vigorously. Being interested primarily in conceptual relationships, we always use hypotheses strong enough to exclude every kind of irregularity.

Picard's theorem for $y'=f(x, y)$. Let us review briefly the main features of Picard's familiar treatment of the differential equation $y'=f(x, y)$ ([6], Ch. XV; [7], Secs. 6.9–6.11). Let x range over the interval $I: \alpha<x<\beta$. Let $f(x, y)$ have these properties:

1. *$f(x, u)$ is continuous in x and u together and $|f(x, u)| \leqq M$ for $x \in I$ and all u;*
2. *$|f(x, v)-f(x, u)| \leqq K|v-u|$ for $x \in I$ and all u and v (Lipschitz condition).*

Let a and b be given numbers such that $a \in I$. The problem (P) is to find a differentiable function $y(x)$ such that

$$\text{(P)} \qquad y'(x) = f(x, y(x)) \qquad (x \in I) \text{ and } y(a) = b.$$

The first step is to pass by integration to an integral equation satisfied by any solution of (P):

$$\text{(E)} \qquad y(x) = b + \int_a^x f(t, y(t))dt.$$

Then observe that, vice versa, a continuous function satisfying (E) will solve (P). Thus (E) and (P) are equivalent.

One then solves (E) by an iterative process, putting $y_0(x)=b$, and, for $n=0, 1, 2, \cdots$.

$$y_{n+1}(x) = b + \int_a^x f(t, y_n(t))dt.$$

The functions $y_n(x)$ are the partial sums of the infinite series

$$\text{(S)} \qquad y_0(x) + \sum_{n=1}^{\infty} (y_n(x) - y_{n-1}(x)).$$

The terms of this series can easily be estimated. First of all,

$$|y_1(x) - y_0(x)| = \left| \int_a^x f(t, b)dt \right| \leqq M|x - a|.$$

By using the properties of $f(x, u)$, one then obtains by induction the estimate

$$|y_{n+1}(x) - y_n(x)| \leqq MK^n \frac{|x - a|^{n+1}}{(n + 1)!} \qquad (n = 1, 2, \cdots).$$

Being dominated by the series for $MK^{-1}e^{K|\beta-\alpha|}$, the series (S) converges (absolutely and) uniformly to a function $y(x)$ which is continuous on I. One sees readily that $y(x)$ satisfies (E) and therefore also solves (P), and is, moreover, the unique solution of (P).

Let us examine certain features of the foregoing process. We are working, let us first observe, with functions y of the independent variable x that are continuous on I. To abbreviate, let C denote the class of those functions; that is, $y \in C$ if and only if y is continuous on I. If $y \in C$ then the function whose value at x is $b + \int_a^x f(t, y(t))dt$ belongs to C. Let us denote this function by Fy; thus, for $x \in I$,

$$Fy(x) = b + \int_a^x f(t, y(t))dt.$$

We therefore have a correspondence or transformation $F: y \rightarrow Fy$, defined on C into C, carrying y onto Fy. The equation (E) asks for a self-corresponding element of C, an element y such that $y = Fy$, a fixed point of F.

The natural algebraic structure of C as a real linear space permits us to carry out formally each step in the iterative process by which (E) is solved; it is clear, indeed, that each y_n exists and belongs to C. The algebraic structure, however, gives us no means of deciding whether the y_n are leading us anywhere at all, let alone toward a solution of (E). A notion of convergence is needed, that is, a topology.

The convergence actually used in our proof was uniform convergence on I. One is tempted to try to introduce an equivalent metric topology. A scalar-valued function of one or more vectors in a linear space is called a functional. Consider now the functional

$$\rho(u, v) = \sup_x |u(x) - v(x)|$$

with u and v in C. This symmetric functional is never negative, vanishes if and only if $u = v$, and obeys the triangle inequality. There is this difficulty, however, that if u is bounded and v is unbounded then $\rho(u, v)$ is not finite. If we wish a set on which ρ is always finite, and wish also to keep the algebraic structure, we must insist on keeping the zero of the linear space and require therefore that

$$\rho(u, 0) = \sup_x |u(x)| < \infty;$$

in other words we must use only that subclass B of C consisting of those functions which are bounded as well as continuous on I. B consists of precisely those elements of C which are at a finite distance ρ from the origin.

Using the same operations of addition and multiplication by a scalar, we easily verify that B is itself a vector space; B is therefore called a subspace of C. On B, moreover, $\rho(u, v)$ is a metric such that $\rho(u_n, u)\to 0$ as $n\to\infty$ if and only if $u_n(x)\to u(x)$ uniformly on I. Thus B is a metric linear space. It is obvious, too, that the iterative process for solving (E) took place entirely in B; $y_0\in B$, and if $u\in B$ then $Fu\in B$.

One final point must be clarified. The functions y_n turned out to be a Cauchy sequence uniformly on I. We may now express this by saying that y_n is a Cauchy sequence in B; in other words $\rho(y_m, y_n)\to 0$ as m and $n\to\infty$. We must be able to conclude that then there exists a y *in* B such that $\rho(y_n, y)\to 0$ as $n\to\infty$. Here we appeal to a theorem of analysis which says that if a sequence of bounded continuous functions on I is a Cauchy sequence uniformly on I then there exists a bounded continuous function to which the sequence converges uniformly on I. In other words B is a metric linear space which is complete in the sense that every Cauchy sequence of elements of B converges to an element of B.

Summarizing, we may say that the integral equation (E) asks for a fixed point of the transformation $y\to Fy$ defined on the complete metric linear space B into itself, and that the Picard process locates a fixed point by an iterative process.

Algebraic considerations. Having seen that a complete metric linear space arises quite naturally in certain problems in analysis, let us return for a closer look at some algebraic properties of the linear space $R(D)$ of all real-valued functions defined on an arbitrary set D.

If a property is preserved under the formation of linear combinations then the subset of R consisting of all functions with that property will form a subspace of R. It is fortunate indeed that many properties of interest in analysis are in fact preserved under the formation of linear combinations. This is true, for example, of such regularity properties as boundedness, continuity, Hölder continuity (with fixed exponent), smoothness, and integrability. It is true of measurability, and of evenness and oddness, but not of convexity. It is true of the property of vanishing on a given subset of D. It is true of the property of satisfying a linear homogeneous differential or integral equation.

These examples lead us to the notion of a linear transformation. Let X and Y be real linear spaces and let L be a mapping $x\to y=Lx$ on X into Y that is linear: $L(a_1x_1+a_2x_2)=a_1Lx_1+a_2Lx_2$. If X and Y are finite-dimensional, and bases are introduced in them, then a matrix L will exist such that in the equation $Lx=y$ the column vector x is multiplied by the matrix L to yield the column vector y.

Let us now fix $y\in Y$ and seek to solve the equation $Lx=y$ for x; in the finite-

dimensional case we are trying to solve a system of linear equations $\sum_{j=1}^{n} l_{\alpha j}x_j = y_\alpha$ $(\alpha = 1, \cdots, m)$.

If $y=0$ (homogeneous case) the trivial solution $x=0$ is always available. If $y=0$ implies $x=0$ then L is said to be nonsingular; otherwise L is said to be singular. If L is singular then the set of solutions of $Lx=0$ is a subspace of X called the nullspace of L and denoted by N_L. If $y\neq 0$ then the equation $Lx=y$ is consistent (has a solution) if and only if y is in the range of L; in the finite dimensional case the condition is that y be in the column space of the matrix L, so that L and the "augmented matrix" (with extra column y) have the same rank. In any case we may easily verify that *either* L is nonsingular and for each y in the range of L there is a unique x such that $Lx=y$ *or* L is singular and $Lx=y$ if and only if $x=x_0+n$ where $Lx_0=y$ and $Ln=0$, that is, $n\in N_L$.

In geometric language, the set of all solutions is obtained by translating the subspace N_L by any particular vector x_0 satisfying the equation. For want of a better name, let us for present purposes only call a set arising from translation of a subspace a flat. The result then is that the set of solutions of $Lx=y$ is the flat obtained by translating N_L by any particular solution x_0.

Generally speaking, now, any linear condition can be expressed by a linear equation $Lx=y$ where L is defined on an appropriate domain into an appropriate range. As a first example, let us consider a linear differential operator on an interval I:

$$x \to Lx = a_n(t)x^{(n)}(t) + \cdots + a_1(t)x'(t) + a_0(t)x(t)$$

with continuous coefficients and $a_n(t)\neq 0$ on I. For certain problems an appropriate domain is the space C^n of functions with n continuous derivatives on I, and the range is the space C^0 of functions continuous on I. It is known that the nullspace N_L is n-dimensional; a basis for N_L consists of n linearly independent solutions $x_1(t), \cdots, x_n(t)$ of the equation $Lx=0$, and a parametric representation

$$x_c(t) = c_1x_1(t) + \cdots + c_nx_n(t)$$

of N_L is commonly called the "complementary function." It may also be shown, by variation of parameters, that the range of L is all of C^0. It follows from our general remarks, therefore, that for any y in C^0 the nonhomogeneous equation $Lx=y$ is consistent and its solutions constitute the flat $x(t)=x_p(t)+x_c(t)$, where $Lx_p(t)=y(t)$; $x_p(t)$ is commonly called a "particular integral" ([8], Ch. 3, Sec. 6).

The same general principle underlies the "displacement of nonhomogeneity" in the problem of finding a function subject to several linear conditions, some of which are nonhomogeneous ([9], p. 236). Suppose, for example, that the conditions fall into two sets, which may be written symbolically as

$$Lu = H, \qquad lu = h,$$

where H and h are given. For example, L might be a differential operator, while

the equation $lu=h$ might express initial or boundary conditions. Suppose further that one set of conditions can be met, for example, that $lu_0=h$. Put $u=u_0+v$, getting for v the conditions

$$Lv = H - Lu_0, \qquad lv = 0.$$

Thus v must be sought in the flat determined by translating by u_0 the nullspace of conditions l.

For a final example, let us return to the space $R(D)$ of all real-valued functions defined on a domain D. Let A be a nonvoid proper subset of D, let $y(P)$ be defined only on A, and consider the nonhomogeneous problem of finding those $x\in R(D)$ such that $x(P)=y(P)$ for $P\in A$. This problem amounts to finding all extensions of y from A to D. Since $y\in R(A)$, let us consider a transformation L on $R(D)$ into $R(A)$ defined by $x\rightarrow Lx$, where Lx is the element $x(P)$ of $R(A)$, sometimes called the restriction of x to A. In this case the subspace N_L of $R(D)$ consists of all functions defined on D so as to vanish on A, and the solution of the extension problem is the flat consisting of N_L translated by any particular extension of y (for example, one obtained by putting $x=0$ on the complement of A).

These remarks apply to any problem in which a desired function is required, among various constraints, to take on given values at specified points. In the most elementary problem in the calculus of variations, for example, one is interested in all sufficiently regular functions $x(t)$ taking on prescribed values at, say, $t=a$ and $t=b$; from one such function x_0 one obtains all others by translating by x_0 the space of all sufficiently regular functions vanishing at a and at b. Initial and boundary value problems for ordinary and hyperbolic or parabolic partial differential equations, as well as the Dirichlet and Neumann problems for elliptic equations all require functions to take on specified values at specified places. In all these cases, therefore, the desired function must be sought within an appropriate flat.

Another algebraic idea has important consequences ([1], Ch. VII, Sec. 12; [2], Sec. 13; [4], Ch. I, Sec. 10). A real-valued linear function $x^*(x)$ defined on a real linear space X is called a linear functional; thus x^* is a linear transformation on X into the one-dimensional space of scalars. In a finite-dimensional space, in a fixed coordinate system, each linear functional determines a unique linear form, and vice-versa. In the space C^0 of all functions $x(t)$ continuous for $\alpha\leqq t\leqq\beta$, $x^*(x)=\int_\alpha^\beta x(t)dt$ is a linear functional. In the space $R(D)$, for each fixed $P\in D$, $x_P^*(x)=x(P)$, the value of x at P, is a linear functional on $R(D)$; the extension problem just discussed seeks those $x\in R(D)$ such that $P\in A$ implies $x_p^*(x)=y(P)$.

The class of all linear functionals, with addition and multiplication by real numbers again defined pointwise, is a linear space X^* called the dual of X.

If $x\in X$ and $x^*\in X^*$ then x and x^* are said to be normal to each other if and only if $x^*(x)=0$. If Y is a subspace of X then the set of all $x^*\in X^*$ such that

$y \in Y$ implies $x^*(y)=0$ is easily seen to be a subspace of X^*; this subspace will be denoted by Y^N. Similarly, to any subspace Z of X^* there corresponds the subspace Z^N of X (not of X^{**}) consisting of all $x \in X$ normal to every $x^* \in Z$.

We have seen that the solutions of a system of linear homogeneous algebraic equations $Lx=0$ constitute the nullspace N_L of the transformation L. If in a given coordinate system the equations require certain linear forms to vanish

$$x_\alpha^*(x) = \sum_{j=1}^{n} l_{\alpha j} x_j = 0 \qquad (\alpha = 1, \cdots, m)$$

and if Z is the subspace of X^* spanned by the x_α^*, then the solutions constitute the space Z^N of X; thus $Z^N = N_L$.

Let us conclude these algebraic observations by mentioning some of the difficulties that arise in the infinite-dimensional case. In the finite-dimensional case, for example, it is always true that

$$U^{NN} = U \quad \text{and} \quad (U \cap V)^N = U^N + V^N.$$

In the infinite-dimensional case, if U and V are subspaces of X^*, then only the inclusions

$$U \subset U^{NN} \qquad U^N + V^N \subset (U \cap V)^N$$

can be established generally ([5], p. 50 (Prop. 10); p. 48 (Prop. 8); p. 60 (9c)).

In the finite-dimensional case, again, X and X^* and X^{**} are isomorphic. It is a consequence of the definition of dimension in the infinite case that if C denotes the cardinal of the real number system, then $\dim X^* = C^{\dim X} > \dim X$, so that isomorphism between X and X^* or X^{**} is out of the question ([4], p. 247).

The symmetry in the relationships between normal subspaces is restored, and dimensional questions are again brought under control, by adjustments in the notion of the dual space required by topological considerations, to which we now turn.

Topological considerations. A topological linear space X is a linear space with a topology in which the operations of addition and multiplication by a scalar are continuous. We cannot discuss the general theory of these spaces, which has been developed extensively in recent years [10]. If the topology in X is induced by a metric $\rho(u, v)$ then one speaks of a metric linear space.

In order to see that the most general metric linear space may be mildly pathological let us ask which linear functionals x^* are continuous in the sense that $\rho(x_n, x) \to 0$ implies $x^*(x_n) \to x^*(x)$. Let M denote the metric linear space of measurable functions $x(t)$ on the unit interval with the metric

$$\rho(x, y) = \int_0^1 \frac{|x(t) - y(t)|}{1 + |x(t) - y(t)|} \, dt.$$

The only continuous linear functional on M is the functional $x^*=0$, *i.e.*, $x^*(x)=0$ for all x ([11], p. 234).

Let us recall that a subset of a linear space is convex if it contains the entire segment joining any pair of its elements. Now it has been shown that nontrivial continuous linear functionals exist in a topological linear space if and only if the origin is contained in an open convex proper subset ([12], p. 18; [13]). In a metric topology, a sufficiently small open sphere with center at the origin will meet this condition provided the sphere is convex; in the space M, therefore, all sufficiently small spheres containing the origin are not convex.

Difficulty arises on an even more elementary level when one tries to connect the metric $\rho(u, v)$ with the notion of length of a vector. It is natural to investigate the effect of defining the length of u to be $\rho(u, 0)$. One would wish the length of $v-u$ to be $\rho(u, v)$, and would wish the ratio of the length of au to the length of u to be $|a|$. The metric in the space M does not have these properties.

It is perhaps not entirely unnatural that in a vector space the notion of length of a vector is the decisive metric feature. A real-valued functional $\|u\|$ is said to be a ***norm***, and its value at u is called the length of u if the following conditions are satisfied:

1. $\|0\|=0$, and if $u\neq 0$ then $\|u\|>0$;
2. $\|au\|=|a|\,\|u\|$;
3. $\|u+v\|\leqq\|u\|+\|v\|$.

In the space of bounded continuous functions on a domain D the functional $\|u\|=\sup_{x\in D}|u(x)|$ is a norm; many other examples will occur.

It is easy to see that, if $\|u\|$ is a norm then $\rho(u, v)=\|v-u\|$ is a metric. The resulting metric topology is called the normed topology. Clearly $u_n\to v$ if and only if $\rho(u_n, v)=\|u_n-v\|\to 0$. It has been shown that, in a topological linear space, a norm can be defined yielding an equivalent topology if the origin has a bounded convex symmetric neighborhood ([12], p. 18; [14]).

In order to be able to carry out limiting operations freely we require that our normed linear space be complete; that is, if $\|u_m-u_n\|\to 0$ as $m, n\to\infty$ then there exists an element v of the space such that $\|u_n-v\|\to 0$ as $n\to\infty$. If a normed linear space arises which is not complete, it can always be completed in a unique way by a process very similar to the Cauchy-Cantor process for generating the reals from the rationals ([15], p. 196; [16], p. 94 (Th. 2); [18], p. 30). A complete normed linear space is usually called a Banach space.

Many of the topological difficulties encountered in infinite-dimensional Banach spaces may be traced to the fact that the unit sphere $\|x\|\leqq 1$ is not compact. A locally compact Banach space is finite-dimensional ([11], p. 84).

The advantages of a norm appear clearly in connection with the dual space. The (topological) dual X^* of a *topological* linear space X is now defined to consist of all linear functionals x^* that are *continuous* on X. It was observed above that X^* may be trivial; M^* is. If X is a normed linear space, however, the situation is very satisfying.

Let us observe once and for all that any linear transformation L on one Ba-

nach space into another (and in particular any linear functional x^*) is continuous if and only if the nonnegative real number

$$\|L\| = \sup_{x \neq 0} \frac{\|Lx\|}{\|x\|} \qquad \left(\text{or } \|x^*\| = \sup_{x \neq 0} \frac{|x^*(x)|}{\|x\|}\right)$$

is finite ([16], p. 134 (Th. 2)). Then the transformation (or functional x^*) is said to be bounded and $\|L\|$ (or $\|x^*\|$) is called its norm.

Thus X^* is the space of bounded linear functionals, and it turns out that $\|x^*\|$ does have, as its name implies, the properties of a norm on X^*([16], p. 149). Whether or not X is complete in its norm $\|x\|$, moreover, X^* is complete in the norm $\|x^*\|$ ([16], p. 149 (Th. 1)). Thus the dual X^* of a Banach space X comes with a built-in normed topology with respect to which it is itself a Banach space.

Finally, X^* is definitely nontrivial, as indicated by the fact that if $x_0 \neq 0$ then there exists an $x^* \in X^*$ such that $x^*(x_0) = \|x_0\|$ and $\|x^*\| = 1$ ([16], p. 146 (Th. 5); [12], p. 19 (Th. 2.9.3)).

As an example, let us mention the space L^p ($p > 1$) of functions $x(t)$ on a finite interval I with the property that $|x(t)|^p$ is integrable over I in the Lebesgue sense, with the norm $\|x\|$, where $\|x\|^p = \int_I |x(t)|^p dt$. The space L^{p^*} dual to L^p is isomorphic and isometric with L^q, where $p + q = pq$. It can be shown, in fact, that for each x^* in L^{p^*} there exists a unique $y \in L^q$ such that, for all $x \in L^p$, $x^*(x) = \int_I y(t)x(t)dt$ ([16], p. 141 (Th. 3); [11], p. 64). The space L^2, to which we shall return later, is self-dual.

The spaces L^p have many pleasant properties. For example, $(L^{p^*})^* = (L^q)^* = L^p$. In order to mention one rather disconcerting property of L^p for $p \neq 2$, let us recall that subspaces X and Y of a linear space Z are said to be complementary if $X \cap Y = 0$ and for each $z \in Z$ there exist $x \in X$ and $y \in Y$ such that $z = x + y$. It is an algebraic fact that every subspace has many complements ([4], pp. 27–8; [5], p. 37 (Prop. 5)). The question here concerns the existence of complementary closed subspaces. It has been shown that if $p \neq 2$ then L^p has a closed subspace S with no closed complement [17]. As a corollary, there is no bounded projection of L^p onto S.

The Picard process led us to a Banach space. After reviewing briefly some basic algebraic and topological ideas we have reached the same point again, with perhaps a clearer idea of the structure of a Banach space. Our object in the remaining sections is to illustrate a very few of the uses of Banach spaces and of a very important special class of Banach spaces, namely, Hilbert spaces.

Banach spaces. Let us mention first a theorem on the iterative solution of equations $x = Tx$, where $x \rightarrow Tx$ is a mapping (nonlinear, to avoid triviality) on a Banach space X into itself ([18], p. 58). Suppose that, for some $r > 0$, $\|x\| \leqq r$ implies $\|Tx\| \leqq r$; thus T maps into itself the closed sphere of radius r and center at the origin. Suppose further (Lipschitz condition) that there is a number θ with $0 < \theta < 1$, such that $\|x\| \leqq r$ and $\|y\| \leqq r$ imply $\|Ty - Tx\| \leqq \theta \|y - x\|$. Then

there is a unique x such that $\|x\| \leqq r$ and $x = Tx$. One proves this theorem by first choosing x_0 with $\|x_0\| \leqq r$, then, for $n=0, 1, 2, \cdots$, defining $x_{n+1} = Tx_n$, then showing that x_n is a Cauchy sequence, and showing finally that $x = \lim_n x_n$ satisfies the equation and is unique. The analogy with Picard's process is clear.

Before applying this theorem let us note that there are many other fixed-point theorems. The one just cited necessarily yields a unique result (with $\|x\| \leqq r$) and therefore can not be applied if there is more than one solution (with $\|x\| \leqq r$).

One specific application of an iterative process of the kind just described has resulted in a treatment of the following problem:

$$\begin{cases} u_{xx} - u_{tt} - 2ru_t - \alpha u = b + \epsilon u^3 & (-\infty < x < \infty, 0 < t) \\ u(x, 0+) = f(x), \ u_t(x, 0+) = g(x) & (-\infty < x < \infty), \end{cases}$$

where f has two continuous derivatives and g has one. This is the initial value problem for the transverse displacement of an infinite string or for the voltage in an infinite transmission line, with applied force $-b$. The nonlinear term ϵu^3 is analogous to the nonlinear term in Duffing's equation. One first "solves" the differential equation as though it were a nonhomogeneous "telegraph equation," and uses the initial conditions, arriving at an equivalent single nonlinear integral equation $u = Tu$, where

$$\begin{aligned} Tu(x, t) &= \tfrac{1}{2}e^{-rt}\int_{-t}^{t} [rf(x+y) + g(x+y)]J_0(\omega\sqrt{t^2 - y^2})dy \\ &+ \tfrac{1}{2}e^{-rt}\frac{\partial}{\partial t}\int_{-t}^{t} f(x+y)J_0(\omega\sqrt{t^2 - y^2})dy \\ &- \tfrac{1}{2}\int_0^t e^{-rs}\int_{-s}^{s}[b(x+y, t-s) + \epsilon u^3(x+y, t-s)]J_0(\omega\sqrt{s^2 - y^2})\, dyds; \end{aligned}$$

here J_0 is the Bessel function of order zero and $\omega^2 = \alpha - r^2$.

Let us keep the present discussion simple by assuming that $r > 0$. Then the Banach space appropriate for an iterative solution of the equation $u = Tu$ is the space CC of functions $w(x, t)$ that are bounded and continuous in x and t together for $-\infty < x < \infty$ and $t \geqq 0$, with $\|w\| = \sup_{x,t} |w(x, t)|$. One arrives in this way at a rather complete discussion of the initial value problem. It is possible to show, moreover, that if the applied force is p-periodic in t, $b(x, t+p) = b(x, t)$, then the initial data f and g can be chosen so that the corresponding solution is also p-periodic in t [**19**].

This method of dealing with the infinite string has the advantage that the semi-infinite string with fixed endpoint and the finite string with both endpoints fixed can be regarded as special cases. To treat the semi-infinite string with fixed endpoint, at $x = 0$, one goes into the subspace of CC consisting of functions that are odd in x. To treat the finite string with fixed endpoints, at $x = 0$ and $x = \pi$, one goes into the subspace of CC consisting of functions that are odd and

2π-periodic in x. These subspaces are easily seen to be invariant under the integral operators that are used. If begun in the appropriate subspace, the iterative process will therefore take place entirely within the subspace, and, since the subspaces are closed, will deliver a result also lying there.

Another remark concerns the passage from a differential equation with accompanying initial (or boundary) conditions to a single integral equation. It is true that getting the problem into a single package may be advantageous. More to the point, however, is the general experience that many integral operators, including most of the linear ones that seem to arise in practice, are bounded and therefore continuous, while a differential operator, in the nature of things, is necessarily unbounded.

For example, let C^0 denote the space of functions $x(t)$ continuous for $0 \leqq t \leqq 1$, with $\|x\| = \sup_t |x(t)|$, let $K(s, t)$ be continuous for $0 \leqq s, t \leqq 1$, with $k = \sup_{s,t} |K(s, t)|$, and define a linear transformation $x \to Tx$ on C^0 into C^0 by putting

$$Tx(t) = \int_0^1 K(t, s)x(s)ds.$$

Since $|Tx(t)| \leqq k\|x\|$, it follows that $\|T\| \leqq k$ and T is bounded with norm not exceeding k. On the other hand, the transformation defined on the subspace C^1 of C^0, consisting of those functions with derivatives lying in C^0, by $x \to Dx$, where $Dx(t) = dx(t)/dt$, is not bounded, for if $x_n(t) = t^n$, then $\|x_n\| = 1$ and $\|Dx_n\| = \sup_t nt^{n-1} = n$.

Another reason for preferring integral to differential operators is their smoothing effect. Thus the transformation $x \to Jx$, where $Jx(t) = \int_0^t x(u)du$ maps C^0 into the subspace C^1, whose elements are more regular than those of C^0, while $x \to Dx$ is defined (for our purposes) only on the subspace C^1 and yet the range of D covers all of C^0; in fact, for any $x \in C^0$, $Jx \in C^1$ and $DJx = x$.

The notion of the dual transformation of a given linear transformation is illuminating in the general as well as in the finite-dimensional case ([**12**], Sec. 2.13; [**16**], Ch. 7, Sec. 13). Let L map a Banach space X linearly and continuously into a Banach space Y, and let R_L denote the range LX of L. Being a subspace of the Banach space Y, R_L is complete, and therefore a Banach space, if and only if R_L is closed ([**16**], p. 95). Let X^* and Y^* denote the Banach spaces dual respectively to X and Y; thus Y^* is the space of bounded linear functionals y^* defined on Y. Now observe that, with y^* fixed in Y^*, the number $y^*(Lx)$ depends linearly on x; moreover, since

$$|y^*(Lx)| \leqq \|y^*\|\|L\|\|x\|,$$

$y^*(Lx)$ is a bounded linear functional of x and therefore is an element x^* of X^*. This element x^* of X^* depends linearly on y^*. The linear transformation on Y^* into X^* carrying y^* onto x^* is called the dual L^* of L; the equations defining L^* are therefore $x^* = y^*L^*$, where $y^*L^*(x) = y^*(Lx)$.

In the finite-dimensional case, using suitable bases, let us think of the vectors of X and Y as column vectors and of the vectors of X^* and Y^* as row vectors. Then the applicable equations between matrices are $Lx=y$ and $y^*L^*=x^*$, where the matrix L^* is identical with the matrix L ([**4**], p. 58). If one wishes to treat x^* and y^* also as column vectors then the second equation becomes $x^{*t}=L^t y^{*t}$, where L^t is the transpose of L; this is the basis for the conventional identification of the dual with the transpose.

Recalling now in the general case that the nullspace of L is denoted by N_L, we may display the relationships ([**16**], Ch. 10, Sec. 3; [**5**], Sec. 4, No. 9) between the ranges and nullspaces of L and L^* as follows:

$$N_L = (R_{L^*})^N; \qquad N_{L^*} = (R_L)^N;$$

$$R_L \subset (N_{L^*})^N, \text{ and conversely if } R_L \text{ is closed;}$$

$$R_{L^*} \subset (N_L)^N, \text{ and conversely if } R_{L^*} \text{ is closed.}$$

In order to obtain a typical application, let us suppose that R_L is closed (as it always is, of course, in the finite-dimensional case). Then $R_L=(N_{L^*})^N$, and we have arrived in the general case at a consistency condition familiar in the finite-dimensional case for the equation $Lx=y$; if R_L is closed then $y\in R_L$ if and only if y is normal to every solution of the homogeneous dual ("transposed") equation $y^*L^*=0$.

As a final example, let us mention very briefly the notion of the differential of a transformation $x\to Tx$ on an open subset S of X into Y ([**12**], Sec. 4.3; [**18**], p. 127). Suppose that, with x fixed in S, and $x+u$ in S, it is possible to express the increment $T(x+u)-Tx$ as

$$T(x + u) - Tx = L(x; u) + \|u\| R(x; u),$$

where $u\to L(x, u)$ and $u\to R(x, u)$ map X into Y in such a way that L is linear in u and $R(x, u)\to 0$ as $u\to 0$. Then L is said to be the differential of T at x.

As a simple instance of this idea, let X be the space C^1 of functions $x(t)$ that are continuous and have continuous first derivatives on the interval $a\leqq t\leqq b$, let S be the flat consisting of those $x\in C^1$ such that $x(a)=\alpha$ and $x(b)=\beta$, and let U be the subspace of C^1 consisting of those x such that $x(a)=0=x(b)$; thus $x\in S$ and $u\in U$ imply $x+u\in S$. Let Y be the system of real numbers, let $F(t, x, x')$ be a sufficiently regular function of three variables, and define T on S into Y by setting

$$x \to Tx = \int_a^b F(t, x(t), x'(t))dt.$$

One sees readily that

$$L(x; u) = \int_a^b u\left[\frac{\partial F}{\partial x} - \frac{d}{dt}\frac{\partial F}{\partial x'}\right]dt,$$

where the arguments of F_x and $F_{x'}$ are t, $x(t)$, and $x'(t)$. The coefficient of u is the Euler expression for the integrand F.

The differential has been used [20] to develop implicit function theorems for equations of the form $T(x; y)=0$, where T maps a product space $X\times Y$ into X, so as to obtain local solutions $x(y)$ of the equations. It is essential for this purpose to have a known pair (x^0, y^0) such that $T(x^0; y^0)=0$. If the differential $L(x^0, y^0; u)$ with respect to x is nonsingular, and certain more detailed conditions are met, then for y near y^0 one can find $x(y)$ such that $T(x(y); y)=0$ and $x(y)\to x^0$ as $y\to y^0$. In the finite-dimensional case the matrix of the differential turns out to be the Jacobian matrix.

Hilbert spaces. A very important special class of Banach spaces consists of the Hilbert spaces ([16], [21], [22], [23], [24]). Forgetting topological notions for a moment, let us suppose that we have a linear space X endowed with a scalar product, that is, with a symmetric bilinear functional, ordinarily written (x, y), such that the associated quadratic form is positive: $(x, x)>0$ unless $x=0$. In this situation one can show readily that the functional $\|x\|=\sqrt{(x, x)}$ is a norm on X. With this norm, after completion, if necessary, X is called a Hilbert space.

In the space C^0 of continuous functions $x(t)$ on the interval $0\leqq t\leqq 1$, for example, one might define

$$(x, y) = \int_0^1 x(t)y(t)dt,$$

thus arriving at the norm $\|x\|$, where $\|x\|^2=\int_0^1 x^2(t)dt$. With this norm, C^0 is not complete; its completion is the Hilbert space L^2 of all functions whose squares are Lebesgue-integrable on the unit interval ([18], pp. 32–3 (Ex. 3.1)).

We digress briefly to observe that not every norm can be generated by a scalar product. If there were to be a scalar product (x, y) inducing a given norm $\|x\|$, then (x, y) could be calculated from the norm by the formula

$$(x, y) = \tfrac{1}{4}[\|x + y\|^2 - \|x - y\|^2].$$

It follows immediately from the properties of the norm that the functional $f(x, y)$ on the right is symmetric, and that $f(x, x)=\|x\|^2$ is positive. When will $f(x, y)$ be bilinear, and therefore a scalar product? We give the earliest of the many conditions in the literature ([25], [26]): It must be true of any vectors x and y that

$$\|x + y\|^2 + \|x - y\|^2 = 2\|x\|^2 + 2\|y\|^2$$

(parallelogram law). One may verify by obvious examples that the space C^0 mentioned above with the norm $\|x\|=\sup_t |x(t)|$, while assuredly a Banach space, does not obey the parallelogram law.

An essential property of a Hilbert space H is that H is isomorphic and isometric with H^*; this commonly expressed by saying that H is self-dual. The

basic theorem ([16], p. 138; [24], p. 31 (Th. 3)) states that for each x^* in H^* there exists in H a unique y, depending on x^*, such that $\|y\| = \|x^*\|$ and, for each x in H, $x^*(x) = (y, x)$. Vice versa, for each fixed $y \in H$, the functional (y, x) is a bounded linear functional of x, and therefore an element x^* of H^*. Let us say that x^* and y are "associated."

If H is finite dimensional then in each coordinate system there exists a symmetric positive matrix (g_{ij}) such that $(y, x) = \sum_{i,j} y^i g_{ij} x^j$, where y^i and x^j are components of y and x, and for each x^* there exist numbers x_j^* such that $x^*(x) = \sum_j x_j^* x^j$. Thus x^* and y will be associated if $x_j^* = \sum_i y^i g_{ij}$, a relationship which may be expressed in the terminology of tensor analysis by saying that x_j^* and y^i are covariant and contravariant components of the same vector ([27], pp. 56–7; [28], p. 39).

Now let L be a bounded linear mapping $x \rightarrow y = Lx$ on H into H. The dual mapping L^* is defined on H^* into H^*, by the equation $x^* = y^*L^*$ where $y^*L^*(x) = y^*(Lx)$. By virtue of the 1-1 correspondence between H and H^* just mentioned, L^* induces a bounded linear mapping L' on H into H; L' is called the adjoint of L. This mapping on H into H has the property that, for every x, $(y, Lx) = (L'y, x)$; this equation may be used, in fact, to define L' on H into H. The adjective "adjoint" is sometimes applied in general to what has here been called the dual L^* (on Y^* into X^*) of a linear mapping L on a linear space X into a linear space Y; there may be some merit in reserving the adjective for the mapping L' on a Hilbert space H into H induced by L^* on H^* into H^* ([16], p. 205).

For example, let L denote a linear ordinary differential operator, and L' its adjoint in the sense usual in the theory of differential equations. Without giving details, which are by no means trivial, let us note that, with the scalar product

$$(x, y) = \int_a^b x(t)y(t)dt,$$

the equation $(x, Ly) = (L'x, y)$ amounts to Green's formula

$$\int_a^b (xLy - yL'x)dt = 0,$$

which is valid for functions satisfying appropriate boundary and other conditions ([6], p. 255; [8], p. 86).

Another consequence of the presence of a scalar product is the notion of orthogonality: x and y are said to be orthogonal if $(x, y) = 0$. It is clear that if the functional x^* and the vector y are associated, so that, for all x, $x^*(x) = (y, x)$, then x^* and x are normal if and only if y and x are orthogonal. Now if Z is a subspace of X then the class of all vectors orthogonal to every vector of Z is a closed subspace $Z^\perp$ of X, and it is evident that the class of functionals associated with vectors of $Z^\perp$ is the subspace of X^* previously called Z^N. In this sense the notion of normality reduces in a Hilbert space to that of orthogonality.

It is clear that if Z is a closed subspace of H then Z and $Z^{\perp}$ are complementary closed subspaces of H, so that the difficulty noticed earlier in L^p if $p \neq 2$ does not arise for $p=2$. The presence of the distinguished complement $Z^{\perp}$ is helpful in many investigations.

A Euclidean space of finite dimension d is a Hilbert space, and an essential part of its furniture is a numerous family (if $d \geqq 2$) of orthonormal systems. In any Hilbert space H, one says that a class V of vectors is an orthonormal system if $u \in V$ and $v \in V$ imply

$$(u, v) = \begin{cases} 1 & \text{if } u = v \\ 0 & \text{if } u \neq v. \end{cases}$$

One says that H is separable if H has a countable subset that is dense in H. It is not hard to show that H is separable if and only if H has a maximal orthonormal system that is countable ([16], p. 95 (Sec. 3)).

Let H be separable, and let v_n $(n=1, 2, \cdots)$ be a maximal orthonormal sequence ([16], p. 113 (Sec. 11); [21], pp. 7–16; [24], Sec. 14). Then each $x \in H$ has a Fourier expansion

$$x = \sum_{n=1}^{\infty} (x, v_n)v_n,$$

where the equation is understood to mean that

$$\left\| x - \sum_{n=1}^{N} (x, v_n)v_n \right\| \to 0$$

as $N \to \infty$. Moreover,

$$(x, y) = \sum_{n=1}^{\infty} (x, v_n)(y, v_n);$$

in particular, $\|x\|^2 = \sum_{n=1}^{\infty} (x, v_n)^2$.

Now the space l_2 of real sequences $s = (s_1, s_2, \cdots, s_n, \cdots)$ such that $\|s\|^2 = \sum_{n=1}^{\infty} s_n^2 < \infty$ is easily seen to be a separable Hilbert space; the scalar product is $(s, t) = \sum_{n=1}^{\infty} s_n t_n$. If H is any separable Hilbert space, with maximal orthonormal sequence v_n, then the mapping $x \to s$ on H into l_2 given by $x \to s = ((x, v_n))$ is easily seen to be 1-1, isomorphic, and also isometric since $\|x\|^2 = \|s\|^2$. Thus every separable Hilbert space is in this sense equivalent to l_2, which is therefore the prototype of the infinite-dimensional separable Hilbert space. In fact, l_2 was the first Hilbert space to be investigated.

In the space L^2 of functions $x(t)$ on the interval $-\pi \leqq t \leqq \pi$ with

$$\|x\|^2 = \int_{-\pi}^{\pi} x^2(t)dt < \infty,$$

the integral being taken in the Lebesgue sense, the system of functions

$$\frac{1}{\sqrt{2\pi}}, \quad \frac{1}{\sqrt{\pi}}\cos nt, \quad \frac{1}{\sqrt{\pi}}\sin nt \qquad (n = 1, 2, \cdots)$$

is a maximal orthonormal system. The Fourier expansion with respect to this system is the familiar Fourier series. Systems of properly normalized orthogonal polynomials on other intervals yield similar examples ([**29**], [**30**]). The normalized eigenfunctions of any self-adjoint linear boundary-value problem for an ordinary differential equation on a finite interval constitute a maximal orthonormal set in L^2 on that interval ([**8**], Ch. 7).

Let us mention an example ([**31**], Th. 50, 52) in the space L^2 on the interval $0<t<\infty$ with the norm $\|x\|^2=\int_0^\infty x^2(t)dt$. For $x\in L^2$ define the Fourier-Plancherel sine transform as follows:

$$F_s x(t) = \underset{a\to\infty}{\text{l.i.m.}} \sqrt{\frac{2}{\pi}} \int_0^a x(u) \sin tu du,$$

where the limit is taken in the L^2 norm. It turns out that the mapping $x\to F_s x$ is an isomorphic isometric mapping of L^2 onto itself. Moreover, F_s is involutory: If $y=F_s x$ then $x=F_s y$.

Let us bring these remarks to a close by mentioning very briefly the problem of "diagonalizing" a symmetric (*i.e.*, self-adjoint) transformation L on a Hilbert space H into H ([**1**], Ch. IX, Sec. 10; [**2**], Sec. 79; [**22**], Sec. 92). If H is finite-dimensional then there exist n real numbers $\lambda_1 \leqq \cdots \leqq \lambda_n$ and n mutually orthogonal subspaces $H_1, \cdots, H_n$ such that if $x\in H$ then $x=x_1+\cdots+x_n$ with unique $x_i\in H_i$, and $Lx_i=\lambda_i x_i$ $(i=1, \cdots, n)$. In a coordinate system adapted to the subspaces H_i, the matrix of L has the numbers λ_i down the diagonal and zeros elsewhere, and this representation of L is called its spectral representation. In this coordinate system, $(Lx, x)=\sum_i \lambda_i(x_i, x_i)=\sum_i \lambda_i\|x_i\|^2$.

In order to arrive at a formulation that can be extended readily to the infinite-dimensional case, we let G_t denote the direct sum of those H_i having $\lambda_i<t$, let E_t denote the transformation projecting H onto G_t, and define $\alpha=\inf (Lx, x)$ and $\beta=\sup (Lx, x)$, both taken for $\|x\|=1$. Then for each $x\in H$ the function $\sigma_x(t)=(E_t x, x)$ has these properties:

1. $\sigma_x(t)=0$ for $t\leqq\alpha$, and $\sigma_x(t)=(x, x)$ for $t>\beta$;
2. $\sigma_x(t)$ is nondecreasing and continuous from the left;
3. $\sigma_x(t)$ is constant for $\lambda_i<t<\lambda_{i+1}$ and $\sigma_x(\lambda_i+)-\sigma_x(\lambda_i)=\|x_i\|^2$.

Using the Stieltjes integral, we may then write

$$(*) \qquad (Lx, x) = \int_\alpha^\beta t d\sigma_x(t) = \int_\alpha^\beta t d(E_t x, x),$$

a relation sometimes expressed by writing $L=\int_\alpha^\beta t dE_t$.

One can show ([**22**], Sec. 107) for any bounded self-adjoint (symmetric) L on any H into H that there exists a one-parameter family of projections E_t such that the function $\sigma_x(t)=(E_t x, x)$ has the first and second of the properties listed

above, and has the third property in an appropriately modified sense. One can still show, moreover, that (*) holds.

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KHINTCHINE'S CONSTANT

DANIEL SHANKS AND J. W. WRENCH, JR., David Taylor Model Basin

Representation by series. The absolute constant of Khintchine [1], [2], [3] may be defined by the slowly convergent infinite product

$$K = \prod_{n=1}^{\infty}\left(1 + \frac{1}{n(n+2)}\right)^{\ln n/\ln 2}, \tag{1}$$

and its logarithm is, therefore, given by the infinite series

$$\ln 2 \ln K = \sum_{n=2}^{\infty} \ln n \ln \frac{(n+1)^2}{n(n+2)}. \tag{2}$$

In view of the equation

$$2 = \prod_{n=1}^{\infty} \frac{(n+1)^2}{n(n+2)}, \tag{3}$$

we may substitute in the right member of equation (2) first

$$\ln 2 \ln \frac{3}{2} = \sum_{n=2}^{\infty} \ln 2 \ln \frac{(n+1)^2}{n(n+2)},$$

then

$$\ln \frac{3}{2} \ln \frac{4}{3} = \sum_{n=3}^{\infty} \ln \frac{3}{2} \ln \frac{(n+1)^2}{n(n+2)},$$

and finally, by induction, we obtain [4]

$$\ln 2 \ln K = \sum_{n=2}^{\infty} \ln \left(\frac{n}{n-1}\right) \ln \left(\frac{n+1}{n}\right), \tag{4}$$

which can be written alternatively

$$\ln 2 \ln K = -\sum_{n=2}^{\infty} \ln \left(1 - \frac{1}{n}\right) \ln \left(1 + \frac{1}{n}\right). \tag{5}$$

Now the Cauchy product of the Maclaurin series for $\ln(1-x)$ and $\ln(1+x)$ is readily found to be

$$\begin{aligned} &-\ln(1-x)\ln(1+x) \\ &= x^2 + \left(1 - \frac{1}{2} + \frac{1}{3}\right)\frac{x^4}{2} + \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}\right)\frac{x^6}{3} + \cdots, \end{aligned} \tag{6}$$

and consequently equation (5) may be written

$$\ln 2 \ln K = \sum_{n=2}^{\infty} n^{-2} + \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3}\right) \sum_{n=2}^{\infty} n^{-4} \tag{7}$$
$$+ \frac{1}{3}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5}\right) \sum_{n=2}^{\infty} n^{-6} + \cdots .$$

This transformation permits the computation of $\ln 2 \ln K$ by use of the appropriate values of the accurately known Riemann zeta function, $\zeta(s) = \sum_{n=1}^{\infty} n^{-s}$. Furthermore, nearly geometric convergence has been attained in equation (7), with the ratio of successive terms tending to a limit of $1/4$.

More rapid convergence is obtained by a second application of equation (3), this time in the form

$$\ln 2 = \sum_{n=2}^{\infty} n^{-2} + \frac{1}{2} \sum_{n=2}^{\infty} n^{-4} + \frac{1}{3} \sum_{n=2}^{\infty} n^{-6} + \cdots . \tag{8}$$

Accordingly, the right member of equation (7) can be rearranged to read

$$\ln 2 \ln K = \ln 2 - \frac{1}{2\cdot 3}\left(\ln 2 - \sum_{n=2}^{\infty} n^{-2}\right) \tag{9}$$
$$- \frac{1}{4\cdot 5}\left(\ln 2 - \sum_{n=2}^{\infty} n^{-2} - \frac{1}{2}\sum_{n=2}^{\infty} n^{-4}\right)$$
$$- \frac{1}{6\cdot 7}\left(\ln 2 - \sum_{n=2}^{\infty} n^{-2} - \frac{1}{2}\sum_{n=2}^{\infty} n^{-4} - \frac{1}{3}\sum_{n=2}^{\infty} n^{-6}\right) - \cdots .$$

Still more rapid convergence is obtained by deferring the substitution of (6) until after several terms of the series in (5) have been summed. Thus, we can write

$$\ln 2 \ln K = - \sum_{n=2}^{N} \ln\left(1 - \frac{1}{n}\right) \ln\left(1 + \frac{1}{n}\right) + \sum_{n=N+1}^{\infty} n^{-2} \tag{7a}$$
$$+ \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3}\right) \sum_{n=N+1}^{\infty} n^{-4} + \cdots .$$

The generalization of (8), namely

$$\ln\left(1 + \frac{1}{N}\right) = \sum_{n=N+1}^{\infty} n^{-2} + \frac{1}{2} \sum_{n=N+1}^{\infty} n^{-4} + \frac{1}{3} \sum_{n=N+1}^{\infty} n^{-6} + \cdots , \tag{8a}$$

permits us to deduce from (7a) the following rapidly convergent series

$$\ln 2 \ln K = - \sum_{n=2}^{N} \ln\left(1 - \frac{1}{n}\right) \ln\left(1 + \frac{1}{n}\right) + \ln\left(1 + \frac{1}{N}\right) \tag{9a}$$
$$- \frac{1}{2\cdot 3}\left(\ln\left(1 + \frac{1}{N}\right) - \sum_{n=N+1}^{\infty} n^{-2}\right) - \cdots .$$

Representation by integrals. It is of interest to sum these series in closed form, and this can be accomplished in terms of a definite integral. For, if the coefficients of the terms of the series in (7) are written as follows

$$
\begin{aligned}
\ln 2 \ln K = &\left(\ln 2 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \cdots\right)\sum_{n=2}^{\infty} n^{-2} \\
&+ \frac{1}{2}\left(\ln 2 + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \cdots\right)\sum_{n=2}^{\infty} n^{-4} \\
&+ \frac{1}{3}\left(\ln 2 + \frac{1}{6} - \frac{1}{7} + \frac{1}{8} - \cdots\right)\sum_{n=2}^{\infty} n^{-6} + \cdots,
\end{aligned} \tag{10}
$$

then, in view of equation (8) and of the relation

$$
\frac{1}{2n} - \frac{1}{2n+1} + \frac{1}{2n+2} - \cdots = \int_0^1 \frac{x^{2n-1}}{1+x}\,dx, \tag{11}
$$

equation (10) can be written

$$
\ln 2 \ln K = (\ln 2)^2 + \int_0^1 \frac{dx}{x(1+x)}\left[\sum_{n=2}^{\infty}\frac{x^2}{n^2} + \frac{1}{2}\sum_{n=2}^{\infty}\frac{x^4}{n^4} + \cdots\right]
$$

or

$$
\ln 2 \ln\left(\frac{K}{2}\right) = \int_0^1 \frac{dx}{x(1+x)}\left[-\ln\left(1-\frac{x^2}{4}\right)\left(1-\frac{x^2}{9}\right)\left(1-\frac{x^2}{16}\right)\cdots\right]. \tag{12}
$$

The infinite product in the right member of this equation is well known, with relation to the sine function, and we can accordingly write

$$
\ln 2 \ln\left(\frac{K}{2}\right) = \int_0^1 \frac{dx}{x(1+x)} \ln\left[\frac{\pi x(1-x^2)}{\sin \pi x}\right]. \tag{13}
$$

An alternative form of this result, involving the gamma function, is

$$
\ln 2 \ln\left(\frac{K}{2}\right) = \int_0^1 \frac{dx}{x(1+x)} \ln\left[\Gamma(2+x)\Gamma(2-x)\right]. \tag{14}
$$

Numerical results. As an illustration of the computational effectiveness of the preceding series, equation (9a) with $N=2$ was employed to evaluate ln 2 ln K to more than 65 decimal places. The numerical series may be written

$$
\begin{aligned}
\ln 2 \ln K = \ln\frac{3}{2} + \ln 2 \ln \frac{3}{2} - &\left\{\frac{1}{2\cdot 3}\sum_{k=2}^{\infty}\frac{S_{2k}''}{k}\right. \\
&\left. + \frac{1}{4\cdot 5}\sum_{k=3}^{\infty}\frac{S_{2k}''}{k} + \frac{1}{6\cdot 7}\sum_{k=4}^{\infty}\frac{S_{2k}''}{k} + \cdots\right\},
\end{aligned} \tag{15}
$$

where, for simplicity in writing, S_{2k}'' has been used to represent $\sum_{n=3}^{\infty} n^{-2k}$.

Elaborate tables by R. Liénard [5] contain values of $\sum_{n=2}^{\infty} n^{-2k}$ that are correct to 50 decimal places. These data were checked and then extended to at least 65 decimal places, corresponding to $k=1$ (1) 68, by one of the present writers.

Omnibus checks on successive stages of the calculation consisted of using the relations

$$\sum_{k=1}^{\infty} S''_{2k} = \frac{5}{12}, \tag{16}$$

$$\sum_{k=1}^{\infty} \frac{S''_{2k}}{k} = \ln \frac{3}{2}, \tag{17}$$

$$\sum_{r=2}^{\infty} \sum_{k=r}^{\infty} \frac{S''_{2k}}{k} = \frac{5}{12} - \ln \frac{3}{2}. \tag{18}$$

Rounded 65-place approximations to S''_{2k}, when substituted in these checking relations, yielded discrepancies of 3.3×10^{-65}, 0.4×10^{-65}, and 31×10^{-65}, respectively. Subsequent introduction of two additional guard figures in the calculations led to the following approximation, rounded to 66 decimal places.

$\ln 2 \ln K$

$= 0.68472\ 47885\ 63157\ 12329\ 91461\ 48755\ 77762\ 04606\ 75416\ 33744\ 88366\ 06289\ 86781\ 6 \cdots.$

Corresponding to this result, the following approximations to Khintchine's constant and its natural logarithm were calculated:

$\ln K = 0.98784\ 90568\ 33810\ 78966\ 92547\ 27147\ 07295\ 43261\ 99254\ 96088\ 67354\ 27755\ 30068\ 7 \cdots,$

$K = 2.68545\ 20010\ 65306\ 44530\ 97148\ 35481\ 79569\ 38203\ 82293\ 99446\ 29530\ 51152\ 34556 \cdots.$

References

1. A. Khintchine, Metrische Kettenbruchprobleme, Compositio Math., vol. 1, 1934, p. 376.

2. D. H. Lehmer, Note on an absolute constant of Khintchine, this MONTHLY, vol. 46, 1939, pp. 148–152.

3. D. Shanks, Note MTE 164, Math. Tables Aids Comput., vol. 4, 1950, p. 28. (The second formula herein has a typographical error, and should read $s=\ln 2+\sum_{r=2}^{\infty} \cdots$. In an informal communication, D. H. Lehmer sent one of the authors a value of Khintchine's constant calculated correctly to 17 decimal places from this formula.)

4. D. Shanks, Non-linear transformations of divergent and slowly convergent sequences, J. Math. Phys., vol. 34, 1955, p. 40.

5. R. Liénard, Tables Fondamentales à 50 décimales des Sommes S_n, u_n, $\sum_n$, Centre de Documentation Universitaire, Paris, 1948.

NOTE ON THE DISTRIBUTIVE LAWS

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1. A ring is usually defined to be a system R with two compositions, called addition and multiplication, such that

I. R is an abelian group with respect to addition;
II. R is a semigroup with respect to multiplication;
III. R satisfies the distributive laws:

$$x(y+z) = xy + xz, \qquad (y+z)x = yx + zx, \qquad \text{for all } x, y, z \in R.$$

In this note, we discuss first a system S arising from the above definition of a ring by replacing III by the weaker distributive laws:

IV. In S there exist fixed elements e_1 and e_2 such that

$$x(y+z) = xy + xz - e_1, \qquad (y+z)x = yx + zx - e_2, \qquad \text{for all } x, y, z \in S.$$

Such a system is called a w-ring. In this section, we give several lemmas which hold in a w-ring S.

LEMMA 1. $e_1 = 00 = e_2$.

Proof. Setting $x=y=z=0$ in the first equation of IV, we have $00 = 00 + 00 - e_1$. Thus we have $e_1 = 00$. That $e_2 = 00$ can be proved dually.

This common element $e_1 = e_2$ is called the defining element of the w-ring S and in this section we denote it by e.

LEMMA 2. $x0 = e = 0x$ *for all* $x \in S$.

Proof. We have $x0 = x(0+0) = x0 + x0 - e$. Therefore $x0 = e$ and dually we have $0x = e$.

LEMMA 3. $xe = e = ex$ *for all* $x \in S$.

Proof. We have $xe = x(00) = (x0)0 = e$ and dually we have $ex = e$.

LEMMA 4. $x(y-z) = xy - xz + e$, $(y-z)x = yx - zx + e$ *for all* $x, y; z \in S$.

Proof. We have $xy = x((y-z)+z) = x(y-z) + xz - e$ and so we have $x(y-z) = xy - xz + e$. The second equality can be proved dually.

LEMMA 5. *The set* $\{x \mid xy = yx = e$ *for all* $y \in S\}$ *is a subgroup of the additive group of* S *and it contains the element* e.

Proof. In order to prove the first assertion, it suffices to show that if $py = yp = e$ and $qy = yq = e$ for all $y \in S$, then $(p-q)y = y(p-q) = e$ for all $y \in S$. Indeed, we have $(p-q)y = py - qy + e = e - e + e = e$ and, dually, we have $y(p-q) = e$. The second assertion of the lemma follows from Lemma 3.

In a w-ring S the order of the defining element e with respect to the additive group of S is called the order of S. Evidently, a w-ring S is a ring if and only if $e=0$ and hence if and only if the order of S is 1.

2. In this section we discuss the existence of w-rings. Let n be any positive integer or ∞. Taking an abstract element e we construct the cyclic additive group G_n generated by e and with order n. We define multiplication in G_n as follows:

$$xy = e \qquad \text{for all } x, y \in G_n.$$

Then we have the following

THEOREM 1. *The system G_n constructed above is a w-ring with order n and defining element e. Conversely, any w-ring S of order n has a subsystem which is isomorphic to the system G_n.*

Proof. By definition, G_n is an abelian group with respect to addition. For all $x, y, z \in G_n$ we have $(xy)z=e=x(yz)$ so that G_n is a semigroup with respect to multiplication. Also, for all $x, y, z \in G_n$ we have $x(y+z)=e$, $xy+xz-e=e+e-e=e$, and hence $x(y+z)=xy+xz-e$. Dually, we have $(y+z)x=yx+zx-e$. Thus G_n is a w-ring with defining element e. Moreover, by definition, G_n has order n.

Next we consider an arbitrary w-ring S of order n. By Lemma 5 the set $\{x \mid xy=yx=e \text{ for all } y \in S\}$ contains the cyclic additive subgroup G_n' generated by the defining element e of S. It is clear that, for all $x, y \in G_n'$, we have $xy=e$. Therefore G_n' is closed with respect to addition and multiplication and so is subsystem of S. It is easy to see that G_n' is isomorphic to G_n.

3. Let us consider a w-ring S with defining element e. We now define a new composition $\circ$ as follows:

$$x \circ y = xy - e. \tag{1}$$

Then S becomes a ring with respect to the original addition and the new multiplication $\circ$. In fact, we have $(x \circ y) \circ z=(xy-e) \circ z=(xy-e)z-e$ and so, by Lemmas 4 and 3, we have $(x \circ y) \circ z=xyz-ez+e-e=xyz-e$. On the other hand, $x \circ (y \circ z)=x \circ (yz-e)=x(yz-e)-e=xyz-xe+e-e=xyz-e$. Hence we have $(x \circ y) \circ z=x \circ (y \circ z)$ and thus S is a semigroup with respect to the composition $\circ$. Also we have $x \circ (y+z)=x(y+z)-e=xy+xz-e-e=(xy-e)+(xz-e)=x \circ y+x \circ z$ and dually we have $(y+z) \circ x=y \circ x+z \circ x$. Hence S is a ring with respect to addition and the multiplication $\circ$. Moreover, in this ring we have $e \circ x=x \circ e=0$ for all $x \in S$. In fact, $e \circ x=ex-e=e-e=0$ and dually $x \circ e=0$.

Conversely, let us consider a ring S with multiplication denoted by $\circ$. In S we take an element e such that $e \circ x=x \circ e=0$ for all $x \in S$ and we define a new product as follows:

$$xy = x \circ y + e, \tag{2}$$

Then S becomes a w-ring with respect to the original addition and the new multiplication and e is the defining element of this w-ring. In fact, $(xy)z = (x \circ y+e)z = (x \circ y+e) \circ z+e = x \circ y \circ z+e \circ z+e = x \circ y \circ z+e$. On the other hand, $x(yz) = x(y \circ z+e) = x \circ (y \circ z+e)+e = x \circ y \circ z+x \circ e+e = x \circ y \circ z+e$. Therefore we have $(xy)z = x(yz)$, that is, S is a semigroup with respect to the new multiplication. Also we have $x(y+z) = x \circ (y+z)+e = x \circ y+x \circ z+e = (x \circ y+e)+(x \circ z+e)-e = xy+xz-e$ and dually $(y+z)x = yx+zx-e$.

Summarizing these results, we have the

THEOREM 2. *Let S be a w-ring with defining element e. If we define a new product in S by* (1), *then S is a ring with respect to the original addition and the multiplication $\circ$ and in this ring, $e \circ x = x \circ e = 0$ for all $x \in S$. Conversely, let S be a ring with multiplication denoted by $\circ$. If we define a new product in S by* (2), *then S is a w-ring with respect to the original addition and the new multiplication and e is the defining element of this w-ring.*

4. In this section we consider a system T arising from the definition (Sec. 1) of a ring by replacing III by the composite distributive law:

V. $(u+v)(x+y) = ux+uy+vx+vy$ for all $u, v, x, y \in T$.

Such a system is called a c-ring. We first give several lemmas which hold in a c-ring T. In this section we denote 00 by e.

LEMMA 6. $e+e+e=0$.

Proof. Setting $u=v=x=y=0$ in V, we have $e=(0+0)(0+0)=e+e+e+e$ and hence $e+e+e=0$.

LEMMA 7. $x0=0x=e$ *for all* $x \in T$.

Proof. We have $x0=(x+0)(0+0)=x0+x0+e+e$. Cancelling $x0$, we have $x0+e+e=0=e+e+e$. Cancelling $e+e$, we obtain $x0=e$. Dually we have $0x=e$.

LEMMA 8. $x(y+z)=xy+xz-e$, $(y+z)x=yx+zx-e$ *for all* $x, y, z \in T$.

Proof. We have $x(y+z)=(x+0)(y+z)=xy+xz+0y+0z=xy+xz+e+e = xy+xz-e$. Dually, we have $(y+z)x=yx+zx-e$.

COROLLARY. *If a c-ring has an element a such that $a0=0$, then T is a ring.*

In fact, in this case $e=0$ by Lemma 7 and then T is a ring by Lemma 8.

We now obtain the following

THEOREM 3. *A c-ring is a w-ring of order* 3 *or* 1 *according as* $00 \neq 0$ *or* $00=0$ *and* 00 *is the defining element of the w-ring. Conversely, a w-ring of order* 3 *or* 1 *is a c-ring.*

Proof. The first assertion follows from Lemmas 6 and 8. Now let S be a w-ring of order 3 or 1. Then $(u+v)(x+y) = (u+v)x+(u+v)y-e = (ux+vx-e)$

$+(uy+vy-e)-e=ux+vx+uy+vy-(e+e+e)=ux+vx+uy+vy$. Thus S is a c-ring.

COROLLARY. *G_3 in Section 2 is a c-ring which is not a ring.*

Appendix. John L. Kelley, *General Topology*, p. 18, gives a definition of a ring as a system which we call a c-ring. The corollaries to Theorem 3 and Lemma 8 above therefore provide a solution to Problem 4784, this MONTHLY, vol. 65, 1958, p. 289. Another solution appears on page 318 of this issue.

THE POLYNOMIAL CORRELATION COEFFICIENT

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Let it be assumed that the variates y and x are related in such a way that the regression of y on x is a polynomial curve. The x-variates are given and are hence fixed. Let the two variates be given in standard units with respect to the sample. This means that their arithmetic means and their sums of squares, respectively, are equal to

$$\bar{y} = \bar{x} = 0 \quad \text{and} \quad \sum x^2 = \sum y^2 = n,$$

where n is the number of pairs of values. Let it also be assumed that the x-variates have their first $2N$ moments equal to those for a normal variable, which means that $\alpha_{2i+1}=0$, $\alpha_{2i}=(2i)!2^i(i!)$, for the fixed interval in which x is defined.

Let the equation of the predicting polynomial at the point x_j be

$$f(x_j) = \sum_{i=0}^{N} a_i x_j^i.$$

For $N=3$, $y=f(x)=a_0+a_1x+a_2x^2+a_3x^3$.

The square of the correlation coefficient between y and $f(x)$ is equal to

$$\mathbf{r}^2 = 1 - \sum (f-y)^2/n$$

for large values of n. By maximizing $\mathbf{r}^2$, four equations arise from which the constants are found to be

$$a_0 = -\gamma/\sqrt{2}, \qquad a_1 = (5r - \sqrt{15}\delta)/2,$$
$$a_2 = \gamma/\sqrt{2}, \qquad a_3 = (\sqrt{15}\delta - 3r)/6,$$

where r is the linear correlation coefficient between observed values of y and x, γ is the correlation coefficient between observed values of y and x^2, and δ is the correlation coefficient between observed values of y and x^3. (Note that the moments, $\sum xy$, $\sum x^2y$, and $\sum x^3y$, have respectively the values nr, $n\gamma\sqrt{2}$, and $n\delta\sqrt{15}$).

The average of the first derivatives of the estimating curve at the given values of x is equal to

$$\bar{f}'(x) = a_1 + 3a_3 = (5r - \sqrt{15}\delta)/2 + (\sqrt{15}\delta - 3r)/2 = r,$$

which indicates that the linear correlation coefficient, r, between y and x is the average of the first derivatives of the predicting curve at the given values of the independent variates. The number of values of the derivative at each x-value depends upon the number of y-values. The averages of the second and third derivatives of the predicting curve at the given values of x are, respectively, $\bar{f}'' = 2a_2$, $\bar{f}''' = 6a_3$.

On substituting the values of a_1, a_2, and a_3 in the expression for $\mathbf{r}^2$ one obtains*

$$\mathbf{r}^2 = (\bar{f}')^2/1! + (\bar{f}'')^2/2! + (\bar{f}''')^2/3! = \sum (\bar{f}^{(i)})^2/i!,$$

which expresses the square of the nonlinear correlation coefficient between observed values of y and x as a function of the averages of the derivatives of the estimating curve at the given values of x.

By use of characteristic functions of bivariate frequency distributions Wicksell† developed an expression for the regression of the mean which is an infinite series involving cumulants and derivatives of the distribution function of the independent variable. The expression for $\mathbf{r}^2$ here discussed also involves moments and averages of the derivatives of $f(x)$.

An example. The following table lists, in standard units, data pertaining to annual volume in cubic feet of Eastern White Pines in relation to their ages. The standard units were calculated before the observations were arranged in this table. The ages are approximately normally distributed. The broken line connects the means of the volume increment arrays as recorded in this table.

It is assumed that the annual increment, y, is a polynomial function of the age x, *i.e.*, $y = a_0 + a_1x + a_2x^2$. The quantities $\sum x^2$ and $\sum y^2$ are not quite equal to 161, the number of pairs, owing to the rounding-off of decimals in the standard units. By the method of least squares the best fitting parabola to the data, before they were arranged in the table, is

$$y = 0.399321 + 0.460474x - 0.398911x^2.$$

By an analysis of variance this parabola was shown to be a significantly better fit to the data than a straight line. A third-degree polynomial was found to be not a significantly better fit than the parabola.

The nonlinear correlation coefficient between y and x, as found from the standard error of estimate, $\sigma_e = \sigma_y\sqrt{(1 - r_{xy}^2)}$, is $\mathbf{r}_{yx} = 0.699$. The nonlinear correlation coefficient as found by the derivatives, or by formula (10) below, is

* For $N = 3$, $\mathbf{r}^2 = (5/2)r^2 - \sqrt{15}\, r\delta + (5/2)\delta^2 + \gamma^2$.

† S. D. Wicksell, Analytical theory of regression, Medd. Lunds Astr. Obs., Series 2, No. 69, 1934. M. G. Kendall, The Advanced Theory of Statistics, London, 1946, Vol. II, pp. 142–145.

$\mathbf{r}_{yx}=0.728$. The two values are approximately the same, showing that the non-linear correlation coefficient can be found by a function of the derivatives of the polynomial at the given values of the independent variable.

ANNUAL VOLUME INCREMENT IN CUBIC FEET IN RELATION TO THE AGE OF EASTERN WHITE PINES (STANDARD UNITS)

AGES (MIDPOINTS)

ANNUAL VOLUME INCREMENT—y	-2.46	-1.97	-1.48	$-.98$	$-.49$	0	.49	.98	1.48	1.97	2.46
1.75– 2.25							2	1			
1.25– 1.75					2	4	5	3			
.75– 1.25					5	4	5	2	2		
.25– .75				5	5	9	6	6	1	1	
$-.25$– .25			1	4	5	7	3	1	6	2	
$-.75$– $-.25$			3	3	8	7	5	3	1	2	2
-1.25– $-.75$			2	4	2		2	2	1		
-1.75–-1.25		2	2	3	1			1			
-2.25–-1.75		1	3								
-2.75–-2.25		1									
-3.25–-2.75	1	1									
-3.75–-3.25	1										
	2	5	11	19	28	31	28	19	11	5	2
Average	-3.25	-2.00	-1.14	$-.39$	.11	.35	.54	.42	.09	$-.10$	$-.50$

The general case. Let the equation of the predicting curve, about which the observed values of y fall be

$$f(x) = \sum_{i=0}^{N} a_i x^i. \tag{1}$$

Let the fixed x values be chosen so that

$$\sum_{j=1}^{n} x_j^i/n = C_i, \tag{2}$$

where $C_{2i+1}=0$ and $C_{2i}=(2i)!/2^i(i!)$. The square of the correlation coefficient between y and $f(x)$ is

$$\mathbf{r}^2 = 1 - \sum (f - y)^2/n,$$

since $\sum y^2 = n$.

To maximize $\mathbf{r}^2$ with respect to the a_i that appear in f, we equate to zero the

partial derivatives with respect to each a_i. Hence, if we set $f_j = f(x_j)$ we have

$$0 = \sum_{j=1}^{n} [2(f_j - y_j)(\partial f_j / \partial a_i)] = 2 \sum_{j=1}^{n} (f_j - y_j) x_j^i$$

or $\sum_j x_j^i f_j = \sum_j x_j^i y_j$.

From the above $\mathbf{r}^2 = \sum fy/n$, since $\sum y^2 = n$, $\sum (f^2) = \sum fy$ and $f_j = y|_{x=x_j}$. Hence

$$\begin{aligned} \mathbf{r}^2 &= \sum_{i=0}^{N} a_i \sum_{j=1}^{n} [x_j^i f_j]/n \\ &= \sum_{i=0}^{N} a_i \sum_{j=1}^{n} \left[\sum_{k=0}^{N} a_k x_j^{i+k} \right] \Big/ n = \sum_{i,k}^{N} a_i a_k C_{i+k}. \end{aligned} \tag{4}$$

Let the average of the kth derivative of f at the given values of x be

$$g_k = (1/n) \sum_j (d^k f/dx^k)_j = \sum_{i=k}^{N} a_i \binom{i}{k} k! C_{i-k}. \tag{5}$$

Let the quantity R be defined by

$$R^2 = \sum_{k=1}^{N} g_k^2/k! = \sum_{k=1}^{N} \left[\sum_{i=k}^{N} a_i \binom{i}{k} k! C_{i-k} \right]^2 \Big/ k! \tag{6}$$

or

$$R^2 = \sum_{i,j,k=1}^{N} a_i a_j \binom{i}{k} \binom{j}{k} k! C_{i-k} C_{j-k}.$$

To establish the fact that $\mathbf{r}^2 = R^2$ it is necessary to prove the identity

$$\sum_{k=1}^{N} \binom{i}{k} \binom{j}{k} k! C_{i-k} C_{j-k} = C_{i+j}. \tag{7}$$

When $i+j$ is odd, both members of (7) vanish. It remains to prove two identities for the cases in which i and j are both even or both odd:

$$\sum_k \binom{2i}{2k} \binom{2j}{2k} (2k)! C_{2i-2k} C_{2j-2k} = C_{2i+2j} \tag{7a}$$

and

$$\sum \binom{2i+1}{2k+1} \binom{2j+1}{2k+1} (2k+1)! C_{2i-2k} C_{2j-2k} = C_{2i+2j+2}. \tag{7b}$$

These last two equations reduce to

$$\sum_k \frac{(2i)!(2j)!2^{2k}}{(2k)!(i-k)!(j-k)!2^{i+j}} = \frac{(2i+2j)!}{2^{i+j}(i+j)!} \tag{8a}$$

and

$$\sum_k \frac{(2i+1)!(2j+1)!2^{2k+1}}{(2k+1)!(i-k)!(j-k)!2^{i+j+1}} = \frac{(2i+2j+2)!}{2^{i+j+1}(i+j+1)!}, \tag{8b}$$

respectively. Let us consider the coefficients of $x^{2i}y^{2j}$ in $(x^2+2xy+y^2)^{i+j}$ and in $(x+y)^{2i+2j}$; they are, respectively,

$$\sum_k \frac{(i+j)!2^{2k}}{(i-k)!(j-k)!(2k)!} \quad \text{and} \quad \frac{(2i+2j)!}{(2i)!(2j)!}; \tag{9a}$$

whence

$$\sum_k \left[\frac{(2i)!(2j)!2^{2k}}{(2k)!(i-k)!(j-k)!2^{i+j}}\right] = \frac{(2i+2j)!}{2^{i+j}(i+j)!}.$$

Similarly the coefficients of $x^{2i+1}y^{2j+1}$ in $(x^2+2xy+y^2)^{i+j+1}$ and in $(x+y)^{2i+2j+2}$ are, respectively,

$$\sum_k \frac{(i+j+1)!2^{2k+1}}{(2k+1)!(i-k)!(j-k)!} \quad \text{and} \quad \frac{(2i+2j+2)!}{(2i+1)!(2j+1)!}; \tag{9b}$$

whence

$$\sum_k \frac{(2i+1)!(2j+1)!2^{2k+1}}{(2k+1)!(i-k)!(j-k)!2^{i+j+1}} = \frac{(2i+2j+2)!}{2^{i+j+1}(i+j+1)!}.$$

Therefore the square of the correlation coefficient between y and $f(x)$ is equal to

$$\mathbf{r}^2 = \sum_{i=1}^{N} [\bar{f}^{(i)}(x)]^2/i! \tag{10}$$

This expresses the square of this correlation coefficient as a function of the squares of the averages of the derivatives of the polynomial at the given values of x.

CORRECTION

L. Mirsky, *Diagonal elements of orthogonal matrices*, this MONHTLY, vol. 66, 1959, pp. 19–22. Professor Louis Brand has drawn attention to a misprint on page 20, line 4, where $-d_1, \cdots, -d_n$ should be changed to $-d_1, d_2, \cdots, d_n$.

MATHEMATICAL NOTES

EDITED BY ROY DUBISCH, Fresno State College

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RECURRENCE RELATIONS FOR SOLUTIONS OF PELL'S EQUATION

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It is well known that if x_1, y_1 be the smallest positive integral solution of the Pellian equation [1]:

$$x^2 - Dy^2 = 1, \tag{1}$$

where D is a positive integer which is not a square, then the general solution x_r, y_r in positive integers is given by

$$x_r + y_r\sqrt{D} = (x_1 + y_1\sqrt{D})^r, \tag{2}$$

where $r=1, 2, \cdots$ [2]. The smallest solution x_1, y_1 of (1) is easily obtainable in terms of the continued fraction for $\sqrt{D}$ [3] but (2) is unwieldy for the computation of other solutions and it is more convenient to use recurrence relations which can readily be shown to follow from (2).

From (2) are obtained

$$x_{r+s} + y_{r+s}\sqrt{D} = (x_r + y_r\sqrt{D})(x_s + y_s\sqrt{D}), \tag{3}$$

and

$$x_{nr} + y_{nr}\sqrt{D} = (x_r + y_r\sqrt{D})^n. \tag{4}$$

Equating rational and irrational coefficients in (3) gives the general recurrence relations:

$$x_{r+s} = x_r x_s + D y_r y_s, \qquad y_{r+s} = x_r y_s + y_r x_s. \tag{5}$$

Particular cases of (5) are

$$x_{r+1} = x_1 x_r + D y_1 y_r, \qquad y_{r+1} = x_1 y_r + y_1 x_r; \tag{6}$$

$$x_{2r} = -1 + 2x_r^2, \qquad y_{2r} = 2x_r y_r; \tag{7}$$

$$x_{3r} = -3x_r + 4x_r^3, \qquad y_{3r} = (-1 + 4x_r^2)y_r. \tag{8}$$

Equation (6) enables one to go, with a minimum of computation, from x_1, y_1 to x_2, y_2 to x_3, y_3, and so on. Equations (7) and (8) are particular cases of general expressions for x_{nr} and y_{nr} and y_{nr}/y_r in terms of x_r, for $n=1, 2, \cdots$. These are obtained by changing the notation in (1) and (4), either by putting $x_r=\cos\rho$, $y_r=\sin\rho$, $x_{nr}=\cos n\rho$, $y_{nr}=\sin n\rho$, $\sqrt{D}=\sqrt{-1}=i$, to get (9) and (10), respectively, or by putting $x_r=\cosh\rho$, $y_r=i\sinh\rho$, $x_{nr}=\cosh n\rho$, $y_{nr}=i\sin n\rho$, $\sqrt{D}=i$,

to get (11) and (12), respectively.

$$\cos^2 \rho + \sin^2 \rho = 1, \tag{9}$$

$$\cos n\rho + i \sin n\rho = (\cos \rho + i \sin \rho)^n, \tag{10}$$

$$\cosh^2 \rho - \sinh^2 \rho = 1, \tag{11}$$

$$\cosh n\rho - \sinh n\rho = (\cosh \rho - \sinh \rho)^n. \tag{12}$$

Hence it follows that x_{nr} can be expressed in terms of x_r in the same way that $\cos n\rho$ can be expressed in terms of $\cos \rho$, or that $\cosh n\rho$ can be expressed in terms of $\cosh \rho$, since these expressions are independent of D. Likewise, y_{nr}/y_r can be expressed in terms of x_r in the same way that $\sin n\rho/\sin \rho$ can be expressed in terms of $\cos \rho$. These expressions are well known [**4**]. The results are most conveniently expressed in the form:

$$\begin{aligned} x_{nr} &= 2^{n-1} x_r - \frac{n}{1!} 2^{n-3} x_r^{n-2} + \cdots \\ &+ (-1)^s \frac{n(n-s-1)\cdots(n-2s+1)}{s!} 2^{n-2s-1} x_r^{n-2s} + \cdots, \end{aligned} \tag{13}$$

$$\begin{aligned} \frac{y_{nr}}{y_r} &= 2^{n-1} x_r^{n-1} - \frac{(n-2)}{1!} 2^{n-3} x_r^{n-3} + \cdots \\ &+ (-1)^s \frac{(n-s-1)\cdots(n-2s)}{s!} 2^{n-2s-1} x_r^{n-2s-1} + \cdots. \end{aligned} \tag{14}$$

Whereas (1) always has positive integral solutions the similar equation

$$u^2 - Dv^2 = -1, \tag{15}$$

is only soluble in positive integers u, v for values of D which satisfy certain other conditions [**3**]. In such cases, if u_1, v_1 is the smallest positive integral solution of (15), then all solutions are again given by an equation like (2). Consequently, equations like (3), (4), (5) and (6) also apply to solutions of (15), if they exist. Whereas, by (1), $Dy_r^2 = x_r^2 - 1$, it follows from (15) that $Dv_r^2 = u_r^2 + 1$, so that (7) and (8) now become, respectively:

$$u_{2r} = 1 + 2u_r^2, \qquad v_{2r} = 2u_r v_r; \tag{16}$$

$$u_{3r} = 3u_r + 4u_r^2, \qquad v_{3r} = (1 + 4u_r^2)v_r. \tag{17}$$

The general relations between u_{nr} or v_{nr}/v_r and u_r are

$$\begin{aligned} u_{nr} &= 2^{n-1} u_r + \frac{n}{1!} 2^{n-3} u_r^{n-2} + \cdots \\ &+ \frac{n(n-s-1)\cdots(n-2s+1)}{s!} 2^{n-2s-1} u_r^{n-2s} + \cdots, \end{aligned} \tag{18}$$

$$\frac{v_{nr}}{v_r} = 2^{n-1}u_r^{n-1} + \frac{(n-2)}{1!}2^{n-3}u_r^{n-3} + \cdots \tag{19}$$
$$+ \frac{(n-s-1)\cdots(n-2s)}{s!}2^{n-2s-1}u_r^{n-2s-1} + \cdots .$$

I am indebted to Professor R. L. Goodstein, who has previously given (7) [5], and who kindly pointed out to me that the value of x_{nr}, which I obtained otherwise, is the same polynomial in x_r that cosh $n\rho$ is in cosh ρ or that cos $n\rho$ is in cos ρ.

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4. E. W. Hobson, A Treatise on Plane Trigonometry, Cambridge, 1928, p. 108.
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NOTE ON AN IDENTITY OF CAYLEY*

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Cayley [1] proved an identity relating a third-order determinant and the corresponding permanent,

$$\overset{+}{\left|\begin{matrix} \frac{1}{a_1} & \frac{1}{a_2} & \frac{1}{a_3} \\ \frac{1}{b_1} & \frac{1}{b_2} & \frac{1}{b_3} \\ \frac{1}{c_1} & \frac{1}{c_2} & \frac{1}{c_3} \end{matrix}\right|}^{+} \left|\begin{matrix} \frac{1}{a_1} & \frac{1}{a_2} & \frac{1}{a_3} \\ \frac{1}{b_1} & \frac{1}{b_2} & \frac{1}{b_3} \\ \frac{1}{c_1} & \frac{1}{c_2} & \frac{1}{c_3} \end{matrix}\right| = \left|\begin{matrix} \frac{1}{a_1^2} & \frac{1}{a_2^2} & \frac{1}{a_3^2} \\ \frac{1}{b_1^2} & \frac{1}{b_2^2} & \frac{1}{b_3^2} \\ \frac{1}{c_1^2} & \frac{1}{c_2^2} & \frac{1}{c_3^2} \end{matrix}\right| \tag{1}$$

provided that

$$\left|\begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{matrix}\right| = 0, \tag{2}$$

and no element in the determinant (2) is zero.

In (1) the first factor on the left is a permanent of order 3, the general definition of a permanent of order n being

$$\overset{+}{|}\, a_{ij} \,\overset{+}{|} = \sum a_{1p}a_{2q}\cdots a_{nr}, \tag{3}$$

---

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where the sum on the right is over all $n!$ permutations $pq \cdots r$ of $1\ 2 \cdots n$, (Muir [2]). The expansion of the permanent (3) thus follows that of a determinant except all signs are taken as plus.

In this note we show that identity (1) can be extended to the corresponding products of the fourth order according to the theorem:

THEOREM. *If $|a_{ij}|$ is a determinant of order $n \leqq 4$, and if the rank of the matrix $[a_{ij}]$ is less than 3 (and all $a_{ij} \neq 0$), then*

$$\overset{+}{\Bigg|}\frac{1}{a_{ij}}\overset{+}{\Bigg|}\ \Bigg|\frac{1}{a_{ij}}\Bigg| = \Bigg|\frac{1}{a_{ij}^2}\Bigg|. \tag{4}$$

The theorem is obviously true if $n=1$ or 2, and the case $n=3$ is Cayley's result (1). There remains the case $n=4$. In this case we make use of the formula of Muir [2] giving the product of a general permanent and determinant,

$$\overset{+}{\left|\begin{matrix} a_{11} \cdots a_{1n} \\ a_{21} \cdots a_{2n} \\ \cdot \quad \cdot\ \cdot\ \cdot\ \cdot \\ a_{n1} \cdots a_{nn} \end{matrix}\right|}\ \left|\begin{matrix} x_{11} \cdots x_{1n} \\ x_{21} \cdots x_{2n} \\ \cdot \quad \cdot\ \cdot\ \cdot\ \cdot \\ a_{n1} \cdots x_{nn} \end{matrix}\right| = \sum \pm \left|\begin{matrix} a_{11}x_{i1} \cdots a_{1n}x_{in} \\ a_{21}x_{j1} \cdots a_{2n}x_{jn} \\ \cdot \quad \cdot\ \cdot\ \cdot \quad \cdot \\ a_{n1}x_{k1} \cdots a_{nn}x_{kn} \end{matrix}\right|, \tag{5}$$

the summation being over all $n!$ permutations $ij \cdots k$ of $1\ 2 \cdots n$, and the $+$ and $-$ sign is used according as $ij \cdots k$ is an even or odd permutation of $1\ 2 \cdots n$ respectively.

Now in (5), (with $n=4$), put $x_{ij}=a_{ij}$, and then replace a_{ij} by $1/a_{ij}$ (assuming all $a_{ij} \neq 0$). Then the 24 terms on the right side of (5) can be combined into five terms, as follows:

$$\overset{+}{\Bigg|}\frac{1}{a_{ij}}\overset{+}{\Bigg|}\ \Bigg|\frac{1}{a_{ij}}\Bigg| = [abcd] + 2[adbc] + 2[dbac] + 2[dacb] + 2[cabd], \tag{6}$$

where

$$[pqrs] \equiv \left|\begin{matrix} \dfrac{1}{a_1p_1} & \dfrac{1}{a_2p_2} & \dfrac{1}{a_3p_3} & \dfrac{1}{a_4p_4} \\ \dfrac{1}{b_1q_1} & \dfrac{1}{b_2q_2} & \dfrac{1}{b_3q_3} & \dfrac{1}{b_4q_4} \\ \dfrac{1}{c_1r_1} & \dfrac{1}{c_2r_2} & \dfrac{1}{c_3r_3} & \dfrac{1}{c_4r_4} \\ \dfrac{1}{d_1s_1} & \dfrac{1}{d_2s_2} & \dfrac{1}{d_3s_3} & \dfrac{1}{d_4s_4} \end{matrix}\right|,$$

and on the right side of (6) we have placed $a_j=a_{1j}$, $b_j=a_{2j}$, $c_j=a_{3j}$, $d_j=a_{4j}$.

The first determinant $[abcd]$ on the right side of (6) is the same as the determinant on the right side of (4). The other four determinants on the right side of (6) are all zero on making use of the rank assumption of matrix $[a_{ij}]$.

To see this, consider, for example, the determinant $[adbc]$. This can be expressed as

$$(7)\qquad [abdc] = \begin{vmatrix} \frac{1}{a_1^2} & \frac{1}{a_2^2} & \frac{1}{a_3^2} & \frac{1}{a_4^2} \\ \frac{1}{b_1d_1} & \frac{1}{b_2d_2} & \frac{1}{b_3d_3} & \frac{1}{b_4d_4} \\ \frac{1}{c_1b_1} & \frac{1}{c_2b_2} & \frac{1}{c_3b_3} & \frac{1}{c_4b_4} \\ \frac{1}{d_1c_1} & \frac{1}{d_2c_2} & \frac{1}{d_3c_3} & \frac{1}{d_4c_4} \end{vmatrix} = A \begin{vmatrix} \frac{b_1c_1d_1}{a_1^2} & \frac{b_2c_2d_2}{a_2^2} & \frac{b_3c_3d_3}{a_3^2} & \frac{b_4c_4d_4}{a_4^2} \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \\ b_1 & b_2 & b_3 & b_4 \end{vmatrix},$$

where $A=(b_1c_1d_1)^{-1}(b_2c_2d_2)^{-1}(b_3c_3d_3)^{-1}(b_4c_4d_4)^{-1}$.

By expanding the second determinant in (7) by the elements of its first row, it follows that $[adbc]=0$ since we are assuming that matrix

$$[a_{ij}] = \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & b_4 \\ c_1 & c_2 & c_3 & c_4 \\ d_1 & d_2 & d_3 & d_4 \end{bmatrix}$$

is of rank less than 3.

A similar proof shows $[dbac]=[dacb]=[cabd]=0$. The proof of the theorem now follows since as previously stated the remaining determinant on the right side of (6) $[abcd]=|1/a_{ij}^2|$.

As an illustration of the theorem we may take $|a_{ij}|$ to be

$$|a_{ij}| = \begin{vmatrix} a & a+d & a+2d & a+3d \\ a+4d & a+5d & a+6d & a+7d \\ a+8d & a+9d & a+10d & a+11d \\ a+12d & a+13d & a+14d & a+15d \end{vmatrix}.$$

The author thanks the referee for his helpful suggestions.

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ON THE REDUCIBILITY OF SOME LINEAR DIFFERENTIAL OPERATORS*

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The differential equations

$$[x^n D^{2n} - 1]y = 0, \tag{1}$$

$$[x^{2n} D^n - 1]y = 0, \tag{2}$$

have been solved previously by Lommel [1a] and Steen [2a], respectively, but not in a direct manner. By showing that the operators $x^n D^{2n}$ and $x^{2n} D^n$ are reducible, we immediately obtain the solutions of (1) and (2) in terms of solutions of a second-order and a first-order equation, respectively. We also obtain reducible generalizations of the above operators.

The linear polynomial operator $P(D, x)$ is defined to be reducible if it can be expressed in the form

$$P(D, x) \equiv F[G(D, x)], \tag{3}$$

where $G(D, x)$ is a linear polynomial operator of order less than that of $P(D, x)$ (F will then have to be a polynomial of degree $\geqq 2$). It then follows that the solution to the equation $P(D, x)y = ay$ is given by the solution of the lower order equation $G(D, x)y = \lambda y$, where $F(\lambda) = a$.

We now show that the operator $x^n D^{2n}$ is reducible to

$$x^n D^{2n} \equiv [xD^2 - (n-1)D]^n. \tag{4}$$

Letting $x = e^z$ and $xD = \theta = d/dz$, it follows that

$$x^n D^{2n} = e^{-nz}\theta(\theta - 1) \cdots (\theta - 2n + 1), \tag{5}$$

$$[xD^2 - (n-1)D]^n = [e^{-z}\theta(\theta - n)]^n = e^{-z}\theta(\theta - n)e^{-z}\theta(\theta - n) \cdots e^{-z}\theta(\theta - n). \tag{6}$$

By using the exponential shift theorem

$$F(\theta)e^{\int \psi dz} = e^{\int \psi dz} F(\theta + \psi),$$

we can transform (6) into (5) by shifting all the e^{-z} terms to the extreme left.

The solution to (1) is now immediately given by the solution of the second order equation

$$[xD^2 - (n-1)D]y = \lambda_r y, \tag{7}$$

where $\{\lambda_r\}$ are the n roots of $\lambda^n = 1$. Equation (7) is a modified† Bessel equation whose solution is $y = x^{n/2} Z_n(2\sqrt{-\lambda_r x})$. Hence, the general solution of (1) is

* Presented to the American Mathematical Society, August 29, 1957.

† We have used the term *modified* Bessel equation to apply to any second order equation whose solution can be expressed "simply" in terms of Bessel functions other than $AJ_v(x) + BY_v(x)$.

$$y = x^{n/2} \sum_{r=0}^{n-1} \left\{ A_r J_n(2\sqrt{-\lambda_r x}) + B_r Y_n(2\sqrt{-\lambda_r x}) \right\}. \tag{8}$$

Another application is in solving the Nicholson [1b] type equation

$$D^2 z^4 D^2 u = z^2 u.$$

Since $D^m z^m$ and $z^n D^n$ commute, the equation can be rewritten as

$$(z^2 D^4 - 1) z^2 u = 0,$$

and since $z^2 D^4$ is reducible, the equation is easily solved.

A dual identity to (4) is given by

$$x^{2n} D^n \equiv [x^2 D - (n-1)x]^n \tag{9}$$

which can be established in a similar manner. Consequently, the general solution of (2) is

$$y = \sum_{r=1}^{n} A_r x^{n-1} e^{-\lambda_r / x} \tag{10}$$

where $\{\lambda_r\}$ are the n roots of $\lambda^n = 1$.

As an interesting aside, it follows from our two identities that two non-commutative solutions of the operator equation $A^n B^n C^n = (ABC)^n$ are $A = x^{-1}$, $B = x^2 D - (n-1)x$, $C = D$, or $A = x$, $B = xD^2 - (n-1)D$, $C = D^{-1}$.

Next we will show that the identity

$$z^{n+1}\left(\frac{1}{z}\frac{d}{dz}\right)^n \equiv \frac{1}{z^{n+1}}\left[\left(z^3 \frac{d}{dz}\right)^n z^{2-2n}\right]; \tag{11}$$

derived by Glaisher [1c], is equivalent to identity (9). If we let $x = 1/z^2$, (11) is transformed into $x^{2n} D^n \equiv x^{n-1}(x^2 D)^n x^{1-n}$.

By the exponential shift theorem, $x^{n-1}(x^2 D)^n x^{1-n} \equiv [x^2 D - (n-1)x]^n$.

Generalizations of the previous identities are given in the following four equations:

$$x^{rn} D^{(r+1)n} \equiv [(xD + 1 - n)(xD + 1 - 2n) \cdots (xD + 1 - rn)D]^n, \tag{12}$$

$$x^{(r+1)n} D^{rn} \equiv [x(xD + 1 - n)(xD + 1 - 2n) \cdots (xD + 1 - rn)]^n, \tag{13}$$

$$D^{(r+1)n} x^{rn} \equiv [(xD + 1 + n)(xD + 1 + 2n) \cdots (xD + 1 + rn)D]^n, \tag{14}$$

$$D^{rn} x^{(r+1)n} \equiv [x(xD + 1 + n)(xD + 1 + 2n) \cdots (xD + 1 + rn)]^n. \tag{15}$$

We can now solve the equation

$$[x^{3n} D^{2n} - 1] y = 0, \tag{16}$$

in terms of Bessel functions. From (13) it follows that the solution is given by the solution of

$$x(xD + 1 - n)(xD + 1 - 2n) y = \lambda y,$$

where $\lambda^n = 1$. This latter equation is the modified Bessel equation [2b]

$$\left[x^2D^2 + 3(1 - n)xD + (1 - n)(1 - 2n) - \frac{\lambda}{x}\right]y = 0,$$

whose solution is $y = x^{(3n/2)-1}Z_n\{2\sqrt{-\lambda/x}\}$.

Remark. It would be of interest to find necessary and/or sufficient conditions for a linear differential operator to be irreducible.

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A REMARK ON ENNEPER'S MINIMAL SURFACE*

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In a paper of 1864 A. Enneper wrote down the formulas of a special minimal surface,

$$S: \begin{cases} x = u + uv^2 - \frac{1}{3}u^3 \\ y = v + u^2v - \frac{1}{3}v^3 \\ z = u^2 - v^2 \end{cases} \tag{1}$$

which, in the sequel, has been extensively studied in the literature and used for illustrative purposes.† Equations (1) give the parametric representation. However, after elimination of the parameters u and v, S also can be recognized as an algebraic surface of ninth degree:

$$\left[y^2 - x^2 + \frac{4}{3}z + \frac{4}{9}z^3\right]^3 = 3z\left[y^2 - x^2 + \frac{8}{9}z - z\left(x^2 + y^2 + \frac{8}{9}z^2\right)\right]^2. \tag{2}$$

The surface contains the origin $x = y = z = 0$, and its normal there is parallel to the z-axis. Consequently, it is possible, in a neighborhood of the origin, to bring the equation of S into the form $z = z(x, y)$. We are concerned in the present note with the determination of a circle in the xy-plane in which this special representation can be maintained. More precisely, we shall prove the following assertion:

The part of Enneper's surface containing the origin can be brought into the form $z = z(x, y)$ *over the circle* $x^2 + y^2 < (64/243)$.

An interesting conclusion can be drawn from this result. Suppose a minimal

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† See for instance A. Enneper [3], pp. 107–108; G. Darboux [1], pp. 316–320; T. Radó [7], p. 40; L. P. Eisenhart [2].

surface $z=z(x, y)$ is defined over the circle $x^2+y^2<R^2$. Denote its Gaussian curvature at $x=y=0$ by K_0. According to an inequality of E. Heinz* $|K_0| \leqq C\cdot R^{-2}W_0^{-2}(W=\sqrt{1+z_x^2+z_y^2})$. Here C is a universal constant. Its value is unknown. The best estimate so far is $C\leqq 49/4$, see [6]. It is easy to compute for Enneper's surface the values $K_0=-4$, $W_0=1$. Thus the following estimate from below for C holds: $C\geqq(256/243)=1.05$.

In order now to prove our assertion we have to investigate the mapping $u, v\rightarrow x, y$, defined by the first two equations of (1). In domains where this mapping is one-to-one, u and v can be expressed in terms of x, y and substituted into the last equation of (1). Introducing polar coordinates $u \quad \rho\cos\theta$, $v=\rho\sin\theta$ we obtain

$$
(4)\qquad
\begin{aligned}
x &= a(\rho, \theta)\cdot\cos\theta, & a(\rho, \theta) &= \rho\left(1+\rho^2-\frac{4}{3}\rho^2\cos^2\theta\right);\\
y &= b(\rho, \theta)\cdot\sin\theta, & b(\rho, \theta) &= \rho\left(1+\rho^2-\frac{4}{3}\rho^2\sin^2\theta\right).
\end{aligned}
$$

Denote by C_ρ the image curve in the xy-plane of the circle $u^2+v^2=\rho^2$. Its curvature, up to a positive factor, is given by the expression $x_\theta y_{\theta\theta}-x_{\theta\theta}y_\theta$. A straightforward computation yields

$$
(5)\qquad
\begin{aligned}
x_\theta y_{\theta\theta}-x_{\theta\theta}y_\theta &= \rho^2(1-2\rho^2\cos 4\theta-3\rho^4)\\
&\geqq \rho^2(1-2\rho^2-3\rho^4) = \rho^2(1+\rho^2)(1-3\rho^2).
\end{aligned}
$$

Hence the curves C_ρ are convex as long as $\rho<1/\sqrt{3}$. The following method—a "generalized ellipse construction"—for the construction of the curves C_ρ is suggested. First draw, for fixed ρ, the two curves

$$
(6)\qquad A_\rho\colon \begin{cases} x = a(\rho, \theta)\cdot\cos\theta \\ y = a(\rho, \theta)\cdot\sin\theta \end{cases};\qquad B_\rho\colon \begin{cases} x = b(\rho, \theta)\cdot\cos\theta \\ y = b(\rho, \theta)\cdot\sin\theta \end{cases}.
$$

B_ρ can be obtained by rotation of A_ρ about an angle of 90 degrees. The radius vector in the xy-plane which forms the angle θ with the x-axis intersects A_ρ in a point P_ρ and B_ρ in a point Q_ρ. The new point whose abscissa is that of P_ρ and whose ordinate is that of Q_ρ will be a point on C_ρ.

We now proceed to prove two properties of the curve A_ρ (and, *a fortiori*, B_ρ):

1. The curve A_ρ is convex for $\rho<1/\sqrt{3}$. The curvature of A_ρ, up to a positive factor, is given by $x_\theta y_{\theta\theta}-x_{\theta\theta}y_\theta=a^2+2a_\theta^2-aa_{\theta\theta}$. Substituting the function $a(\rho, \theta)$ from (4_1), we obtain

$$
(1+\rho^2)\left(1+\frac{11}{3}\rho^2\right)+\frac{8}{3}\rho^2(\rho^2-3)\cos^2\theta-\frac{16}{3}\rho^4\cos^4\theta,
$$

* See E. Heinz [4]; E. Hopf [5]; J. Nitsche [6].

an expression which, as long as $\rho^2-3\leqq 0$, is not smaller than

$$(1+\rho^2)\left(1+\frac{11}{3}\rho^2\right)+\frac{8}{3}\rho^2(\rho^2-3)-\frac{16}{3}\rho^4=\left(\frac{1}{3}-\rho^2\right)(3-\rho^2).$$

Therefore the curvature of A_ρ is positive for $\rho<1/\sqrt{3}$.

2. A_{ρ_1} lies inside A_{ρ_2} for $\rho_1<\rho_2<1/\sqrt{3}$. In order to show this we observe for one thing that

$$a(\rho,\theta)\geqq\rho\left(1+\rho^2-\frac{4}{3}\rho^2\right)=\rho\left(1-\frac{1}{3}\rho^2\right)>0\quad\text{for}\,\rho<1/\sqrt{3}.$$

On the other hand we have

$$a_\rho(\rho,\theta)=1+3\rho^2-4\rho^2\cos^2\theta\geqq 1-\rho^2>0\qquad\text{for}\ \rho<1.$$

From the properties ascertained above it follows that the curves C_ρ form a nested family of convex curves for $\rho<1/\sqrt{3}$. Furthermore C_ρ does not enter the interior of the intersection of the interior of A_ρ and that of B_ρ. Hence we can conclude that the circle $\sqrt{x^2+y^2}<\rho(1+\rho^2-4\rho^2/3)=\rho(1-\rho^2/3)$ is contained in the interior of C_ρ. Our statement follows for $\rho\to 1/\sqrt{3}$.

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CURVE-FITTING MATRICES

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In a recent note, Wilf [1] explicitly calculated inverses required for polynomial curve-fitting up through the case of 6 points. For machine usage, it would seem desirable to have readily calculable inverses for arbitrary order n. One method for doing this was demonstrated by Macon and Spitzbart [2]. Another procedure for accomplishing this is indicated below.

Using a notation similar to that of Wilf, we desire a polynomial

$$p(x)=a_0+a_1x+\cdots+a_{n-1}x^{n-1}$$

such that

$$p(x_i) = y_i, \qquad i = 1, \cdots, n.$$

This reduces to the matrix form $V_n a = y$, where $a' = (a_0 a_1 \cdots a_{n-1})$, $y' = (y_1 y_2 \cdots y_n)$, and the ith row of V_n is $(1\ x_i \cdots x_i^{n-1})$. After computing the inverse of V_n, the problem is solved by forming $a = V_n^{-1} y$.

Let

$$f(x) \equiv (x + x_1) \cdots (x + x_n) = x^n - s_1 x^{n-1} + \cdots + (-1)^n s_n.$$

The numbers s_k can be generated recursively by calculating

$$s_{i,j} = s_{i,j-1} - x_j s_{i-1,j-1}, \qquad i = 1, \cdots, j,$$

letting j run from 1 to n, and defining $s_{0,j-1} = 1$, $s_{j,j-1} = 0$, where $s_k = s_{k,n}$. The quantity s_n is not required. Thus

$$s_{1,1} = s_{1,0} - x_1 s_{0,0} = 0 - x_1(1) = -x_1,$$

$$\begin{aligned} s_{1,2} &= s_{1,1} - x_2 s_{0,1} = -x_1 - x_2(1) = -x_1 - x_2, \\ s_{2,2} &= s_{2,1} - x_2 s_{1,1} = 0 - x_2(-x_1) = x_1 x_2, \end{aligned}$$

$$\begin{aligned} s_{1,3} &= s_{1,2} - x_3 s_{0,2} = -x_1 - x_2 - x_3(1), \\ s_{2,3} &= s_{2,2} - x_3 s_{1,2} = x_1 x_2 - x_3(-x_1 - x_2), \\ s_{3,3} &= s_{3,2} - x_3 s_{2,2} = 0 - x_3(x_1 x_2), \end{aligned}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

To compute the elements appearing in the successive columns of the inverse, calculate

$$q_{i,j} = s_i + x_j q_{i-1,j}, \qquad r_{i,j} = q_{i,j} + x_j r_{i-1,j},$$

$i = 1, \cdots, n-1$, where j runs from 1 to n, and $q_{0,j}$ and $r_{0,j}$ are defined to be 1. In expanded form

$$\begin{aligned} q_{1,j} &= s_1 + x_j q_{0,j} = s_1 + x_j, \\ q_{2,j} &= s_2 + x_j q_{1,j} = s_2 + x_j s_1 + x_j^2, \\ q_{3,j} &= s_3 + x_j q_{2,j} = s_3 + x_j s_2 + x_j^2 s_1 + x_j^3, \end{aligned}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\begin{aligned} r_{1,j} &= q_{1,j} + x_j r_{0,j} = s_1 + 2x_j, \\ r_{2,j} &= q_{2,j} + x_j r_{1,j} = s_2 + 2x_j s_1 + 3x_j^2, \\ r_{3,j} &= q_{3,j} + x_j r_{2,j} = s_3 + 2x_j s_2 + 3x_j^2 s_1 + 4x_j^3, \end{aligned}$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

Finally, defining d_j as the reciprocal of $r_{n-1,j}$, the jth column of the inverse is the transpose of the vector $(d_j q_{n-1,j} d_j q_{n-2,j} \cdots d_j q_{1,j} d_j)$.

If we multiply the resulting matrix on the left by the matrix whose ith row

is $(1\ x_i \cdots x_i^{n-2} x_i^{n-1})$, the element appearing in the ith row and jth column of the product is

$$\begin{aligned}
&d_j(q_{n-1,j} + x_i q_{n-2,j} + \cdots + x_i^{n-2} q_{1,j} + x_i^{n-1}) \\
&= d_j[s_{n-1} + x_j s_{n-2} + \cdots + x_j^{n-2} s_1 + x_j^{n-1} \\
&\qquad + x_i s_{n-2} + \cdots + x_i x_j^{n-3} s_1 + x_i x_j^{n-2} \\
&\qquad + \cdots \\
&\qquad + x_i^{n-2} s_1 + x_i^{n-2} x_j \\
&\qquad + x_i^{n-1}] \\
&= \begin{cases} d_j(-1)^{n-1} \dfrac{f(-x_j) - f(-x_i)}{(-x_j) - (-x_i)} = 0, & j \neq i, \\ d_i(-1)^{n-1} f'(-x_i) = d_i r_{n-1,i} = 1, & j = i, \end{cases}
\end{aligned}$$

since

$$\begin{aligned}
0 &= f(-x_i) = (-1)^n (x_i^n + s_1 x_i^{n-1} + \cdots + s_{n-1} x_i + s_n), \\
0 &= f(-x_j) = (-1)^n (x_j^n + s_1 x_j^{n-1} + \cdots + s_{n-1} x_j + s_n), \\
0 &\neq (-1)^{n-1} f'(-x_i) = n x_i^{n-1} + (n-1) s_1 x_i^{n-2} + \cdots + s_{n-1}.
\end{aligned}$$

Evidently, all the identities involved in the above discussion could be proved by induction.

The referee suggested by way of contrast the following procedure. Assuming the form

$$p(x) = a_0 + a_1(x - x_1) + \cdots + a_{n-1}(x - x_1)^{n-1},$$

we observe first that

$$a_i = p^{(i)}(x_1)/i!. \tag{1}$$

Also, since Lagrangian interpolation is exact here, we may write

$$p(x) = \sum_j \frac{p_n(x)}{(x - x_j) p_n'(x_j)} y_j,$$

where $p_n(x) = \prod_k (x - x_k)$. Thus

$$p'(x) = \sum_j \frac{y_j}{p_n'(x_j)} \frac{(x - x_j) p_n'(x) - p_n(x)}{(x - x_j)^2}$$

and

$$y_i' \equiv p'(x_i) = \sum_j C_{ij} y_j, \tag{2}$$

where

$$C_{ij} = \begin{cases} \dfrac{p_n'(x_i)}{(x_i - x_j)p_n'(x_j)}, & i \neq j, \\ \dfrac{p_n''(x_j)}{2p_n'(x_j)}, & i = j. \end{cases}$$

Since the quantities C_{ij} depend only on the x's, relation (2) continues to hold if we replace y_i by y_i', and y_i' by y_i''. Thus $y' = Cy$, $y'' = Cy' = C^2y$, *etc.* Comparing this information with (1), it follows that the $(k+1)$st row of the desired inverse, the one to obtain the a's for the present case, is the first row of the kth power of the matrix C divided by $k!$. The first row, by inspection, has a one in the first position and zeros elsewhere.

Although this procedure is conceptually interesting from a computational standpoint, it appears that the number of steps involved in computing C and C^2 already exceeds the total number required in the method originally discussed above.

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CLASSROOM NOTES

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ON THE METHOD OF VARIATION OF PARAMETERS

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Given the linear differential equation of second order,

$$y'' + P(x)y' + Q(x)y = R(x), \tag{1}$$

let the complementary solution, y_c, be given by $y_c = C_1u + C_2v$, where u and v are two linearly independent solutions of (1) with $R \equiv 0$. Let the particular integral, y_p, be given by $y_p = Au + Bv$, where A and B are functions of x to be determined. Then the variation of parameters method yields the system of two equations,

$$A'u + B'v = 0, \tag{2}$$

$$A'u' + B'v' = R. \tag{3}$$

Since the Wronskian, $W = uv' - vu' \neq 0$, we find that

$$y_p = -u\int \frac{vR}{W}\,dx + v\int \frac{uR}{W}\,dx. \tag{4}$$

It is recalled that (2) is arbitrary, that (3) is determined from (1), and that the general solution of (1) is $y=y_c+y_p$. We note that (2) and (3) remain the same if $y_p=(A+\alpha)u+(B+\beta)v$, where α and β are *specified* constants. In practice, one chooses $\alpha=\beta=0$.

Let $f=f(x, u, v)$ be a function of class C^1 in x. If we introduce f into the right-hand side of (2), then (3) assumes a new form. Assuming a particular integral, $Y_p=(A+\alpha)u+(B+\beta)v$, where α and β are constants whose values will be specified later, the new system of equation is

$$A'u + B'v = f, \qquad A'u' + B'v' = R - f' - Pf. \tag{5}$$

We will now show that (5), accompanied with a proper specification of α and β, yields the particular integral $Y_p\equiv y_p$, as given by (4).

We find that $WA'=-vR+vf'+fv'+Pvf$, $WB'=uR-(uf'+fu'+Puf)$, $Y_p=y_p+g(x)$, where

$$g(x) = u\int \frac{Pvf + (vf)'}{W}\,dx - v\int \frac{Puf + (uf)'}{W}\,dx + \alpha u + \beta v. \tag{6}$$

Given f, there exists suitable choices of α and β such that $g(x)\equiv 0$. Suppose $f\equiv 0$. Then, if $\alpha=\beta=0$, $g(x)\equiv 0$. We show now that there exists no $f\not\equiv 0$ such that $Pvf+(vf)'\equiv 0$ and $Puf+(uf)'\equiv 0$ simultaneously. Suppose now that such an $f\not\equiv 0$ exists. Then, since $W'+PW\equiv 0$, we have $(vf)'W-W'vf\equiv 0$ and $(uf)'W-W'uf\equiv 0$; thus $vf\equiv k_1W$ and $uf\equiv k_2W$, where $k_1\neq 0$, $k_2\neq 0$ are fixed constants. From this, we have that $k_1u\equiv k_2v$, which contradicts the assumption that u and v are linearly independent solutions of (1). Thus, given $f\not\equiv 0$, only one of the following three conditions must hold: (i) $vf\equiv k_1W$, $k_1\neq 0$; (ii) $uf\equiv k_2W$, $k_2\neq 0$; or (iii) neither (i) nor (ii) holds.

Let $f=K_1h+K_2q+K_3r$, where the K_i are constants, of which none, some, or all may be zero, and where $h=h(x, u, v)\not\equiv 0$, $q=q(x, u, v)\not\equiv 0$, and $r=r(x, u, v) = \sum_{i=1}^{N} r_i(x, u, v)\not\equiv 0$, $r_i\not\equiv 0$, are functions of class C^1 in x satisfying the following conditions: $vh\equiv aW$, $uq\equiv bW$, and $v\sum r_i\not\equiv a_\nu W$, $u\sum r_i\not\equiv b_\nu W$, $\nu=1, 2, \cdots, 2^N-1$ where $a\neq 0$, $b\neq 0$, $a_\nu\neq 0$, and $b_\nu\neq 0$ are constants, and where $\sum$ indicates summation over the set s_ν which is always a nonempty subset of the set of integers, $(1, \cdots, N)$, and which is one of the (2^N-1) sets determined by enumerating the combinations, $\binom{N}{k}$, $k=1, \cdots, N$. (For $N=3$, there are seven sets, s_ν: (1), (2), (3), (1, 2), (1, 3), (2, 3), (1, 2, 3)). Noting that if $\sigma\equiv q$ or $\sigma\equiv r$, then $Pv\sigma+(v\sigma)'\not\equiv 0$, and that if $\delta\equiv h$ or $\delta\equiv r$, then $Pu\delta+(u\delta)'\not\equiv 0$, we find, using the relation, $W'+PW\equiv 0$, that

$$\int \frac{Pv\sigma + (v\sigma)'}{W}\,dx = \frac{v\sigma}{W}, \qquad \int \frac{Pu\delta + (u\delta)'}{W}\,dx = \frac{u\delta}{W}\cdot \tag{7}$$

Since $vh\equiv aW$ and $uq\equiv bW$, we have $Pvh+(vh)'\equiv 0$ and $Puq+(uq)'\equiv 0$. Using these relations and (7), we find that

$$
\begin{aligned}
u\int \frac{Pvf+(vf)'}{W}\,dx &= uK_2\int \frac{Pvq+(vq)'}{W}\,dx + uK_3\int\frac{Pvr+(vr)'}{W}\,dx \\
&= (K_2uqv)/W + (K_3uvr)/W = bvK_2 + (K_3uvr)/W,
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
v\int \frac{Puf+(uf)'}{W}\,dx &= vK_1\int \frac{Puh+(uh)'}{W}\,dx + vK_3\int\frac{Pur+(ur)'}{W}\,dx \\
&= (K_1vhu)/W + (K_3uvr)/W = auK_1 + (K_3uvr)/W.
\end{aligned} \tag{9}
$$

Substitution of (8) and (9) into (6) yields $g(x)=(\alpha-aK_1)u+(\beta+bK_2)v$. Thus, if $\alpha=aK_1$, $\beta=-bK_2$, then $g(x)\equiv 0$. This completes the proof. Since each constant K_i has two states, $K_i=0$ or $K_i\neq 0$, the function f may occur in $2^3=8$ ways. In only two cases, when $f\equiv 0$ or $f=K_3r$, can one choose $\alpha=\beta=0$.

SOLIDS OF REVOLUTION

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Except for an occasional isolated example, most calculus books have no discussion of solids generated by the rotation of an area about a line parallel to neither axis. This note attempts to summarize in a brief, informal way, some techniques for determining such volumes.

It will be assumed that the area to be rotated can be summed in terms of the usual elements in rectangular and polar coordinates, and that the given line, $ax+by+c=0$, (in polar coordinates, $r\cos(\theta-\omega)-p=0$), lies outside the area, although it may have certain contacts of even order with it.

Given the centroid, $(\overline{X}, \overline{Y})$, (in polar coordinates, $(\overline{R}, \overline{\Theta})$), of an element of area, and its perpendicular distance, D, to a line, it follows from the Theorem of Pappus that the elementary volume of rotation is $2\pi DdA$. In the following table are listed centroids and distances for the basic elements of area.

Element of Area	Centroid	Distance to Line
ydx	$(\overline{X},\overline{Y}) = \left[x, \dfrac{y_1(x)+y_2(x)}{2}\right]$	
xdy	$(\overline{X},\overline{Y}) = \left[\dfrac{x_1(y)+x_2(y)}{2}, y\right]$	$D=\lambda\overline{X}+\mu\overline{Y}-p$
$dxdy$	$(\overline{X},\overline{Y}) = (x, y)$	
$\dfrac{1}{2}(r_1^2-r_2^2)d\theta$	$(\overline{R},\overline{\Theta}) = \left[\dfrac{2}{3}\,\dfrac{r_1^2(\theta)+r_1(\theta)r_2(\theta)+r_2^2(\theta)}{r_1(\theta)+r_2(\theta)}, \theta\right]$	$D=\overline{R}\cos(\overline{\Theta}-\omega)-p$
$rdrd\theta$	$(\overline{R},\overline{\Theta}) = (r, \theta)$	

point at which $\partial P/\partial y \neq 0$, then there exists a sequence of points $(x_i, y_i) \to (x_0, y_0)$ at each of which $\partial P/\partial y \neq 0$. Moreover, in a neighborhood ΔR of (x_0, y_0) in which $\partial Q/\partial x$ maintains the same sign, $\partial P/\partial y$ and $\partial Q/\partial x$ cannot be equal at infinitely many of the (x_i, y_i), for continuity would demand their equality at (x_0, y_0). Thus, there exists at least one point $(\bar{x}, \bar{y})$ at which $\partial P/\partial y \neq \partial Q/\partial x$ with each term $\neq 0$, and relation (1) would hold for the partials evaluated at $(\bar{x}, \bar{y})$ and ΔC the boundary of a sufficiently small neighborhood of $(\bar{x}, \bar{y})$. The remaining case $\partial Q/\partial x = 0$ but $\partial P/\partial y \neq 0$ at (x_0, y_0) can be treated in a similar manner. Consequently, in each case there exists a point such that for a sufficiently small neighborhood ΔN of it and all simple closed curves ΔC in ΔN, taken counterclockwise and enclosing an area ΔA, $0 = (1/\Delta A)\oint_{\Delta C} Pdx + Qdy \approx (-\partial P/\partial y + \partial Q/\partial x)$ evaluated at the point, and $(-\partial P/\partial y + \partial Q/\partial x)$ maintains the same sign in ΔN. As the approximation can be made as close as desired by choosing a suitably small ΔN about the point, we have a contradiction to the inequality of $\partial P/\partial y$ and $\partial Q/\partial x$ in D.

Conversely, suppose that $P(x, y)$ and $Q(x, y)$ have continuous first partials in a simply connected domain D and that $\partial P/\partial y = \partial Q/\partial x$ in D. From the geometric interpretation above, $\oint_C Pdx + Qdy$, for any simple closed curve C in D, is the algebraic sum of the areas of two projections. Now, if the region in the XY-plane bounded by C is subdivided into subregions ΔR with ΔA the area of a typical one, then the areas of the corresponding projections ΔR_1 and ΔR_2 will be approximately given by $|\partial P/\partial y|\Delta A$ and $|\partial Q/\partial x|\Delta A$, respectively, as in the paragraph above. Thus, the areas of the two projected subregions differ at most by an infinitesimal of the third order. Moreover, for a given orientation of the boundary of ΔR, the boundaries of the corresponding ΔR_1 and ΔR_2 will be traversed in opposite directions. Consequently, by considering the counterclockwise line integral around C as the sum of counterclockwise line integrals around the boundaries of the subregions, the interior line integrals will cancel; and, hence, it follows that $\oint_C Pdx + Qdy = 0$, guaranteeing independence of path in D.

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CORRECTION

R. D. Larsson, *General solutions of linear ordinary differential equations*, this MONTHLY, vol. 65, 1958, pp. 523–525. In lines 1 and 2 on page 525, $F(x, k_2)$ should be replaced by $cF(x, k_2)$.

MATHEMATICAL EDUCATION NOTES

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MAA VISITING LECTURESHIP PROGRAM FOR SECONDARY SCHOOLS 1958–59

At the meeting of the Board of Governors in Boston on August 25, 1958, it was announced that the Mathematical Association of America had received a National Science Foundation grant of $47,700 to sponsor a visiting lectureship program for secondary schools as a pilot program for this school year and the school year of 1959–60. The program is being carried on in secondary schools in six regions during the second semester of 1958–59. In general, the plan follows that of the Association's visiting lectureship program for colleges, successfully operated since 1954. The general aims of the visiting lectureship program for secondary schools are listed as the following:

a. To strengthen and stimulate the mathematics programs of secondary schools;
b. To encourage co-operation between college and secondary school mathematics staffs;
c. To provide the mathematics staff and the students in secondary schools with an opportunity for personal contacts with productive and creative mathematicians;
d. To aid in the motivation of secondary school students to consider careers in mathematics and the teaching of mathematics.

It is expected that the visiting lecturers will spend one day in a school, except in California, where most of the visits will be on a half-day basis. In the larger cities the lecturer may plan to stay several days and visit a number of secondary schools, both in the city and in the neighboring smaller communities. It is hoped that teachers from a number of schools may meet the lecturer in a joint meeting and that it also may be possible for students or selected groups of students from several schools to meet together in assembly to hear the lecturer.

The lecturers who have agreed to work with the program during the second semester 1958–59 are as follows:

Richard D. Anderson, Louisiana State University
John D. Baum, Oberlin College
William E. Briggs, University of Colorado
James A. Cooley, University of Tennessee
James Clifton Eaves, University of Kentucky
William H. Fagerstrom, Pan American College
William Thomas Guy, Jr., University of Texas
Cletus O. Oakley, Haverford College
Henry W. Syer, Kent School, Kent, Connecticut
Verne J. Varineau, University of Wyoming

The six regions for 1958–59, and the committee representative of each region, are:

Colorado-Wyoming	B. W. Jones
Indiana	Marie S. Wilcox
Louisiana-Mississippi	Houston T. Karnes
Massachusetts-New Hampshire	W. E. Ferguson
Northern California	Roy Dubisch
Tennessee	F. A. Ficken

Most of the lecturers will work on a part-time rather than a leave-of-absence basis, during the second semester of this year. Dr. Anderson will spend most of his time in Louisiana and Mississippi; Dr. Baum and Dr. Syer have been assigned primarily to Massachusetts and New Hampshire; and Dr. Cooley and Dr. Oakley to Tennessee. Several of the lecturers will work in Indiana and Colorado-Wyoming. In Northern California the program is to be carried out in complete cooperation with the Northern California Section of the Association. The plan of operation there will be similar to that followed by the Northern California Section last year. The lecturers in Northern California will be local people who will travel to schools short distances from their colleges. The California lecturers are not listed above, and there may be a few other lecturers not listed who will be available for short trips in some of the other regions.

To date, there is indication of good cooperation from school people, including state departments of education. In one state at least—Tennessee—the state department of education has mailed the announcements of the program to all high schools.

SOME EDUCATIONAL ACTIVITIES IN PHYSICS

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Events of the past year have brought physicists, as well as their colleagues in the other natural sciences and mathematics, increased opportunities to improve the teaching of their subject at all levels of education. The history of these educational efforts in physics runs far back into the pre-Sputnik era, but a brief report on several recent developments may be of interest to mathematicians.

First, a note about the organizations involved in activities reported here is appropriate. The American Institute of Physics (AIP) is a federation of five principal physics societies in the United States: the American Physical Society, the Optical Society of America, the Acoustical Society of America, the Society of Rheology, and the American Association of Physics Teachers (AAPT). At its headquarters in New York City, AIP carries on those activities which can best be done by a central office in support of research and teaching in physics and the furthering of an appreciation of physics among the general public. Publication of journals in physics, public relations, and education and manpower are the three areas in which AIP is active. Its Education Department, established within the last year, includes the writer and secretarial assistants as a permanent core staff plus other professional personnel who join the AIP staff

on leave from physics departments to work on special educational projects. By June of this year, the AIP Education Department will have a full-time staff of eight. The AIP Education Department works closely with all of the Member Societies. Since the AAPT has a special interest in education in physics, however, many of the educational projects described here are joint AAPT-AIP activities, administered by AIP under policies jointly determined by AAPT and AIP.

The Visiting Scientists Program in Physics under a National Science Foundation grant resembles in its objectives the program of visits by mathematicians to colleges and need not be discussed in detail here. However, the physics program relies upon a rather large number of visitors—about eighty this year—each making only one or two visits to colleges. Another part of the program of visits by physicists provides for visits to high schools to assist teachers and stimulate interest in physics. About three hundred visiting days will be devoted to secondary schools this year.

The Project on the Design of Physics Buildings was launched this year under a grant from the Educational Facilities Laboratories. A preliminary survey of the building plans of physics departments in colleges and universities revealed that almost a quarter of a billion dollars will be spent in the next decade by educational institutions to meet increasing enrollments and expanding research programs in physics. This project will provide information for physicists, architects, and administrators about the best design features of present physics building facilities in schools and colleges as well as estimates of how the changing space requirements in physics can be met in the future. Professor R. Ronald Palmer of Beloit College will direct this project. The project will also have the services of a full-time architect and of consultants. Since science buildings on college campuses are often shared by several academic departments, the project staff will seek the advice of mathematicians and scientists in other areas as the work proceeds. That the design of physics building facilities is an educational problem can be attested by anyone who has struggled with a poorly-designed lecture room or teaching laboratory.

Equipment for illustrating the principles of physics is important to physics teachers at all levels of education. At the secondary school level, AAPT and AIP are seeking to provide assistance to school districts which wish to improve their stock of physics teaching equipment. Advice by physicists during their visits to schools is but one of the ways being explored. At the college level, AAPT and AIP are conducting an apparatus drawings project in which complete shop drawings of new apparatus developed in physics departmental shops will be made available to other physics departments for local construction. Further to stimulate interest in new apparatus, an AAPT apparatus competition was held in New York on January 29–31, 1959. Prizes were awarded on items of equipment which were judged most likely to advance the teaching of introductory college physics.

Physicists are greatly interested in recent developments in the teaching of

college mathematics, both because these developments are interesting in their own right and because they affect the teaching of college physics. At present there is a great need for physicists to become better informed about these developments and to discuss them with their colleagues in mathematics. AAPT and the Mathematical Association of America are in the process of appointing a joint committee to stimulate discussion of the new mathematics program by physicists and mathematicians by means of an exchange of journal articles and by colloquia at national meetings.

Numerous other educational projects in physics are well advanced: publication of a booklet to stimulate interest among students in physics and mathematics courses in high schools, publication of a booklet for the improvement of the high school physics course, evaluation of the effectiveness of the teaching of physics by television and film, stimulation of research in liberal arts colleges, revision of the introductory college physics course, a review of the physics content of elementary school science, awards to secondary schools outstanding for their work in science, physics tests, and a book of advanced laboratory experiments.

These are some of the educational projects being conducted by AAPT and AIP. In addition, physicists are interested in other experimental approaches to the improved teaching of physics, which are described elsewhere, such as the physics course for high schools being developed by the Physical Science Study Committee, the physics films made by Professor Harvey White for high schools, and the Continental Classroom "Atomic Age Physics" which Professor White is conducting on television.

TEXAS COMMISSION TO STUDY MATHEMATICS

JOHN WAGNER, The University of Texas

On November 14, 1955, the State Board of Education for the State of Texas authorized a study of graduation requirements in accredited high schools. This study was in recognition of the widespread interest in Texas in the aims and accomplishments of public schools.

In addition to recommendations on graduation requirements the Board and a special Advisory Committee made a far-reaching recommendation. The suggestion was that a detailed study be made of the approved courses which are offered in Texas schools and that the State Board specify the objectives and general content of each course.

A Commission to Study the Mathematics Curriculum in Texas Elementary and Secondary Schools has been appointed and should have its studies completed during the calendar year of 1959.

The original membership of the Mathematics Study Commission was composed as follows: six public school teachers (two elementary, two junior high and two senior high), three college teachers (two of whom shall be instructors of pure and applied mathematics—Dr. D. E. Edmondson, Southern Methodist

University and Dr. W. T. Guy, Jr., The University of Texas—and one of whom shall be an instructor in mathematics education—Dr. Joyce Benbrook, University of Houston), three public school principals (one elementary, one junior high, and one senior high), two curriculum directors or supervisors, and two school board members. Recently the State Board of Education appointed three more members, a high school teacher, a supervisor, and a college instructor of mathematics education, Mr. Don Cude, Southwest State College.

Consultants to the Commission have included Mr. Ralph Stafford, Texas Education Agency, and Mr. John Wagner, Extension Division of The University of Texas.

The purposes of the Study include the following:

1. To review the content of the present mathematics curriculum and to consider the changes in content which have taken place in recent years in order to meet present-day needs for all children.
2. To study experimental programs for accelerated students of high mathematics ability and to consider recommendations which will allow these students to be given advanced standing in mathematics.
3. To consider a program which will meet the mathematical needs of pupils who are not college bound.
4. To consider successful practices now being used for the early indentification and proper development of students with high aptitude in mathematics and to recommend ways for encouraging greater use of these practices.
5. To assist the State Department of Education in the preparation of curriculum materials which will give direction to those who wish to teach the combined Plane and Solid Geometry course, the Advanced Mathematics course, and other courses in secondary mathematics in which a change in emphasis is taking place.
6. To seek the cooperation of business and industry in providing for students enriched mathematics experiences both in school and out of school.

SCIENTIFIC MANPOWER BULLETIN

Scientific Manpower Bulletin of the National Science Foundation, Number 9, October, 1958, is devoted to a report on foreign-language knowledge of American scientists, 1954–55. Over 97,000 scientists in the National Register of Scientific and Technical Personnel reported a knowledge, with varying degrees of proficiency, of at least one foreign language. This is about three-quarters of the total in the Register 1954–55. German was the language most often reported. The fields of chemistry and chemical engineering contained the largest numbers reporting a language competency. Among mathematicians, 438 indicated some proficiency in the Balto-Slavic languages and 104 reported some proficiency in the Tibeto-Chinese languages.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1361. *Proposed by M. S. Klamkin, AVCO Research and Development*

If A, B, C are angles of a triangle, show that

$$\csc A/2 + \csc B/2 + \csc C/2 \geqq 6.$$

E 1362. *Proposed by Marlow Sholander, Carnegie Institute of Technology*

Consider a vertical girl whose waist is circular, not smooth, and temporarily at rest. Around the waist rotates a hula hoop of twice its diameter. Show that, after one revolution of the hoop, the point originally in contact with the girl has traveled a distance equal to the perimeter of a square circumscribing the girl's waist.

E 1363. *Proposed by Leo Moser, University of Alberta*

Let numbers be written to base b where b has the form $b=r^2+1$. Given r consecutive numbers, the last divisible by r, then the digital root of their sum is $1+2+\cdots+r=r(r+1)/2$.

E 1364. *Proposed by James Serrin, University of Minnesota*

In a plane, let A denote a closed convex curve in contact with a given curve C. Also, let B denote the mirror image of A across the tangent line to C at the point of contact. Suppose that the curvatures of A, B, C permit A and B to roll without slipping along their respective sides of C. Then, as A rolls along C, let A' denote the roulette traced out by the point of A initially in contact with C. Similarly, let B' denote the roulette generated by rolling B on C. Show that the area enclosed by an arch of A' and the corresponding arch of B' is independent of C. In particular, if A is a circle the enclosed area is just six times the area of A.

E 1365. *Proposed by Melvin Hausner, Stevens Institute of Technology*

Let H be the class of polynomials $f(x)$ with rational coefficients such that $f(n)$ is an integer when n is an integer. Prove that a sequence a_n of $+1$'s and -1's is of the form $a_n=(-1)^{f(n)}, f(x)\in H$, if and only if it is a periodic sequence of period 2^k.

SOLUTIONS

An Inequality in Two Variables

E 1331 [1958, 627]. *Proposed by P. L. Duren, Massachusetts Institute of Technology*

Show that, for all $a \geqq 0$ and $b \geqq 1$,

$$ab \leqq e^a + b(\ln b - 1),$$

with equality if and only if $b = e^a$.

Solution by H. D. Lipsich, University of Cincinnati. The inequality is true under the less restrictive conditions, a arbitrary, $b > 0$, as follows. It is well known that $e^u \geqq 1 + u$ for all u, with equality holding iff $u = 0$. Thus for arbitrary x and y, putting $u = x - y$ and rearranging, we have

$$xe^y \leqq e^x + e^y(y - 1),$$

from which the desired result follows on putting $x = a$ and $y = \ln b$.

It is of some interest to see a solution with which the restrictions placed by the proposer are naturally associated. To this end suppose $a \geqq 0$, $b \geqq 1$, and consider the curve $y = e^x$. A glance at the graph shows that the area of the rectangle with sides a and b is less than or equal to the area between the curve and the x-axis from 0 to a, plus the area between the curve and the y-axis from 1 to b. That is,

$$ab \leqq \int_0^a e^t dt + \int_1^b \ln t dt,$$

which, upon integration, yields the desired inequality. It is geometrically clear that equality holds iff the point (a, b) is on the curve.

Also solved by Derry Breault, D. R. Brillinger, J. L. Brown, Jr., R. F. Brown and Joel Levy (jointly), Ian Connell, A. E. Danese, Underwood Dudley, David Eakin, E. L. Ellis, P. G. Engstrom, Michael Goldberg, A. G. Grace, Jr., Bernard Greenspan, Cornelius Groenewoud, P. G. Hodge, Jr., Richard Holt, D. I. Knee and G. M. Leibowitz and N. Metas and L. N. Patterson and G. R. Stoodley (jointly), Morton Kupperman, Joe Lipman, Richard McChesney, D. C. B. Marsh, J. W. Mettler, G. J. Michaelides, C. E. Miller, Joseph Muskat, C. S. Ogilvy, F. R. Olson, W. V. Parker, C. L. Perry, Stanton Philipp, C. F. Pinzka, S. C. Port, K. H. Pyle, L. A. Ringenberg, D. A. Robinson, Jack Roseman, Vencil Skarda, J. A. Tierney, R. J. Wagner, Chih-yi Wang, Dale Woods, W. A. Veech, Julius Vogel, David Zeitlin, and the proposer. Late solutions by Robert Bart, S. H. Greene, Norman Greenspan, Vern Hoggatt, A. R. Hyde, J. D. E. Konhauser, Helen M. Marston, M. D. Mavinkurve, and J. L. Pietenpol.

Two Restricted Trigonometric Identities

E 1332 [1958, 627]. *Proposed by P. L. Chessin, University of Maryland*

If A, B, C are the angles of a triangle and x is such that

$$\cos(x + A)\cos(x + B)\cos(x + C) + \cos^3 x = 0,$$

then

$$\tan x = \cot A + \cot B + \cot C, \tag{1}$$

$$\sec^2 x = \csc^2 A + \csc^2 B + \csc^2 C. \tag{2}$$

Solution by David Zeitlin, Remington Rand Univac. Using

$$\cos(u + v) = \cos u \cos v - \sin u \sin v,$$

the equation simplifies to $\sec^2 x(\tan x - m) = 0$, where

$$m = \cot A + \cot B + \cot C.$$

It follows that $\tan x = m$, which is (1).

For (2) we note that

$$\begin{aligned} \sec^2 x &= 1 + \tan^2 x \\ &= \cot^2 A + \cot^2 B + \cot^2 C + 3 \\ &= (1 + \cot^2 A) + (1 + \cot^2 B) + (1 + \cot^2 C) \\ &= \csc^2 A + \csc^2 B + \csc^2 C. \end{aligned}$$

Also solved by A. N. Aheart, Walter Bloch, D. R. Brillinger, Leonard Carlitz, J. W. Clawson, E. L. Ellis, Mildred Going and Bert Levy (jointly), Cornelius Groenewoud, J. R. Hendricks, P. G. Hodge, Jr., Sidney Kravitz, Joe Lipman, D. C. B. Marsh, G. J. Michaelides, Stewart Nagler, T. L. Reynolds, E. R. Vance, Chih-yi Wang, W. V. Webb, Carole Weiss, Charles Wexler, R. H. Wilson, Jr., and the proposer. Late solutions by Robert Bart, D. A. Breault, A. E. Danese, S. H. Greene, Louise Grinstein, E. L. Hubbard, J. B. Muskat, and Roscoe Woods. Woods pointed out that x is the complement of the Brocard angle of triangle ABC.

A Special Determinant

E 1333 (corrected) [1958, 627]. *Proposed by V. F. Ivanoff, San Carlos, California*

If $s_k = 1^k + 2^k + \cdots + n^k$, show that

$$\begin{vmatrix} s_1 & 1 & 0 & 0 & \cdot & 0 \\ s_2 & s_1 & 2 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ s_{n-1} & s_{n-2} & \cdot & \cdot & \cdot & n-1 \\ s_n & s_{n-1} & \cdot & \cdot & \cdot & s_1 \end{vmatrix} = (n!)^2.$$

Solution by Leonard Carlitz, Duke University. The following more general result may be of interest. Let $x_1, \cdots, x_n$ be arbitrary numbers, let

$$s_k = \overset{k}{x_1} + \overset{k}{x_2} + \cdots + \overset{k}{x_n},$$

let D denote the given determinant under the more general interpretation, and let $p_1, \cdots, p_n$ denote the elementary symmetric functions of the x's. Then it is familiar that

$$p_0s_k - p_1s_{k-1} + p_2s_{k-2} - \cdots + (-1)^{k-1}p_{k-1}s_1 + (-1)^k kp_k = 0$$

for $k=1, \cdots, n$ and where $p_0=1$. We may consider this a system of equations in the unknowns $p_0, -p_1, \cdots, (-1)^{n-1}p_{n-1}$. Solving for p_0 we get $p_0=n!p_n/D$, or $D=n!p_n$.

Also solved by D. R. Brillinger, E. A. Fay, J. H. Hodges, Joe Lipman, W. R. McEwen, D. C. B. Marsh, W. V. Parker, B. M. Stewart, and the proposer.

Property of a Simply Periodic Entire Function

E 1334 [1958, 628]. *Proposed by Burton Randol, The Rice Institute*

If $f(z)$ is any simply periodic entire function, show that there exists a (finite) z_0 such that $f(z_0)=z_0$.

Solution by Arthur Rosenthal, Purdue University. Let f be nonconstant and let $h\neq 0$ be a period of f. By Picard's theorem, the entire function $f(z)-z$ assumes either the value 0 or otherwise the value h at a certain z_1. In the former case we have $f(z_1)=z_1$; in the latter case we have $f(z_1+h)=f(z_1)=z_1+h$. We may therefore take $z_0=z_1$ or $z_0=z_1+h$.

Also solved by Eugene Albert, J. L. Brown, Jr., Ian Connell, D. S. Greenstein, Joe Lipman, Joseph Muskat, Stanton Philipp, W. F. Trench, and the proposer.

Fixed Point of a Contraction Mapping

E 1335 [1958, 628]. *Proposed by C. N. Campopiano, Polytechnic Institute of Brooklyn*

Let $f(z)$ be a complex function of the complex variable z defined for all finite z. Suppose there is a constant k, $0<k<1$, such that

$$|f(z) - f(w)| \leqq k|z - w|$$

for all z, w. Show that there is a unique solution to the equation

$$z = f(z) + a,$$

where a is an arbitrary constant.

I. *Solution by Peter Treuenfels, Brookhaven National Laboratory.* Choose z_0 arbitrarily, and recursively define

$$z_{n+1} = f(z_n) + a. \tag{1}$$

It follows from the hypothesis, by induction on j, that

$$|z_{j+1} - z_j| \leqq k^j|z_1 - z_0|.$$

Since $0<k<1$, the series

$$\sum_{j=0}^{\infty} (z_{j+1} - z_j)$$

converges absolutely. Therefore

$$z_{n+1} = z_0 + \sum_{j=0}^{n} (z_{j+1} - z_j)$$

tends to a limit, say z. The hypothesis on f implies that f is continuous. Hence, passing to the limit in (1), we see that

$$z = f(z) + a. \tag{2}$$

Thus a solution exists. If there were two solutions z' and z'', then, by subtraction, (2) would imply that $z'-z''=f(z')-f(z'')$. But this is compatible with the hypothesis on f only if $z'=z''$.

See (a) L. Collatz, *Einige Anwendungen funktionanalytischer Methoden in der praktischen Analysis*, Z. Angew. Math. Phys., vol. 4, 1953, pp. 327–357, esp. Sec. 2; (b) L. Bers, *Mathematical Aspects of Subsonic and Transonic Gas Dynamics*, New York, 1958, p. 57.

II. *Solution by D. C. B. Marsh, Colorado School of Mines.* For $k<1$ there exist finite values of z, say z_0, such that

$$|f(0) + a|/(1 - k) = |z_0|.$$

Then one has successively,

$$\begin{aligned} |f(z_0) + a| &\leqq |f(z_0) - f(0)| + |f(0) + a| \\ &\leqq k|z_0| + |f(0) + a| \text{ (by hypothesis of problem)} \\ &\leqq |z_0| \text{ (by definition of } z_0). \end{aligned}$$

Thus the region R, $|z| \leqq |z_0|$, will be mapped continuously by $z \to f(z)+a$ into itself. Therefore, by Brouwer's fixed point theorem, there exists at least one Z in R such that $Z=f(Z)+a$.

The existence of two solutions, Z and Z', would imply that $|f(Z)-f(Z')| =|Z-Z'|$, contrary to the hypothesis that $|f(z)-f(w)|<|z-w|$.

Also solved by D. R. Brillinger, J. L. Brown, Jr., Joseph Geiser, D. A. Kearns, Gerald Leibowitz, Joe Lipman, C. S. Ogilvy, Stanton Philipp, W. F. Trench, W. A. Veech, David Zeitlin, and the proposer. Late solutions by C. H. Cunkle and D. A. Freedman.

Geiser found the problem, with solution, in Kolmogoroff, *Elements of the Theory of Functions and Functional Analysis*, vol. 1.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4765 [1957, 746], Corrected. *Proposed by J. L. Massera, Instituto de Matematica y Estadistica, Montevideo, Uruguay*

Let $y=f(x)$ be a real function defined for $x\geqq 0$. If (i) f has a finite upper bound in any finite interval, and (ii) there are two positive numbers h, k such that $x'-x''\geqq h$ implies $f(x')-f(x'')\geqq k$, then, given $\alpha>1$, there is an increasing function $g(x)$ having as many continuous derivatives as we please, such that $g(x-\alpha h)<f(x)<g(x)$ for all $x\geqq \alpha h$.

4840. *Proposed by P. T. Bateman, University of Illinois*

Suppose $a_0>0$, $a_1\geqq 0$, $a_2\geqq 0, \cdots$, and put $A_n=a_0+a_1+\cdots+a_n$. Show that if, as $n\to\infty$, $A_n\to+\infty$ and $a_n/A_n\to 0$, then $\sum a_nx^n$ has radius of convergence unity.

4841. *Proposed by John Lamperti, California Institute of Technology*

Let N be a random variable whose values are nonnegative integers. Independently of each other, each of N balls is placed either in urn A with probability p $(0<p<1)$ or in urn B with probability $1-p$, resulting in N_A balls in urn A and $N_B=N-N_A$ in urn B. Show that the random variables N_A and N_B are independent if and only if N has a Poisson distribution.

4842. *Proposed by John Lamperti, California Institute of Technology*

Let X be a nonnegative random variable, and let Y be uniformly distributed on the interval $(0, X)$. Let $Z=X-Y$. Show that Y and Z are independent if and only if X has the density

$$f(x) = \begin{Bmatrix} a^2xe^{-ax}, & x \geqq 0 \\ 0, & x < 0 \end{Bmatrix}, \qquad \text{for } a \geqq 0.$$

4843. *Proposed by Leopold Flatto, Brooklyn Polytechnic Institute*

Let S be a set everywhere dense in the plane. Does it follow that S is dense on some line segment in the plane?

4844. *Proposed by D. S. Kahn, Princeton University*

Consider the ring Q of polynomials in the partial differential operators $\partial/\partial x_1, \cdots, \partial/\partial x_n$ over the field of rational functions of $x_1, \cdots, x_n$. (1) Determine all (two-sided) ideals of Q. (2) Prove that Q is a primitive ring.

SOLUTIONS

Postulates for a Ring

4784 [1958, 289]. *Proposed by D. W. Jonah, Purdue University*

In John L. Kelley, *General Topology*, p. 18, a definition of a ring is given in which the left and right distributive laws are replaced by the composite distributive law:

$$(u + v)(x + y) = ux + uy + vx + vy.$$

(a) Show by an example that such a system is not necessarily a ring. (b) Show that if such a system contains an element a such that $a0 = 0$ (in particular, if the system has a multiplicative identity) then the system is a ring.

Solution by R. A. Beaumont, University of Washington, Seattle. (a) Let $\{u\}$ be an additive cyclic group of order 3, and define the product of every pair of elements to be u. Then $(u+v)(x+y) = u$ and $ux+uy+vx+vy = u$, so that the composite distributive law is satisfied. Multiplication is associative and commutative, but the system is not a ring since $00 = u \neq 0$.

(b) For all b,

$$b0 = (b + 0)(0 + 0) = b0 + b0 + 00 + 00,$$
$$0b = (0 + 0)(b + 0) = 0b + 00 + 0b + 00,$$

so that $b0 = 0b = -00 - 00$ and, in particular, $a0 = -00 - 00$. Therefore $a0 = 0$ implies $b0 = 0b = 0$ for all b. Thus we have

$$(a + b)c = (a + b)(c + 0) = ac + a0 + bc + b0 = ac + bc,$$
$$a(b + c) = (a + 0)(b + c) = ab + ac + 0b + 0c = ab + ac,$$

so that the system is a ring.

Also solved by Auguste Forge, Fred Galvin, E. R. Gentile, W. G. Leavitt, Joe Lipman, F. D. Parker, G. W. Potnick, D. A. Robinson, Azriel Rosenfeld, Tôru Saitô, J. E. Simpson, Marlow Sholander, Paul Slepian, Robert Spira, M. J. Walsh, Thann Ward, C. R. B. Wright, and the proposer.

Editorial Note. An interesting generalization is given by Saitô. See *Note on the distributive laws* in this issue of the MONTHLY, pp. 280–283.

Vectors in Binary n-space

4794 [1958, 451]. *Proposed by S. W. Golomb, California Institute of Technology*

Let V_n be binary n-space (the collection of n-vectors over the field of two elements). Consider two vectors of V_n to be in the same class if they differ only

by a cyclic permutation of their components. Show that the number of classes is even, except when $n=2$.

Solution by John B. Kelly, Pennsylvania State University. Call two vectors of V_n conjugate if one is obtained from the other by replacing its 0 components by 1 and its 1 components by 0. With each class in V_n we may associate the class of conjugate vectors. Thus, if there are no self-conjugate classes, the number of classes is even. This is clearly the case when n is odd, for the vectors in a self-conjugate class must have the same number of 1 and 0 components, so that the total number of components is even.

Let us assume $n=2^m n'$, where n' is odd. We proceed by induction on m. The theorem is true when $m=0$ by what we have just said.

Let us denote by S_k the number of classes in V_n with k elements. Clearly, if $S_k \neq 0$, then k divides n. Since V_n has 2^n vectors, we have

$$2^n = \sum_{k|n} kS_k = \sum_{k|n/2} kS_k + \sum_{2^m|k} kS_k = \sum_{k|n/2} kS_k + \sum_{d|n'} 2^m dS_{2^m d}. \tag{1}$$

If $k|n/2$, then a vector belonging to a class with k elements in V_n can be obtained only by iterating a vector in $V_{n/2}$. The first summand in (1) is thus equal to the number of vectors in $V_{n/2}$, namely $2^{n/2}$. Hence (1) becomes

$$2^n = 2^{n/2} + 2^m \sum_{d|n'} dS_{2^m d}.$$

This gives

$$\sum_{d|n'} dS_{2^m d} = 2^{(2^{m-1}n'-m)}(2^{n/2}-1).$$

Now $2^{m-1}n'-m>0$ unless $m=1, 2$, and $n'=1$. Thus, with these exceptions

$$\sum_{d|n'} dS_{2^m d} \equiv 0 \bmod 2.$$

Since n' is odd, all divisors of n' are odd and

$$\sum_{d|n'} S_{2^m d} \equiv \sum_{d|n'} dS_{2^m d} \equiv 0 \pmod 2.$$

Thus the total number of classes in V_n with k elements, where $2^m|k$, is even. The total number of classes in V_n with k elements, where $2^m \nmid k$, is the same as the total number of classes in $V_{n/2}$ which is even by our induction hypothesis, save when $n=4$. This corresponds to an exceptional case above. The other exceptional case corresponds to $n=2$, for which the theorem is false. The theorem is readily verified for $n=4$ by explicit construction of the classes of V_4.

Also solved by N. J. Fine, J. H. Hodges and D. Orloff, and the proposer.

Zeros of a Special Sum

4795 [1958, 451]. *Proposed by Y. L. Luke, Midwest Research Institute, Kansas City, Missouri*

Find the zeros of

$$H_n(x) = \sum_{m=0}^{n} \frac{(-2)^m(2n-m)!\Gamma(x+1)}{m!(n-m)!\Gamma(x-m+1)}.$$

Solution by W. A. Al-Salam and L. Carlitz, Duke University. We first show that

$$(*) \qquad H_{n+1}(x) = -2(x-2n-1)H_n(x).$$

Indeed

$$(x-2n-1)H_n(x) = \sum_{m=0}^{n} \frac{(-2)^m(2n-m)!\Gamma(x+1)}{m!(n-m)!\Gamma(x-m+1)}((x-m)-(2n-m+1))$$

$$= \sum_{m=0}^{n} \frac{(-2)^m(2n-m)!\Gamma(x+1)}{m!(n-m)!\Gamma(x-m)} - \sum_{m=0}^{n} \frac{(-2)^m(2n-m+1)!\Gamma(x+1)}{m!(n-m)!\Gamma(x-m+1)}$$

$$= \sum_{m=1}^{n+1} \frac{(-2)^{m-1}(2n-m+1)!\Gamma(x+1)}{(m-1)!(n-m+1)!\Gamma(x-m+1)} - \sum_{m=0}^{n} \frac{(-2)^m(2n-m+1)!\Gamma(x+1)}{m!(n-m)!\Gamma(x-m+1)}$$

$$= -\frac{1}{2}\sum_{m=0}^{n+1} \frac{(-2)^m(2n-m+2)!\Gamma(x+1)}{m!(n-m+1)!\Gamma(x-m+1)} = -\frac{1}{2}H_{n+1}(x).$$

This proves (*).

It is now an immediate sequence of (*) and the fact that $H_0(x)=1$ that

$$H_n(x) = (-2)^n \prod_{m=1}^{n} (x-2m+1),$$

whence the zeros of $H_n(x)$ are $1, 3, 5, \cdots, 2n-1$.

Also solved by Robert Breusch, N. J. Fine, D. C. B. Marsh, and the proposer.

"Preferred Numbers" and a Set of Polynomials

4796 [1958, 451]. *Proposed by Chandler Davis, American Mathematical Society, Providence, R. I.*

Say t_1 is preferred to t_2 provided (i) $0 \leqq t_1 < t_2 \leqq 1$, or (ii) $-1 \leqq t_2 < t_1 < 0$, or (iii) $-1 \leqq t_2 < 0 \leqq t_1 \leqq 1$. Determine a sequence of polynomials P_n such that, whenever t_1 is preferred to t_2, $P_n(t_2) = o(P_n(t_1))$.

Solution by the proposer. Define first

$$R_n(t) = -\int_{-1}^{t} t(1-t)^{n-1}dt \qquad (n = 2, 3, \cdots),$$

real polynomials of degree $n+1$. They satisfy

$$(1)\qquad R_n'(t)\begin{cases}>0 & (t<0)\\ <0 & (0<t<1),\end{cases}$$

$$(2)\qquad R_n(1) = R_n(-1/n),$$

$$(3)\qquad R_n(t) > 0 \qquad (-1 < t \leqq 1).$$

Now define $P_n(t)=(R_n(t))^{n!}$, real polynomials of degree $(n+1)!$. They, too, satisfy (1), (2), (3).

Let t_1 be preferred to t_2. If (iii) holds, then take $N>-1/t_2$; for $n>N$, $0<R_n(t_2)<R_N(t_2)<R_N(-1/N)=R_N(1)<R_n(1)\leqq R_n(t_1)$; whence

$$\left\{\frac{R_n(t_2)}{R_n(t_1)}\right\}^{n!} \leqq \left\{\frac{R_N(t_2)}{R_N(t_1)}\right\}^{n!} = o(1).$$

It is therefore enough to consider (i) and (ii). (Also, for convenience in the next paragraph, exclude $t_2=1$; no generality is thereby lost because, in case $0\leqq t_1<t_2'<t_2=1$, $0<P_n(1)<P_n(t_2')<P_n(t_1)$.)

Now

$$\log\frac{P_n(t_2)}{P_n(t_1)} = \log\{R_n(t_2)\}^{n!} - \log\{R_n(t_1)\}^{n!} = n!(t_2-t_1)\frac{R_n'(T_n)}{R_n(T_n)}$$

for some T_n between t_1 and t_2. This application of the law of the mean is justified because R_n has no zeros in the interval. One substitutes now in the right-hand side the explicit expressions for R_n' and R_n; then, for either $-1\leqq t_2\leqq T_n\leqq t_1<0$ or $0\leqq t_1\leqq T_n\leqq t_2<1$, a straightforward estimate shows that $\log(P_n(t_2)/P_n(t_1))$ approaches $-\infty$ as $n\to\infty$. This completes the proof.

Remark. Actually, $P_n(t_2)=o(P_n(t_1))$ uniformly in t_2, for t_1 preferred to t_2 and t_2 bounded away from t_1. Any sequence of polynomials having this property provides a solution of a problem (the "induced degeneracy difficulty") left open by the proposer in *Estimating eigenvalues*, Proc. Amer. Math. Soc., vol. 3, 1952, pp. 942–947.

Also solved by Jeny Browkin.

Linear Independence

4797 [1958, 452]. *Proposed by D. J. Newman, Massachusetts Institute of Technology*

Prove that all expressions like $7\sqrt{19}/4-3\sqrt{7}+8\sqrt{6}/5$ are irrational. More specifically, prove that the square roots of the square-free integers are linearly independent over the rationals.

Solution by W. F. Furr and L. K. Williams, Southern University, Baton Rouge, La.

If p is any prime then $\sqrt{p}$ is irrational. Assume that r is the smallest integer such that there are r primes $p_1, \cdots, p_r$ and k square-free integers $M_1, \cdots, M_k$ which are products of these primes such that

$$a_1\sqrt{M_1} + \cdots + a_k\sqrt{M_k} = R, \tag{1}$$

where R is rational and the a_i are rational and not all zero. (The M_i are power-products of the p_i with all exponents 1.)

Let $M_{i_1}, \cdots, M_{i_s}$ be the M's which have p_r as a factor and write (1) as

$$a_{i_1}\sqrt{M_{i_1}} + \cdots + a_{i_s}\sqrt{M_{i_s}} = R - a_{j_1}\sqrt{M_{j_1}} - \cdots - a_{j_t}\sqrt{M_{j_t}} \tag{2}$$

where we may take $a_{i_1} \neq 0, \cdots, a_{i_s} \neq 0$.

Now square both sides of (2) and note that we can write the result, after transposing, in the form

$$c_1\sqrt{M_1'} + \cdots + c_k\sqrt{M_k'} = C, \tag{3}$$

where C and the c_i are rational and the c_i are not all zero and, further, the M_i' are power-products of the $r-1$ primes $p_1, \cdots, p_{r-1}$. The contradiction proves the theorem.

Also solved by R. P. Langlands, P. E. Schweitzer and H. F. Mattson, and the proposer.

Generalization of a Theorem of Liouville

4798 [1958, 529]. *Proposed by John McCarthy, Dartmouth College*

Let $\alpha_1, \cdots, \alpha_n$ be algebraic numbers linearly independent over the rationals. Show that there is a positive constant C and an integer N such that if $m_1, \cdots, m_n$ are rational integers not all zero then

$$|m_1\alpha_1 + \cdots + m_n\alpha_n| \geqq \frac{C}{\{|m_1| + \cdots + |m_n|\}^N}.$$

This generalizes Liouville's theorem on the approximation of algebraic numbers by rationals.

Solution by the proposer. Let Q be a positive integer such that $Q\alpha_1, \cdots, Q\alpha_n$ are all algebraic integers, *i.e.*, satisfy algebraic equations with integer coefficients with leading coefficient one. Let $N+1$ be the degree of a normal extension of the field of rationals containing all the α_i, and let G be the Galois group of the extension. Since the sum of algebraic integers is an algebraic integer $Q(m_1\alpha_1 + \cdots + m_n\alpha_n)$ is an algebraic integer and the product

$$Q^{n+1} \prod_{g \in G} g(m_1\alpha_1 + \cdots + m_n\alpha_n)$$

will be a rational integer not zero.

Hence we have

$$|m_1\alpha_1 + \cdots + m_n\alpha_n| \geqq \frac{1}{Q^{N+1}\prod_{g\neq 1} |g(m_1\alpha_1 + \cdots + m_n\alpha_n)|}.$$

Letting A be the maximum of the absolute values of the conjugates of the α_i and taking $C=1/Q^{N+1}A^N$ we get the stated inequality.

Solved also by O. Buchta.

Sequence and Subsequence

4800 [1958, 530]. *Proposed by N. S. Mendelsohn, University of Manitoba*

Let $u(n)$ be a sequence with a recurrence relation of the form

$$u_{n+1} = a_0u_n + a_1u_{n-1} + \cdots + a_ku_{n-k}$$

where $a_0, \cdots, a_k$ are real constants. Show that the sequence obtained by taking every rth term of this sequence satisfies a recurrence of the same length; *i.e.*, if $v_n = u_{rn}$ for $n=0, 1, \cdots k$ there exist constants $A_0, \cdots, A_k$ such that for $n=k+1, k+2, \cdots,$

$$v_{n+1} = A_0v_n + A_1v_{n-1} + \cdots + A_kv_{n-k}.$$

Solution by A. C. Aitken, University of Edinburgh, Scotland. The characteristic equation of the recurrence relation for $u(n)$ is

$$\lambda^k - a_0\lambda^{k-1} - a_1\lambda^{k-2} - \cdots - a_k = 0,$$

and if its roots, λ_j, are distinct, then $u_t = \sum c_j\lambda_j^t$. Hence $v_t = \sum c_j\lambda_j^{rt}$ so that v_t, like u_t, obeys a linear difference equation of order k, the roots of the characteristic equation being λ_j^r.

The case of multiple roots introduces no difficulty.

Also solved by O. Buchta, L. Carlitz, R. C. Lyness, Imanuel Marx, Paul Payette, W. F. Trench and the proposer.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R.V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

COLLEGE TEXTBOOKS

Editorial Note. Every known American publisher of mathematical books was asked to list the mathematical books published during the five-year period 1954–58. The following list was compiled by checking this list against the MONTHLY files and other known book

lists. Every effort has been made to prepare as complete a list as possible. Prices, authors, *etc.* have been triple checked. We fear, however, that there may still be omissions or errors, and apologize in advance for these. If your book was published between 1954–58, but is not listed, complain to your publisher about it!

Titles have been grouped into the following categories: Mathematics of Finance and Business Mathematics; Remedial (Intermediate) Algebra; Trigonometry; College Algebra; College Algebra and Trigonometry; Analytic Geometry; Integrated Freshman Mathematics; Calculus with Analytic Geometry; Calculus; Post-Calculus Engineering Mathematics; Advanced Calculus; Differential Equations; Partial Differential Equations; Series; Applied Mathematics; Complex Variables; Real Variables; Vector Analysis; Modern Abstract Algebra; Theory of Numbers; Numerical Analysis; Computers; Tables; Geometry; Statistics and Related Topics; Probability; Topology and Algebraic Geometry; Logic and Set Theory; Teaching; Unclassified.

In each case the publisher or the author was asked to stipulate into which category his book was placed, and sent a complete mimeographed listing for final checking. Within each category books are classified by age and, within any given year, alphabetically by author. In each case we have tried to give the title, author(s), his institution, publisher, copyright date, number of pages, current price, and the page and year of the MONTHLY in which it was reviewed. A few titles are followed by a brief comment. These do not signify special merit or demerit, but indicate the content when the title is not sufficiently descriptive, or the book is not well known or has not been reviewed in the MONTHLY.

Publishers will usually send an examination copy (which may be kept or returned for full credit) upon request from any school, college, or university mathematics department. We urge you to examine suitable books for yourself. Only you can decide which book best fits your situation.

MATHEMATICS OF FINANCE AND BUSINESS MATHEMATICS

Mathematics of Business Affairs. By Feldman (Beaumont High School, University City, Mo.). Allyn and Bacon, 1958. 244 pages (workbook format), $3.95. To be reviewed.

Mathematics of Investment, Fourth Edition. By Hart (University of Minnesota). D. C. Heath, 1958. 343 pages plus 150 tables, $6.75. Review: p. 377, 1958.

Mathematics in Business. By Lowenstein (Arizona State College). Wiley, 1958. 364 pages, $4.95. To be reviewed.

Business Mathematics, Fourth Edition. By Richtmeyer and Foust (Central Michigan College). McGraw-Hill, 1958. 412 pages, $5.75. To be reviewed.

A revision of a 1936 text.

Essential Business Mathematics, Third Edition. By Snyder (San Francisco City College). McGraw-Hill, 1958. 470 pages, $5.50. To be reviewed.

Fundamentals of Business Mathematics. By Knick (Marquette University). Richard D. Irwin, 1957. 452 pages, $6.50 text, $7.80 trade.

Mathematics of Finance. By Lee (University of No. Carolina). Richard D. Irwin, 1957. 344 pages, $6.50 text, $7.80 trade.

The Mathematics of Investment. By Osborn (University of Texas). Harper, 1957. 162 pages plus 117 pages of tables, $4.25. Review: p. 683, 1957.

Mathematics of Finance. By Stelson (Michigan State University). Van Nostrand, 1957. 327 pages, $5.50. To be reviewed.

The Mathematics of Finance. By Cissell and Cissell (Xavier University). Houghton Mifflin, 1956. 198 pages, $4.50. Review: pp. 206–7, 1957.

Mathematics of Finance, Second Edition. By Hummel and Seebeck (University of Alabama). McGraw-Hill, 1956. 372 pages, $5.00.

A revision of Hummel and Seebeck's 1948 text.

Mathematics of Finance. By Parker (Texas Technological College). Prentice-Hall, 1956. 288 pages, $5.75.

Elements of Business Mathematics for College Students. By Snyder (San Francisco City College). McGraw-Hill, 1956. 249 pages, $3.90.

Mathematics of Business, Accounting, and Finance. By Trefftzs (University of Southern California) and Hills (Los Angeles City College). Harper, 1956. 591 pages, $5.00. Review: pp. 377–8, 1957.

A revision of a 1947 text.

Mathematics of Business. By Zant (Oklahoma State University). Prentice-Hall, 1956. 211 pages, $4.25.

Business Mathematics. By Mira and Hartmann (Manhattanville College of the Sacred Heart and St. John's University). Van Nostrand, 1955. 341 pages, $4.85.

Fundamentals of Business Mathematics, Second Edition. By Van Voorhis and Topp (Fenn College). Prentice-Hall, 1955. 452 pages, $6.95.

Revision of 1948 text.

Mathematics of Finance. By Mira and Hartmann (Manhattanville College of the Sacred Heart and St. John's University). Van Nostrand, 1954. 335 pages, $4.85.

REMEDIAL (INTERMEDIATE) ALGEBRA

Intermediate Algebra. By Bardell and Spitzbart (University of Wisconsin). Addison-Wesley, 1959. 274 pages, $4.75.

Introductory Mathematical Analysis. By Eaves and Wilson (University of Tennessee and Convair, Ft. Worth, Texas). Allyn and Bacon, 1958. 352 pages, $5.00. To be reviewed.

Intermediate Algebra for Colleges. By Fuller (Texas Technological). Van Nostrand, 1958. 258 pages, $3.90. To be reviewed.

Introducing Mathematics. By Helton (Central College). Wiley, 1958. 396 pages, $5.75.

Basic Mathematics. By Kaltenborn, Anderson and Kaltenborn (Memphis State College). Ronald Press, 1958. 350 pages, $4.75. To be reviewed.

The Structure of Arithmetic and Algebra. By Maria (Brooklyn College) Wiley, 1958. 294 pages, $5.90.

An elementary axiomatic development of the real number system written especially for teachers and nonscience students. To be reviewed.

Essential Mathematics for College Students. By Mueller (Towson, Md., State Teachers College). Prentice-Hall, 1958. 288 pages, $3.95.

A Modern Approach to Intermediate Algebra. By Patin (Wilson Junior College). G.P. Putnam's, 1958. 288 pages, $3.75. To be reviewed.

Preparatory Mathematics, Parts I and II. By Georges *et al.* (Wright Jr. College, *etc.*). Edwards, 1957. 208 and 216 pages, $4.00 and $4.50.

Fundamental Mathematics. By Miller (Ohio State University). Henry Holt, 1957. 288 pages, \$3.50.

Intermediate Algebra, Second Edition. By Rees and Sparks (Louisiana State University and Texas Technological). McGraw-Hill, 1957. 306 pages, \$3.90. Review: p. 381, 1957.

A revision of the 1951 text.

Intermediate Algebra for College Students. By White (Stevens Institute of Technology). Allyn and Bacon, 1957. 460 pages, \$4.50.

Over 6000 exercises of standard type.

Basic Mathematics: A Workbook—Form B. By Keller and Zant (Purdue University and Oklahoma State University). Houghton Mifflin, 1956. 255 pages, \$2.20.

A Review of Mathematics for College Students, Revised. By Lapp, Knight, Reitz and revised by Lapp (National Academy of Sciences-National Research Council). Scott, Foresman, 1956. 208 pages, \$1.90 list.

A revision of the 1934 and 1942 workbook.

Elements of Algebra, Second Edition. By Levi (Columbia University). Chelsea, 1956. 168 pages, \$3.25.

A path-breaking text.

Elements of Mathematics, Second Edition. By Roberts and Stockton (University of Connecticut) Addison-Wesley, 1956. 308 pages, \$3.75. Review: pp. 605–6, 1957.

A tear-sheet workbook revision of a 1952 text.

Fundamental Mathematics. By Wade and Taylor (Florida State University). McGraw-Hill, 1956. 380 pages, \$4.95.

First Course in Algebra for Colleges. By Adams (Santa Monica City College). Henry Holt, 1955. 217 pages, \$3.50.

Basic Mathematics for General Education, Second Edition. By Trimble, Peck, and Bolser (Iowa State Teachers College, Ohio Wesleyan University). Prentice-Hall, 1955. 363 pages, \$6.75.

Intermediate Algebra, Third Edition. By Adams (Santa Monica City College). Henry Holt, 1954. 366 pages, \$4.50.

Elements of Mathematics. By Banks (George Peabody College for Teachers). Allyn and Bacon, 1954, 1956. 422 pages, \$5.75.

Mathematics in Agriculture, Second Edition. By McGee (Texas A. and M. College). Prentice-Hall, 1954. 208 pages, \$5.75.

A revision of a 1942 text.

Intermediate Algebra for College Students, Revised Edition. By Peterson (Portland State College). Harper, 1954. 369 pages, \$3.75.

TRIGONOMETRY

Plane Trigonometry. By Niles (United States Naval Academy). Wiley, 1959. 284 pages, \$3.95.

Plane Trigonometry, Second Edition. By Corliss and Berglund (University of Illinois. Navy Pier) Houghton Mifflin, 1958. 397 pages, \$4.00.

Plane Trigonometry. By Rickey and Cole (Louisiana State University). Holt, née Dryden, 1958. 260 pages, $2.90. Review: pp. 639–40, 1958.

Elements of Plane Trigonometry. By Sharp (Emory University). Prentice-Hall, 1958. 298 pages, $4.95. To be reviewed.

Trigonometry, Plane and Spherical. By Hartley (University of Illinois). Odyssey, 1957. 374 pages, $3.80.

Plane Trigonometry, Third Edition. By Nelson and Folley (Wayne State University). Harper, 1956. 195 pages and 135 pages of tables, $4.00.

A revision of 1936 and 1943 texts.

Modern Trigonometry. By Rutledge and Pond (University of Tulsa and Technical Operations, Inc., Washington, D. C.). Prentice-Hall, 1956. 243 pages, $4.95. Review: pp. 55–6, 1957.

Practical Trigonometry. By Underwood and Woodward (Texas Technological). Houghton Mifflin, 1956. 251 pages, $3.25.

Modern Trigonometry. By Brixey and Andree (University of Oklahoma). Henry Holt, 1955. 224 pages, $4.00. Review: pp. 130–132, 1956.

Analytical trigonometry. A teaching guide is available.

Trigonometry. By Dubisch (Fresno State College). Ronald Press, 1955. 395 pages, $5.50, Review: pp. 458–9, 1955.

An outstanding development of analytical trigonometry of real numbers.

Trigonometry. By Perlin (Georgia Institute of Technology). International Textbook, 1955. 334 pages, $3.50.

Plane Trigonometry. By Spitzbart and Bardell (University of Wisconsin). Addison-Wesley, 1955. 205 pages, $3.75. Review: pp. 54–6, 1956.

Plane Trigonometry. By Wylie (University of Utah). McGraw-Hill, 1955. 381 pages, $4.50. Review: pp. 130–2, 1956.

Trigonometry. By Hart (University of Minnesota). D. C. Heath, 1954. 230 pages plus 130 pages of tables, $4.25.

Differs from the author's 1951 *College Trigonometry* only in chapters 1 to 4.

Trigonometry. By Vance (Oberlin College). Addison-Wesley, 1954. 158 pages, $3.75. Review: pp. 458–9, 1955.

Analytic Trigonometry.

COLLEGE ALGEBRA

College Algebra, Second Edition. By Peterson (Portland State College) Harper, 1958. 413 pages, $4.00.

A revision of a 1947 text.

College Algebra, Alternate Edition. By Richardson (Brooklyn College). Prentice-Hall, 1958. 544 pages, $5.95.

College Algebra. By Rietz, Crathorne, Peters (University of Illinois). Henry Holt, 1958. 387 pages, $4.50.

College Algebra, Fourth Edition. By Rosenbach, Whitman, Meserve and Whitman

(Carnegie Institute of Technology, Montclair State College and Johns Hopkins University). Ginn, 1958. 640 pages, $5.25. Review: p. 258, 1934 and p. 647, 1949.

A shorter edition (selections, not condensations) is available under the title *Essentials of College Algebra*, Second Edition. $4.50.

College Algebra. By Cameron and Browne (University of No. Carolina). Henry Holt, 1956. 390 pages, $4.75.

College Algebra for Freshmen. By Fuller (Texas Technological). Van Nostrand, 1956. 343 pages, $3.85.

College Algebra. By Kells (U. S. Naval Academy). Prentice-Hall, 1956. 366 pages, $5.75.

Applies axioms to natural numbers, then to integers, and then to rational numbers, real numbers and complex numbers.

College Algebra. By Apostle (Grinnell College). Henry Holt, 1955. 422 pages, $4.75. Review: pp. 192–3, 1955.

Algebra for College Students. By Whyburn and Daus (University of No. Carolina and U.C.L.A.). Prentice-Hall, 1955. 290 pages, $4.95.

Review: pp. 265–6, 1956.

College Algebra. By Rider (Washington University). Macmillan, 1955. 397 pages, $4.25.

Algebra for College Students. By Britton and Snively (University of Colorado). Rinehart, 1954. 537 pages, $4.50. Review: pp. 192–3, 1955.

College Algebra, Second Edition. By Keller (Purdue University) Houghton Mifflin, 1954. 471 pages, $4.50.

College Algebra, Third Edition. By Rees and Sparks (Louisiana State University and Texas Technological). McGraw-Hill, 1954. 455 pages, $4.75. Review: pp. 192–3, 1955.

A straightforward text; no "frills," but unusually fine graded problems.

COLLEGE ALGEBRA AND TRIGONOMETRY

Integrated Algebra and Trigonometry. By Fisher and Ziebur (Ohio State University). Prentice-Hall, 1958. 427 pages, $6.25. Review: p. 643, 1958.

College Algebra and Trigonometry, Second Edition. By Miller (Cooper Union School of Engineering). Wiley, 1955. 342 pages, $4.50.

College Algebra and Plane Trigonometry. By Spitzbart and Bardell (University of Wisconsin). Addison-Wesley, 1955. 408 pages, $5.25. Review: pp. 54–6, 1956.

Unified Algebra and Trigonometry. By Vance (Oberlin College). Addison-Wesley, 1955. 354 pages, $5.25. Review: pp. 54–6, 1956.

ANALYTIC GEOMETRY

Analytic Geometry. By Purcell (University of Arizona). Appleton-Century-Crofts, 1958. 289 pages, $4.50. To be reviewed.

Analytic Geometry of Three Dimensions, Seventh Edition. By Salmon (University of Dublin). Chelsea, 1958. 496 pages, $4.95. To be reviewed.

Brief Analytic Geometry, Third Edition. By Mason and Hazard (Purdue University). Ginn. 1957. 229 pages, $3.75. Review: p. 381, 1957.

Analytical Conics. By Spain (Sir John Cass College, London). Pergamon Press, 1957. 145 pages, $5.00. Review: p. 301, 1958.

A First Course in Analytic Geometry. By Ayre and Stephens (Western Illinois State College and Knox College). Van Nostrand, 1956. 224 pages, $4.10.

An interwoven presentation of plane and solid analytic geometry.

Analytic Geometry. By Rees (Louisiana State University). Prentice-Hall, 1956. 237 pages, $5.25.

New Analytic Geometry, Revised Edition. By Smith, Gale, Neelley (Carnegie Institute of Technology). Ginn, 1956. 346 pages, $4.00.

Analytic Geometry, Second Edition. By Underwood and Sparks (Texas Technological). Houghton Mifflin, 1956. 282 pages, $3.75.

Analytic Geometry. By Love and Rainville (University of Michigan). Macmillan, 1955. 302 pages, $4.25.

Analytic Geometry. By McCoy and Johnson (Smith College). Rinehart, 1955. 301 pages $3.50. Review: pp. 675–6, 1956.

Analytic Geometry, Second Edition. By Middlemiss (Washington University). McGraw-Hill, 1955. 310 pages, $4.25. Review: pp. 675–6, 1956.

Analytic Geometry. By Sisam and Atchison (Colorado College and Rich Electronic Computer Center). Henry Holt, 1955. 292 pages, $4.00.

Analytic Geometry, Third Edition. By Steen and Ballou (Allegheny College and Middlebury College). Ginn, 1955. 251 pages, $4.00.

Analytic Geometry. By Fuller (Texas Technological). Addison-Wesley, 1954. 205 pages, $4.25. Review: pp. 196–7, 1955.

Conic Sections, Sixth Edition. By Salmon (University of Dublin). Chelsea, 1954. 415 pages, $3.25 cloth; $1.94 paper-bound.

A classic.

Analytic Geometry, Second Edition. By Smith, Salkover and Justice (University of Cincinnati). Wiley, 1954. 306 pages, $4.00. Review: pp. 196–7, 1955.

Covers both the applied and the theoretical sides of analytical geometry.

INTEGRATED FRESHMAN MATHEMATICS

A Modern Introduction to College Mathematics. By Rose (University of Massachusetts). Wiley, 1959. 548 pages, $6.50. To be reviewed.

Introductory College Mathematics. By Wade (Florida State University). Wiley, 1959. 344 pages, $5.50.

Introduction to Mathematical Analysis with Applications to Problems of Economics. By Daus and Whyburn (U.C.L.A. and University of No. Carolina). Addison-Wesley, 1958. 244 pages, $6.50.

Fundamentals of Mathematics. By Richardson (Brooklyn College). Macmillan, 1958. 507 pages, $6.50. To be reviewed.

Mathematics for Science and Engineering. By Alger (General Electric Company). McGraw-Hill, 1957. 360 pages, \$6.95, Text edn. \$5.50.

Very broad coverage, derived partially from the old Steinmetz books. Review: p. 57, 1958.

Introduction to Finite Mathematics. By Kemeny, Snell, and Thompson. (Dartmouth and Ohio Wesleyan University). Prentice-Hall, 1957. 372 pages, \$5.95. Review: pp. 688–9, 1957.

A true trail-blazer, designed particularly as a basic text for behavioral science students, but is being used elsewhere as well. Includes no calculus.

Introductory College Mathematics. By Wagner (University of Massachusetts). McGraw-Hill, 1957. 430 pages, \$5.25. Review: pp. 461–2, 1958.

Mathematical Analysis. By Camp (Macalester College). D. C. Heath, 1956. 610 pages, \$6.75.

A Modern Introduction to Mathematics. By Freund (Arizona State). Prentice-Hall, 1956. 543 pages, \$6.95.

Principles of Mathematics. By Allendoerfer and Oakley (University of Washington and Haverford). McGraw-Hill, 1955. 448 pages, \$5.75. Review: pp. 435–9, 1956.

Another trail-blazer.

Basic Mathematics for Science and Engineering. By Andres, Miser and Reingold (deceased, U.S.A.F., Illinois Institute of Technology). Wiley, 1955. 846 pages, \$7.50.

Introductory Mathematical Analysis. By Georges (Wright Junior College). J. W. Edwards, 1955. 624 pages, \$6.00.

Introduction to College Mathematics. By Hill and Linker (University of No. Carolina). Henry Holt, 1955. 428 pages, \$6.50.

Mathematics and Measurements. By Rassweiler, Merrill, and Harris (University of Minnesota). Row, Peterson, 1955. 251 pages, \$4.50.

Basic Mathematics for General Education, Second Edition. By Trimble, Peck, and Bolser (Iowa State Teachers College, Ohio Wesleyan, Govt. Consultant). Prentice-Hall, 1955. 363 pages, \$6.75.

A First Year of College Mathematics, Second Edition. By Brink (University of Minnesota). Appleton-Century-Crofts, 1954. 725 pages, \$5.75.

Fundamentals of College Mathematics. By Brixey and Andree (University of Oklahoma). Holt, 1954. 609 pages, \$6.95. Review: pp. 193–5, 1955.

Introductory College Mathematics. By Jaeger and Bacon (Pomona College and Stanford University). Harper, 1954. 382 pages, \$4.75.

Introductory College Mathematics. By Leonhardy (Stephens College). Wiley, 1954. 459 pages, \$5.50.

Introduction to College Mathematics, Second Edition. By Newsom and Eves (New York University and University of Maine). Prentice-Hall, 1954. 408 pages, \$6.95. Review: 724–5, 1954.

CALCULUS WITH ANALYTIC GEOMETRY

Analytic Geometry and Calculus. By Hart (University of Minnesota). D. C. Heath, 1957. 716 pages, $7.00.

Calculus with Analytic Geometry. By Johnson and Kiokemeister (Smith College and Mt. Holyoke College). Allyn and Bacon, 1957. 650 pages, $8.25. Review: pp. 640–1, 1958.

Analytic Geometry and Calculus. By Peterson (Portland State College). Harper, 1955. 456 pages, $6.00. Review: pp. 674–5, 1956.

Introductory Calculus with Analytic Geometry. By Begle (Yale University). Holt, 1954. 304 pages, $5.25. Review: pp. 54–6, 1955.

CALCULUS

Calculus. By Leighton (Carnegie Institute of Technology). Allyn and Bacon, 1958. 416 pages, $6.95. To be reviewed.

An Analytical Calculus, Volume IV. By Maxwell (Cambridge University). Cambridge University Press, New York, 1958. 272 pages, $4.00. Review: pp. 536–7, 1958.

Calculus, Second Edition. By Smith, Salkover and Justice (University of Cincinnati). Wiley, 1958. 520 pages, $6.50. Review: p. 377, 1958.

Elements of the Differential and Integral Calculus, New Revised Edition. By Granville, Smith and Longley (Yale University). Ginn, 1957. 556 pages, $5.75.

Calculus. By Britton (University of Colorado). Rinehart, 1956. 584 pages, $6.50. Review: pp. 462–3, 1958.

Calculus. By Morrill (Johns Hopkins University). Van Nostrand, 1956. 537 pages, $6.00·

Differential and Integral Calculus, Second Edition. By Bacon (Stanford University). McGraw-Hill, 1955. 547 pages, $6.50. Review: pp. 202–3, 1956.

Calculus. By Hart (University of Minnesota). D. C. Heath, 1955. 626 pages, $6.75. Review: p. 202–3, 1956.

First Course in Calculus. By Cooley (New York University). Wiley, 1954. 643 pages, $6.50. Review: pp. 52–4, 1955.

Differential and Integral Calculus. By Love and Rainville (University of Michigan). Macmillan, 1954. 526 pages, $5.95. Review: pp. 454–8, 1955.

An Analytical Calculus. By Maxwell (Cambridge University). Cambridge University Press, New York, 1954. Vol. I—200 pp.; Vol. II—200 pp.; Vol. III, 200 pp., $3.50 each.

Calculus: A Modern Approach. By Menger (Illinois Institute of Technology). Ginn, 1954. 372 pages, $5.50. Review: pp. 483–92, 1954.

Calculus. By Merriman (University of Cincinnati). Holt, 1954. 626 pages, $6.95. Review: pp. 454–8, 1955.

Calculus, Third Edition. By Sherwood and Taylor (U.C.L.A.). Prentice-Hall, 1954. 579 pages, $7.95. Review: pp. 454–8, 1955.

POST-CALCULUS ENGINEERING MATHEMATICS

Engineering Mathematics. By Gaskell (Boeing Scientific Research Labs). Holt, née Dryden, 1958. 512 pages, $7.75. To be reviewed.

Applied Mathematics for Engineers and Physicists, Second Edition. By Pipes (U.C.L.A.). McGraw-Hill, 1958. 723 pages, $8.75. Revision of 1946 book. To be reviewed.

Mathematics of Physics and Modern Engineering. By Sokolnikoff and Redheffer (U.C.L.A.). McGraw-Hill, 1958. 828 pages, $9.50. To be reviewed.

Modern Mathematics for the Engineer. By Beckenbach (U.C.L.A.). McGraw-Hill, 1956. 536 pages, $7.50. To be reviewed.

Engineering Analysis (A Survey of Numerical Procedures). By Crandall (M.I.T.). McGraw-Hill, 1956. 417 pages, $9.50. To be reviewed.

Engineering Mathematics. By Miller (New York University). Rinehart, 1956. 417 pages, $6.50. Review: pp. 281–2, 1957.

Mathematics of Engineering Systems (Linear and Nonlinear). By Lawden (College of Technology, Birmingham, England). Wiley, 1955. 380 pages, $5.75.

Higher Mathematics for Students of Chemistry and Physics. By Mellor. Dover, 1955. 641 pages, $2.25.

Advanced Mathematics for Engineers, Third Edition. By Reddick and Miller (Syracuse University and Cooper Union School of Technology). Wiley, 1955. 548 pages, $6.50.

Engineering Analysis. By Ver Planck and Teare (Carnegie Institute of Technology). Wiley, 1954. 344 pages, $6.00.

ADVANCED CALCULUS

Mathematical Analysis, A Modern Approach to Advanced Calculus. By Apostol (California Institute of Technology). Addison-Wesley, 1957. 553 pages, $9.00. Review: pp. 463–4, 1958.

Designed primarily for mathematics majors.

Elements of Pure and Applied Mathematics (International Series). By Lass (California Institute of Technology). McGraw-Hill, 1957. 491 pages, $7.50. Review: p. 517, 1957.

Advanced Real Calculus. By Miller (New York University). Harper, 1957. 185 pages, $5.00. Review: p. 519, 1957.

Advanced Calculus (International Series in Pure and Applied Mathematics). By Buck (University of Wisconsin). McGraw-Hill, 1956. 432 pages, $8.50. Review: pp. 754–5, 1956.

Advanced Calculus—An Introduction to Classical Analysis. By Brand (University of Houston), Wiley. 1955. 574 pages, $8.50. Review: p. 786, 1958.

Advanced Calculus. By Taylor (University of California). Ginn, 1955. 799 pages, $8.75. Review: pp. 733–4, 1956.

The Integral Calculus (two volumes). By Edwards (Queen's College). Chelsea, 1954. 1,922 pages, $15.00.

A comprehensive, detailed text, with many examples.

Advanced Calculus, New Edition. By Woods. Ginn, 1954. 406 pages, $6.25.

DIFFERENTIAL EQUATIONS

Lectures on Ordinary Differential Equations. By Hurewicz (Formerly M.I.T.) Wiley, 1958. 122 pages, $5.00. To be reviewed.

Ordinary Differential Equations. By Kaplan (University of Michigan). Addison-Wesley, 1958. 534 pages, $9.50.

Emphasis on input-output analysis and applications to systems engineering problems.

Elementary Differential Equations. By Rainville (University of Michigan). Macmillan, 1958. 449 pages, $5.50. To be reviewed.

A Short Course in Differential Equations. By Rainville (University of Michigan). Macmillan, 1958. 259 pages, $4.50. To be reviewed.

Applied Differential Equations. By Spiegel (Rensselaer Polytechnic Institute). Prentice-Hall, 1958. 381 pages, $6.95. To be reviewed.

Differential Equations: Geometric Theory. By Lefschetz (Princeton University). Interscience, 1957. 374 pages, $10.50. Review: pp. 784–5, 1958.

Differential Equations Applied in Science and Engineering. By Wayland (California Institute of Technology). Van Nostrand, 1957. 353 pages, $7.50. To be reviewed.

Theory of Ordinary Differential Equations. By Burkill (University of Cambridge). Interscience, 1956. 106 pages, $1.75.

Elementary Differential Equations. By Martin and Reissner. Addison-Wesley, 1956. 260 pages, $6.50.

Differential Equations, Third Edition. By Reddick and Kibbey (Syracuse University). Wiley, 1956. 304 pages, $4.50.

Theory of Ordinary Differential Equations (International Series). By Coddington and Levinson. (U.C.L.A. and M.I.T.). McGraw-Hill, 1955. 429 pages, $8.50.

Differential Equations, Second Edition. By Ford (Formerly of Illinois Institute of Technology). McGraw-Hill, 1955. 288 pages, $5.25.

Introduction to Differential Equations of Physics. By Hopf. Dover, 1955. 160 pages, $1.25.

Differential Equations. By Steen (Allegheny College). Ginn, 1955. 337 pages, $4.75.

Differential Equations with Applications. By Betz, Burcham and Ewing (University of Missouri). Harper, 1954. 310 pages, $4.50. Review: pp. 130–1, 1955.

Ordinary Differential Equations. By Ince. Dover, 1954. 558 pages, $2.55.

A First Course in Ordinary Differential Equations. By Langer (University of Wisconsin). Wiley, 1954. 249 pages, $4.50. Review: p. 437, 1954.

Existence Theorems for Ordinary Differential Equations. By Murray and Miller (Columbia University and New York University). Interscience, 1954. 192 pages, $6.00. Review: p. 668, 1955.

PARTIAL DIFFERENTIAL EQUATIONS

Methods Based on the Wiener-Hopf Technique for the Solution of Partial Differential Equations. By Noble (Royal College of Science and Technology, Glasgow). Pergamon Press, 1958. 280 pages, $10.00.

Elements of Partial Differential Equations. By Sneddon (University of Glasgow). McGraw-Hill, 1957. 345 pages, $7.50. Review: pp. 216–7, 1958.

The Hypercircle in Mathematical Physics. By Synge (Dublin Institute). Cambridge University Press, 1957. 440 pages, $13.50. Review: pp. 217–8, 1958.

Partial Differential Equations of Mathematical Physics. By Bateman. Dover, 1955. 522 pages, $4.95. Not currently available.

Lectures on Partial Differential Equations. By Petrovsky (Akademiia Nauk, Moscow). Interscience, 1955. 255 pages, $6.50.

Partial Differential Equations of Mathematical Physics. By Webster. Dover, 1955. 440 pages, $1.98.

Partial Differentiation. By Gillespie (University of Glasgow). Interscience, 1954. 113 pages, $1.75.

SERIES

Introduction to Fourier Analysis and Generalized Functions. By Lighthill (University of Manchester). Cambridge University Press, 1958. 88 pages, $3.50. To be reviewed.

Orthogonal Functions. By Sansone (University of Florence). Interscience, 1958. 420 pages, Approx. $13.50.

Expansions in series of orthogonal functions, together with preliminary notions of Hilbert space, followed by expansions in Fourier series, in series of Legendre polynomials, and spherical harmonics, and expansions in Laguerre and Hermite series.

Infinite Series and Sequences. By Knopp. Dover, 1956. 186 pages, $1.75.

Trigonometrical Series. By Zygmund. Dover, 1955. 329 pages, $1.50.

Contains analyses of trigonometric, orthogonal, Fourier systems of functions, with descriptions of summability of Fourier series, proximation theory, conjugate theory, divergence of Fourier series.

Introduction to the Theory of Fourier's Series and Integrals. By Carslaw. Dover, 1955. 368 pages, $2.00.

A revision of the author's 1930 book giving an historical introduction, rational and irrational numbers, infinite series, functions of a single variable, definite integral, Fourier series, Fourier integrals, harmonic analysis, and periodogram analysis.

Infinite Series. By Hyslop (University of Glasgow). Interscience, 1954. 133 pages, $1.55.

Mainly real series, but some work on complex series and on infinite products.

APPLIED MATHEMATICS

Operational Mathematics, Second Edition. By Churchill (University of Michigan). McGraw-Hill, 1958. 440 pages, $7.00. To be reviewed.

Methods of Mathematical Physics. Volume II. By Courant and Hilbert (Institute of Mathematical Science). Interscience, in preparation.

Linear Operators in Two Parts—Part I, General Theory, Part II, Spectral Theory. By Dunford and Schwartz. Assisted by Bade and Bartle (Yale University). Interscience, 1958. 872 pages, Part I, $23.00. Part II, in preparation. To be reviewed.

Celestial Mechanics. By Finlay-Freundlich (St. Andrews, Fife). Pergamon Press, 1958. $7.50.

Surveys in Applied Mathematics. Volume I (Midwest Research Institute Series) deals with Elasticity and Plasticity. By Goodier and Hodge (Stanford University and Illinois Institute of Technology). Wiley, 1958. 152 pages, $6.25. To be reviewed.

Surveys in Applied Mathematics. Volume II (Midwest Research Institute Series) deals with Dynamics and Nonlinear Mechanics. By Leimanis and Minorsky (University of British Columbia and Aix-en-Provence, France). Wiley, 1958. 206 pages, $7.75. To be reviewed.

Surveys in Applied Mathematics. Volume III (Midwest Research Institute Series) discusses Mathematical Aspects of Subsonic and Transonic Gas Dynamics. By Bers (New York University). Wiley, 1958. 164 pages, $7.75. To be reviewed.

Surveys in Applied Mathematics. Volume IV (Midwest Research Institute Series) deals with some aspects of analysis and probability. By Kaplansky, Hewitt and Fortet (University of Chicago and University of Washington). Wiley, 1958. 243 pages, $9.00. To be reviewed.

Surveys in Applied Mathematics. Volume V (Midwest Research Institute Series). This volume deals with Numerical Analysis and Partial Differential Equations. By Forsythe and Rosenbloom (Stanford University and University of Minnesota). Two separate papers in one binding. Wiley, 1958. 204 pages, $7.50. To be reviewed.

Calculus of Variations and Its Applications: Proceedings of Symposia in Applied Mathematics Volume VIII. By Graves, Editor. McGraw-Hill for American Mathematical Society, 1958. 215 pages, $7.50. Review: p. 648, 1958.

Reference volume for graduate students and researchers; papers by leading scholars, presented at the symposium held in Chicago in 1956.

Modern Geometrical Optics. By Herzberger (Eastman Kodak Co.). Interscience, 1958· 516 pages, $15.00.

An Introduction to Advanced Dynamics. By McCuskey (Case Institute of Technology). Addison-Wesley, 1959. 269 pages, $8.50.

Mathematics and Wave Mechanics. By Atkin (Northern Polytechnic, London, England). Wiley, 1957. 348 pages, $6.00. To be reviewed.

Integral Equations and Their Applications to Certain Problems in Mechanics, Mathematical Physics and Technology. By Mikhlin, translated from Russian by A. H. Armstrong. Pergamon Press, 1957. 338 pages, $12.50. Review; pp. 644–5, 1958.

Development and Meaning of Eddington's Fundamental Theory. By Noel B. Slater (University of Leeds). Cambridge University Press, New York, 1957. 308 pages, $7.50. Review: p. 134, 1958.

Water Waves: The Mathematical Theory with Applications. By Stoker (New York University). Interscience, 1957. 595 pages, $12.75.

Integral Equations. By Tricomi (University of Turin, Italy). Interscience, 1957. 246 pages, $7.75.

Surveys in Mechanics. By Batchelor (Cambridge University). Cambridge University Press, New York, 1956. 432 pages, $9.50.

Vibration Analysis Tables. By Bishop and Johnson (Cambridge University). Cambridge University Press, New York, 1956. 64 pages, $2.00.

Integral Functions. By Cartwright (Cambridge University). Cambridge University Press, New York, 1956. 130 pages, $3.50.

Electricity. By Coulson (Oxford University). Interscience, 1956. 268 pages, $1.95.

Principles and Techniques of Applied Mathematics. By Friedman (University of California, Berkeley). Wiley, 1956. 315 pages, $8.00.

Principally devoted to linear operators.

The Mathematics of Physics and Chemistry, Second Edition. By Margenau and Murphy (Yale University and New York University). Van Nostrand, 1956. 604 pages, $7.25.

Special Functions of Physics and Chemistry. By Sneddon (University College of North Staffordshire). Interscience, 1956. 171 pages, $1.95.

Discusses the hypergeometric functions, the functions of Legendre, Bessel, Hermite and Laguerre, and the Dirac delta Function.

Mathematical Theory of Elasticity, Second Edition. By Sokolnikoff. (U.C.L.A.). McGraw-Hill, 1956. 476 pages, $9.75.

The Structure of Turbulent Shear Flow. By Townsend (Cambridge University). Cambridge University Press, New York, 1956. 320 pages. $7.50.

Waves: A Mathematical Account of the Common Types of Wave Motion. By Coulson (Oxford University). Interscience, 1955. 171 pages, $1.75.

Integration. By Gillespie (University of Glasgow). Interscience, 1955. 140 pages, $1.55.

First four chapters are devoted to an elementary account of integration. The book then takes up the Riemann integral, infinite integrals (particularly gamma and beta functions), and the Riemann double integral.

Plane Waves and Spherical Means. Applied to Partial Differential Equations. By John (New York University). Interscience, 1955. 180 pages, $5.50.

Theory of functions of a complex variable and advanced calculus are the only prerequisites for this volume.

Hydrodynamic Stability. By Lin. Cambridge University Press, New York, 1955. 160 pages, $4.75.

Physical Mathematics. By Page (National Bureau of Standards). Van Nostrand, 1955. 329 pages, $6.00.

Introduction to Relaxation Methods. By Shaw. Dover, 1955. 400 pages, $2.45.

Static and Dynamic Electron Optics. By Sturrock (Cambridge University). Cambridge University Press, New York, 1955. 240 pages, $6.50.

Linearized Theory of Steady High-Speed Flow. By Ward (College of Aeronautics, Cranfield). Cambridge University Press, New York, 1955. 248 pages, $6.00.

Introduction to Elliptic Functions with Applications. By Bowman (College of Technology, Manchester, England). Wiley, 1954. 115 pages, $2.75.

Discusses Jacobian elliptic functions with the emphasis directed to conformal mapping of polygons.

Thermodynamics. By Fermi. Dover, 1954. 160 pages, $1.75.

Thermodynamic systems, first and second laws of thermodynamics, entropy, thermodynamic potential, phase rule, reversible electric cell, *etc.*

Foundations of Potential Theory. By Kellogg. Dover, 1954. 384 pages, $1.98.

Linear Integral Equations. By Lovitt. Dover, 1954. 253 pages, $1.60.

Linear Equations in Applied Mechanics. By Purday (Harland and Wolff, Ltd.). Interscience, 1954. 254 pages, $4.00.

COMPLEX VARIABLES

Theory of Functions (two volumes). By Caratheodory (University of Munich). Chelsea, 1958 and 1954. 310 and 220 pages, $4.95 each or $9.90 the set.

Functions of Complex Variables. By Franklin (M.I.T.) Prentice-Hall, 1958. 246 pages, $7.25. To be reviewed.

Multivalent Functions. By Hayman (Imperial College of Science and Technology, London). Cambridge University Press, New York, 1958. 168 pages, $4.00. To be reviewed.

Elements of the Theory of Functions. By Knopp. Dover, 1955. 160 pages, $1.35.

Transform Calculus with an Introduction to Complex Variables. By Scott (University of Illinois). Harper, 1955. 330 pages, $7.50. Review: pp. 351–3, 1956.

REAL VARIABLES

Functions of Real Variables and Functions of a Complex Variable. By Osgood (Harvard University). Two volumes in one. Chelsea, 1958. 676 pages, $4.95. To be reviewed.

Foundations of Analysis. By Landau (Goettingen University). Chelsea, 1957. 148 pages, $3.50.

Reprint of 1951 edition.

The Theory of Functions of Real Variables, Second Edition (*International Series*). By Graves (University of Chicago). McGraw-Hill, 1956. 375 pages, $7.50.

Intermediate Analysis. By Olmsted (University of Minnesota). Appleton-Century-Crofts, 1956. 305 pages, $6.00. To be reviewed.

Contributions to the Founding of the Theory of Transfinite Numbers. By Cantor. Dover, 1955. 211 pages, $1.25.

VECTOR ANALYSIS

Vector Analysis. By Brand (University of Houston). Wiley, 1957. 282 pages, $6.00. Review: pp. 297–8, 1958.

Vector and Tensor Analysis. By Coburn (University of Michigan). Macmillan, 1955. 341 pages, $7.00.

Vector Analysis. By Newell (Office of Naval Research). McGraw-Hill, 1955. 216 pages, $5.50. Review: p. 505, 1956.

Vector and Tensor Analysis. By Hay (University of Michigan). Dover, 1954. 208 pages, $1.75.

Vector Methods. By Rutherford (University of St Andrews). Interscience, 1954. 143 pages, $1.75.

MODERN ABSTRACT ALGEBRA

The Theory of Groups. By Hall (Ohio State University). Macmillan, 1959. 416 pages, $8.75.

Selections from Modern Abstract Algebra. By Andree (University of Oklahoma). Holt, 1958. 212 pages, $6.50. Review: p. 158, 1959.

Matrix Calculus, Second Edition Revised. By Bodewig. Interscience, in preparation.

Introduction to Difference Equations. By Goldberg (Oberlin College). Wiley, 1958. 260 pages, $6.75. To be reviewed.

Finite-Dimensional Vector Spaces. By Halmos (University of Chicago). Van Nostrand, 1958. 208 pages, $5.00. To be reviewed.

Elementary Matrix Algebra. By Hohn (University of Illinois). Macmillan, 1958. 352 pages, $7.50. To be reviewed.

Elements of Modern Abstract Algebra. By Miller (New York University). Harper, 1958. 188 pages, $5.00. To be reviewed.

Introduction to Functional Analysis. By Taylor (University of California). Wiley, 1958. 423 pages, $12.50. To be reviewed.

Commutative Algebra, Volume I. By Zariski and Samuel (Harvard University and University of Clermont-Ferrand). Van Nostrand, 1958. 329 pages, $6.95. To be reviewed.

The Theory of Groups, Second Edition. By Zassenhaus (McGill University). Chelsea, 1958. 271 pages, $6.00.

Introduction to the Theory of Equations. By Conkwright (University of Iowa). Ginn, 1957. 222 pages, $4.50.

Linear Algebra for Undergraduates. By Murdoch (University of British Columbia). Wiley, 1957. 239 pages, $5.50. Review: pp. 300–301, 1958.

Vector Spaces and Matrices. By Thrall and Tornheim (University of Michigan and California Research Corporation). Wiley, 1957. 318 pages, $6.75. To be reviewed.

Determinants and Matrices. By Aitken (University of Edinburgh). Interscience, 1956. 151 pages, $1.65.

Fundamental Concepts of Higher Algebra. By Albert (University of Chicago). University of Chicago Press, 1956. 216 pages, $6.50. Review: p. 602, 1957.

Theory of Groups of Finite Order. By Burnside. Dover, 1955. 447 pages, $2.45.
A new printing of Burnside's 1911 text.

Introduction to the Theory of Groups of Finite Order. By Carmichael (University of Illinois).

Dover, 1955. 447 pages, $2.00.

A reprint of the earlier book.

The Theory of Groups (two volumes). By Kurosh (University of Moscow). Chelsea, 1955 and 1956. 271 and 308 pages, $4.95 each or $9.90 the set.

Introduction to Modern Algebra and Matrix Theory. By Schreier and Sperner (Hamburg University). Chelsea, 1955. 386 pages, $6.00.

Introduction to Modern Algebra and Matrix Theory. By Beaumont and Ball (University of Washington and Alabama Polytechnic Institute). Rinehart, 1954. 331 pages, $6.00. Review: pp. 499–500, 1955.

Theory of Equations. By MacDuffee (University of Wisconsin). Wiley, 1954. 120 pages, $3.75. Review: pp. 723–4, 1954.

Certain basic concepts from abstract algebra are woven into the usual work in *Theory of Equations.*

THEORY OF NUMBERS

Introduction to the Theory of Numbers. By Dickson. Dover, 1958. 183 pages, $1.65. Review: p. 650, 1958.

A reprint of the original Dickson volume.

Elementary Number Theory. By Landau (Goettingen University). Chelsea, 1958. 256 pages, $4.95. To be reviewed.

With exercises by Professors Bateman and Kohlbecker.

Introduction to Diophantine Approximation. By Cassels (Cambridge University). Cambridge University Press, New York, 1956. 170 pages, $4.00. Review: pp. 465–6, 1958.

Topics in Number Theory, Volumes I and II. By LeVeque (University of Michigan). Addison-Wesley, 1956. Vol. I, 198 pages, $6.50. Vol. II, 270 pages, $7.50. Review: pp. 445–7, 1957.

Volume I is excellent for the usual first course in Number Theory. Volume II brings together advanced topics in analytic number theory, many of which are not available elsewhere in English texts.

Irrational Numbers. By Niven (University of Oregon). Wiley, 1956. 164 pages, $3.00. Review: pp. 606–7, 1957.

One of the excellent Carus Monograph Series.

The Number-System. By Thurston (University of Bristol, England). Interscience, 1956. 144 pages, $3.00. Review: pp. 295–6, 1958.

The number system is derived formally from the Peano axioms, using Cauchy sequences.

The Theory of Numbers. By Jones (University of Colorado). Rinehart, 1955. 143 pages, $3.75. Review: pp. 52–3, 1956.

An unusually interesting book in elementary number theory with an emphasis on the "discovery method."

Elements of Number Theory. By Vinogradov. Dover, 1955. 227 pages, $1.60.

An Introduction to the Theory of Numbers. By Vinogradov (Akademiia Nauk, Moscow) translated by Popova (University of Aberdeen). Pergamon Press, 1955. 155 pages, $3.00.

Elementary Theory of Numbers (*International Series*). By Griffin (Brooklyn College). McGraw-Hill, 1954. 203 pages, $5.00. Review: p. 132, 1955.

The Method of Trigonometrical Sums in the Theory of Numbers. By Vinogradov (Akademiia Nauk, Moscow). Interscience, 1954. 190 pages, $5.50.

A translation of the later of Vinogradov's two monographs on the subject.

NUMERICAL ANALYSIS

The Calculus of Finite Differences, Fourth Edition. By Boole (Queen's College). Chelsea, 1958. 348 pages, $4.95. To be reviewed.

Linear Programming. By Gass. McGraw-Hill, 1958. 233 pages, $6.50. To be reviewed.

Numerical Analysis. By Kunz (Schlumberger Corp). McGraw-Hill, 1957. 575 pages, $8.00. Review: pp. 464–5, 1958.

Difference Methods for Initial-Value Problems. By Richtmyer (New York University). Interscience, 1958. 250 pages, $7.25. To be reviewed.

Designed for use with today's computers.

Introduction to Numerical Analysis (*International Series*). By Hildebrand (M.I.T.). McGraw-Hill, 1956. 511 pages, $8.50. Review: pp. 128–30, 1957.

Applied Analysis. By Lanczos (Dublin Institute for Advanced Studies). Prentice-Hall, 1956. 608 pages, $6.75. Review: pp. 447–8, 1957.

Methods in Numerical Analysis. By Nielsen (Naval Ordnance Plant, Indianapolis, Indiana). Macmillan, 1956. 382 pages, $6.90. Review: p. 594, 1956.

Numerical Analysis. By Kopal (University of Manchester, England). Wiley, 1955. 556 pages, $12.00. Review: pp. 127–8, 1957.

Numerical Solutions of Differential Equations. By Levy and Baggot (University of Illinois). Dover, 1955. 238 pages, $1.75.

Practical Analysis, Graphical and Numerical Methods. By Willers. Dover, 1955. 422 pages, $2.00.

An Introduction to the Calculus of Finite Differences. By Richardson (Bucknell University). Van Nostrand, 1954. 142 pages, $4.00. Review: pp. 132–3, 1955.

COMPUTERS

Programming Business Computers. By McCracken, Weiss and Lee (General Electric Company). Wiley, 1959. 504 pages, $8.00.

A general introduction to business computers.

Programming the IBM 650 Computer. By Andree (University of Oklahoma). Holt, 1958. 120 pages, $2.95. To be reviewed.

Discusses high-speed programming techniques (flow charting) in general, using the IBM 650 commands. A classroom text with problems.

Mathematics and Logic for Digital Devices. By Culbertson (California State Polytechnic College). Van Nostrand, 1958. 230 pages, $4.85. To be reviewed.

Handbook of Automation, Computation and Control, Volume I. By Grabbe, Ramo and

Wooldridge (Ramo-Wooldridge Corporation). Wiley, 1958. 1020 pages, $17.00. To be reviewed.

Provides specific coverage on aspects of mathematics as applied to control; a compilation of the mathematics of digital computers; the latest techniques and comparisons of different techniques involving computers; an all-in-one treatment of servo theory and operations research; all necessary material on information theory and transmission.

Elementary Mathematical Programming. By Metzger (General Motors Institute). Wiley, 1958. 246 pages, $5.95.

An introduction to mathematical programming with special emphasis on the interests of business and industrial people.

Your Fingers: Nature's Digital Computer. By Blake. William-Frederick Press, 1957. 24 pages, $2.00.

An Introduction to Automatic Computers: A Systems Approach for Business, Second Edition. By Chapin (Stanford Research Institute). Van Nostrand, 1957. 525 pages, $6.75.

Digital Computer Programming. By McCracken (General Electric Company). Wiley, 1957. 253 pages, $6.50. Review: pp. 132–3, 1958.

A general introduction to computers which contains the practical details necessary to work with specific machines. Discusses high speed computer programming in general using the operation codes of a mythical computer TYDAC.

Computing With Desk Calculators. By Varner (Flight Simulation Lab., Convair, San Diego). Rinehart, 1957. 108 pages, $2.00. Review: p. 692, 1957.

The Preparation of Programs for an Electronic Digital Computer, Second Edition. By Wilkes, Wheeler and Gill (Cambridge University, England). Addison-Wesley, 1957. 256 pages, $7.50. Review: pp. 719–20, 1958.

Arithmetic Operations in Digital Computers. By Richards (IBM Engineering Laboratory). Van Nostrand, 1955. 397 pages, $8.00.

TABLES

Six Figure Logarithms, Antilogarithms and Trigonometrical Functions. By Attwood (Trade School, Ford Motor Co. Ltd., Dagenham). Pergamon Press, 1958. 132 pages, $1.25.

Burington Tables. By Burington (Bureau of Ordnance, Navy Dept.). McGraw-Hill née Handbook Publishers.

Mathematical Tables and Formulae. By Camm (Cambridge University). Philosophical Library, 1958. 144 pages, $2.75. To be reviewed.

Handbook of Calculus, Difference and Differential Equations. By Cogan and Norman (Sarah Lawrence, Dartmouth). Prentice-Hall, 1958. 263 pages, $4.50. To be reviewed.

CRC Standard Mathematical Tables, Eleventh Edition revised. Chemical Rubber. 1959. 480 pages, $3.00: To be reviewed.

Tables of Integrals and Other Mathematical Data. By Dwight (M.I.T.). Macmillan, 1957. 288 pages, $3.00.

A Short Table of Integrals, Fourth Edition. By Peirce and Foster. (Polytechnic Institute of Brooklyn). Ginn, 1956. 196 pages, $2.85.

Tables of Functions with Formulae and Curves. By Jahnke and Emde. Dover, 1955. 382 pages, $2.00.

Bateman Tables Project. Erdelyi (Cal. Tech), Editor. The "Bateman Tables Project," (sponsored by California Institute Technology) provides in five convenient volumes the essence of the enormous compilation of functions and *transforms* materials gathered over a period of many years by the late Harry Bateman.

Tables of Integral Transforms, Volume I, McGraw-Hill, 1954. 391 pages, $7.50.

Tables of Integral Transforms, Volume II, McGraw-Hill, 1954. 451 pages, $8.00.

Higher Transcendental Functions, Volume III, McGraw-Hill, 1955. 312 pages, $6.50.

Four Place Tables of Transcendental Functions. By Flügge (Stanford University). Pergamon Press, 1954. 136 pages, $5.00.

Formulas and Theorems for the Functions of Mathematical Physics. By Magnus and Oberhettinger (New York University and University of Wisconsin). Chelsea, 1954. 182 pages, $3.90.

Reprint of 1949 edition.

The Biometrika Tables for Statisticians, Volume I. By Pearson and Hartley (University College, London). Cambridge University Press, New York, 1954. 250 pages, $5.00.

GEOMETRY

Plane Geometry for Colleges. By Adams (Santa Monica City College). Holt, 1958. 214 pages, $3.50. Review: p. 787, 1958.

Modern Geometry: An Integrated First Course. By Adler (New York University). McGraw-Hill, 1958. 215 pages, $6.00. To be reviewed.

Basic Geometry. By Birkhoff and Beatley (Harvard University). Chelsea, 1958. 294 pages, $3.95. To be reviewed.

A corrected edition of the 1940 classic.

College Plane Geometry. By Hemmerling (Bakersfield College) Wiley, 1958. 310 pages, $4.95. To be reviewed.

An Introduction to Euclidean Geometry. By Eaves and Robinson (University of Kentucky and Alabama Polytechnic Institute). Addison-Wesley, 1957. 327 pages, $4.25. To be reviewed.

Review of plane and solid high school geometry for a one-semester remedial course.

Solid Geometry. By Mandelbaum and Conte (Wayne State University and Ramo-Wooldridge Corporation). Ronald Press, 1957. 261 pages, $4.50.

College Geometry. By Miller (Ohio State University). Appleton-Century-Crofts, 1957. 201 pages, $4.50. To be reviewed.

Circles (Volume II in the International Series of Monographs in Pure and Applied Mathematics). By Pedoe (University College, Khartoum, Sudan). Pergamon Press, 1957. 78 pages, $3.75. To be reviewed.

The 13 Books of Euclid's Elements (three volumes). Edited by Heath. Dover, 1956. Vol. I, 448 pages; Vol. II, 448 pages; Vol. III, 560 pages; $2.00 per volume.

The Real Projective Plane. By Coxeter (University of Toronto). Cambridge University Press, New York, 1955. 244 pages, $5.50.

An introductory university textbook in projective geometry which includes a thorough treatment of conics and a rigorous presentation of the synthetic approach to coordinates.

Famous Problems of Elementary Geometry, and other Monographs. By Klein, *et al.* (Goettingen, *etc.*). Chelsea, 1955. 350 pages, $3.25.

Fundamental Concepts of Geometry. By Meserve (New Jersey State Teachers College, Upper Montclair). Addison-Wesley, 1955. 321 pages, $7.50. Review: pp. 673–4, 1956.

Geometry of Four Dimensions. By Manning. Dover, 1954. 348 pages, $1.95. A revision of the 1914 volume.

STATISTICS AND RELATED TOPICS

Elementary Decision Theory. By Chernoff and Moses (Stanford University). Wiley, 1959. 376 pages, $6.25.

A modern introduction to statistics based on decision theory.

Sampling Inspection Tables. By Dodge and Romig (Rutgers University and Paper-Mate Mfg. Co.). Wiley, 1959. 224 pages, $7.00.

Quality Control and Industrial Statistics. Revised Edition. By Duncan (Johns Hopkins University). Richard D. Irwin, 1959. 979 pages, $9.00 text, $10.80 trade.

Information Theory and Statistics. By Kullback (George Washington University). Wiley, 1959. 344 pages, $12.50.

A study of information theory from the viewpoint of a general probabilistic meeting.

An Introduction to Multivariate Statistical Analysis. By Anderson (Columbia University). Wiley, 1958. 374 pages, $12.50. To be reviewed.

Presents basic methods used in multivariate statistical analysis—the analysis of ev-eral variables.

Experimental Designs in Industry. By Chew (No. Carolina State College). Wiley, 1958. 268 pages, $6.00. To be reviewed.

A discussion of designs most applicable in industrial research.

An Introduction to Statistical Mechanics. By Chisholm and DeBorde (University College, Cardiff, and University of Glasgow). Pergamon Press, 1958. 400 pages, $6.00.

Introductory Statistics. By Quenouille. Pergamon Press. 247 pages, $6.00.

Manual of the more common statistical tests.

Planning of Experiments. By Cox (Birbeck College). Wiley, 1958. 308 pages, $6.25. To be reviewed.

An introductory treatment of particular interest to applied statisticians and experimental scientists.

Statistics—An Introduction. By Fraser (University of Toronto). Wiley, 1958. 44 pages, $7.25.

Modern Business Statistics. By Freund and Williams (Arizona State College and University of Tennessee). Prentice-Hall, 1958. 608 pages, $7.50. To be reviewed.

Experimental Design in Psychology and the Medical Sciences. By Maxwell (University of London). Wiley, 1958. 147 pages, $3.75. To be reviewed.

Deals with the relatively small number of basic, yet straightforward, experimental designs—randomized blocks, latin squares, and factorial designs—plus a few others neither so familiar nor so straightforward.

An Introduction to Combinatorial Analysis. By Riordan (Bell Telephone Lab.). Wiley, 1958. 244 pages, $8.50. To be reviewed.

Some Aspects of Multivariate Analysis. By Roy (University of No. Carolina). Wiley, 1958. 214 pages, $8.00. To be reviewed.

Sampling Opinions, An Analysis of Survey Procedure. By Stephan and McCarthy (Princeton University and Cornell University). Wiley, 1958. 451 pages, $12.00. To be reviewed.

Experimental Designs. By Cochran and Cox (Harvard University and University of North Carolina). Wiley, 1957. 617 pages, $8.50. Review: p. 756, 1957.

Nonparametric Methods in Statistics. By Fraser (University of Toronto). Wiley, 1957. 299 pages, $8.50. Review: p. 684–5, 1957.

An Introduction to Genetic Statistics. By Kempthorne (Iowa State College). Wiley, 1957. 545 pages, $12.75. Review: pp. 460–1, 1958.

Games and Decisions: Introduction and Critical Survey. By Luce and Raiffa (Harvard University). Wiley, 1957. 509 pages, $8.75. To be reviewed.

A survey of the central ideas and results of game theory and related decision making models. Not particularly mathematical.

The Essentials of Educational Statistics. By Cornell (Engelhardt and Leggett). Wiley, 1956. 375 pages, $5.75.

Statistical Analysis of Stationary Time Series. By Grenander and Rosenblatt (University of Stockholm and Indiana University). Wiley, 1956. 300 pages, $11.00. Review: pp. 218–9, 1958.

Government Statistics for Business Use. By Hauser and Leonard (University of Chicago). Wiley, 1956. 440 pages, $7.00.

Symposium on Monte Carlo Methods. By Meyer (University of Florida). Wiley, 1956. 382 pages, $7.50.

Papers by twenty-two leading workers in the field, written about their own research and applications.

Thermodynamics and Statistical Mechanics. By Wilson (Cambridge University). Cambridge University Press, New York, 1956. 500 pages, $9.50.

An Introduction to Stochastic Processes. By Bartlett (University of Manchester). Cambridge University Press, New York, 1955. 316 pages, $7.50. Review: p. 134–5, 1956.

Stochastic Models for Learning. By Bush and Mosteller (Columbia University and Harvard University). Wiley, 1955. 365 pages, $9.00.

The first systematic attempt to present a probabilistic analysis of data obtained in learning experiments through the use of stochastic processes.

Applied General Statistics, Second Edition. By Croxton and Cowden (Columbia Uni-

versity, University of North Carolina). Prentice-Hall, 1955. 843 pages, $6.70.

An elementary text with no college mathematics prerequisite.

Statistical Methods. By Mills (Columbia University). Holt, 1955. 864 pages, $7.95.

Statistical Analysis in Chemistry and the Chemical Industry. By Bennett and Franklin (General Electric Company and Leeds University). Wiley, 1954. 724 pages, $9.50.

Theory of Games and Statistical Decisions. By Blackwell and Girshick (University of California). Wiley, 1954. 355 pages, $7.75. Review: pp. 553–4, 1955.

Elements of Statistics. By Fryer (Kansas State College). Wiley, 1954. 262 pages, $4.75. Review: pp. 725–6, 1954.

Limit Distributions for Sums of Independent Random Variables. By Gnedenko and Kolmogorov (Moscow and Lwow Universities), translated from the Russian by Chung. Addison-Wesley, 1954. 264 pages, $12.00.

This volume provides the first appearance in English of much of its contents.

Introduction to Mathematical Statistics. By Hoel (University of California). Wiley, 1954. 331 pages, $5.00.

Mathematics of Statistics, Part I, Third Edition. By Kenney and Keeping (University of Wisconsin and University of Alberta). Van Nostrand, 1954. 335 pages, $5.25.

Business and Economic Statistics. By Spurr, Kellogg, and Smith (Stanford University, Deere and Company and The American University). Irwin, 1954. 590 pages, $6.95 text, and $8.35 trade.

Mathematical Foundations of Statistical Mechanics. By Khinchin. Dover. 1954, 179 pages, $1.35.

Decision Processes. By Thrall, Coombs and Davis (University of Michigan). Wiley, 1954. 332 pages, $5.00. Review: pp. 596–8, 1955.

Containing the material of 23 contributing scientists, this book offers an interdisciplinary approach to decision making under certainty.

The Foundations of Statistics. By Savage (University of Chicago). Wiley, 1954. 294 pages, $7.25.

Selected Papers on Noise and Stochastic Process. By Wax. Dover, 1954. 352 pages, $2.35.

Six basic papers for newcomers in the field of noise characteristics.

PROBABILITY

Probability: An Intermediate Textbook. By Bizley (Institute of Actuaries). Cambridge University Press, New York, 1957. 240 pages, $4.00.

An Introduction to Probability Theory and Its Applications, Volume I. By Feller (Princeton University). Wiley, 1957. 461 pages, $9.00. Review: p. 538, 1958.

Treats probability theory rigorously as a self-contained mathematical subject, and demonstrates how practical problems may be solved through application of this theory.

Elements of Probability Theory and Some of Its Applications. By Cramer (University of Stockholm). Wiley, 1955. 281 pages, $7.00. Review: pp. 132–4, 1956.

A compact discussion of the mathematical theory of probability theory with emphasis on random variables and probability distributions.

Probability Theory. By Loeve (University of California). Van Nostrand, 1955. 532 pages, $12.75.

A text for students having a knowledge of classical analysis.

TOPOLOGY AND ALGEBRAIC GEOMETRY

Convex Surfaces. By Busemann (University of Southern California). Interscience, 1958. 204 pages, $6.00.

Convexity. By Eggleston (Cambridge University). Cambridge University Press, New York, 1958. 160 pages, $4.00. To be reviewed.

Introduction to Algebraic Geometry. By Lang (Columbia University). Interscience, 1958. 272 pages, $7.25. To be reviewed.

Homology Theory on Algebraic Varieties (Volume 6 in the International Series of Monographs in Pure and Applied Mathematics). By Wallace (University of Toronto). Pergamon Press, 1958. 115 pages, $5.50. To be reviewed.

Topological Analysis. By Whyburn (University of Virginia). Princeton University Press, 1958. 124 pages, $4.00. To be reviewed.

Geometric Algebra. By Artin (Princeton University). Interscience, 1957. 224 pages, $7.00. Review: pp. 604–5, 1957.

This text discusses the foundations of affine geometry, and geometry of quadratic forms and the structure of the general linear group. It also contains a discussion of projective and symplectic and orthogonal groups.

Lie Groups. By Cohn. (University of Manchester). Cambridge University Press, 1957. 178 pages, $4.00. Review: p. 646, 1958.

Problems in Euclidean Space—Application of Convexity (Volume 5 in the International Series of Monographs in Pure and Applied Mathematics). By Eggleston (Cambridge University). Pergamon Press, 1957. 165 pages, $6.50. To be reviewed.

Algebraic Geometry and Topology. Fox, Spencer, Tucker (Princeton University). Princeton University Press, 1957. 410 pages, $7.50. To be reviewed.

Introduction to Riemann Surfaces. By Springer (University of Kansas). Addison-Wesley, 1957. 307 pages, $9.50. To be reviewed.

An Introduction to Algebraic Topology (Volume I in the International Series of Monographs in Pure and Applied Mathematics). By Wallace (University of Toronto). Pergamon Press, 1957. 198 pages, $6.50. Review: pp. 466–7, 1958.

The Theory of Lie Derivatives and Its Applications. By Yano (University of Tokyo). Interscience, 1957. 303 pages, $9.00. Review: pp. 294–5, 1958.

Topology. By Patterson (University of St. Andrews). Interscience, 1956. 136 pages, $1.75.

Topological space is approached gradually through Euclidean and metric spaces. The separation axioms, compactness, and connectedness, are then discussed, and also homotopy, with an elementary account of the homology theory of simplicial complexes, as a brief introduction to algebraic topology.

Elementary Topology. By Hall and Spencer (Harpur College and Williams College). Wiley, 1955. 303 pages, $7.00. Review: pp. 591–2, 1956.

General Topology. By Kelley (University of California). Van Nostrand, 1955. 298 pages, $8.75. Review: pp. 668–9, 1955.

Topological Transformation Groups. By Montgomery and Zippin (Princeton University and Queens College). Interscience, 1955. 294 pages, $6.50. Review: pp. 439–40, 1956.

LOGIC AND SET THEORY

Introduction to the Theory of Sets. By Breuer and translated by Fehr (Aachen, Germany; Columbia University Teachers College). Prentice-Hall Inc., 1958. 108 pages, $4.25. To be reviewed.

Introduction to Logic and Sets, Preliminary Edition. By Christian (University of British Columbia). Ginn, 1958. 70 pages, $0.90. To be reviewed.

A Modern Introduction to Logic. By Blyth (Hamilton College). Houghton Mifflin, 1957. 426 pages, $5.50.

Set Theory. By Hausdorff (Greifswald University). Chelsea, 1957. 352 pages, $6.00.

Introduction to Logic. By Suppes (Stanford University). Van Nostrand, 1957. 336 pages, $5.50. Review: pp. 131–2, 1958.

101 Puzzles in Thought and Logic. By Wylie. Dover, 1957. 128 pages, $1.00. Review: p. 648, 1958.

Introduction to Mathematical Logic. By Church (Princeton University). Princeton University Press, 1956. 400 pages, $7.50. Review: pp. 126–7, 1957.

Theory of Sets. By Kamke. Dover, 1956. 152 pages, $1.35.

The Elements of Mathematical Logic. By Rosenbloom (University of Minnesota). Dover, 1955. 214 pages, $1.45.

An Introduction to Symbolic Logic. By Langer. Dover, 1954. 368 pages. $1.75.

TEACHING

Arithmetic for Colleges. By Larsen (Albion College). Macmillan, 1958. 286 pages, $5.50. To be reviewed.

History of Mathematics, Volume I. By Hoffmann (Oklahoma State University). Philosophical Library, 1957. 136 pages, $4.75. Review: p. 692, 1957.

Understanding Arithmetic. By Swain (State University of New York, New Paltz). Rinehart, 1957. 416 pages, $4.75. Review: pp. 296–7, 1958.

An Introduction to Mathematics, Revised Edition. By Boyer (College Preparatory, Department of Public Instruction, Harrisburg, Pa.). Henry Holt, 1956. 528 pages. $5.75.

Teaching of Mathematics. Cambridge University Press, 1956. 275 pages, $3.00.

Arithmetic: Its Structure and Concepts. By Mueller (Towson State, Md., Teachers College). Prentice-Hall, 1956. 279 pages, $6.25.

Arithmetic for Teacher-Training Courses. By Mills and Taylor (Sioux Falls College and Eastern Illinois State College). Holt, 1955. 438 pages, $4.85.

Elementary Mathematics from an Advanced Standpoint: Volume I—Arithmetic, Algebra, Analysis. By Klein. Dover, 1954. 214 pages, $1.75.

Elementary Mathematics from an Advanced Standpoint: Volume II—Geometry. By Klein, Dover, 1955. 214 pages, $1.75.

A reissue of Klein's older book.

Mathematics for the Secondary School. By Reeve (Columbia University), Holt, 1954. 535 pages, $6.95. Review: p. 739, 1955.

The Teaching of Arithmetic, Second Edition. By Spitzer (University of Iowa). Houghton Mifflin, 1954. 416 pages, $4.25.

UNCLASSIFIED

Mathematics in Fun and in Earnest. By Court (University of Oklahoma). Dial Press, 1958. 250 pages, $4.75. To be reviewed.

A delightful "popular book" by one of today's great synthetic geometers.

How to Study, How to Solve (Special Student Edition), Second Edition. By Dadourian (Trinity College). Addison-Wesley, 1958. 48 pages, $0.50.

An Introduction to the Foundations and Fundamental Concepts of Mathematics. By Eves and Newsom (University of Maine and New York University). Rinehart, 1958. 363 pages, $6.75. Review: p. 720, 1958.

Introduction to Mathematical Economics. By Bushaw and Clower (State College of Washington and Northwestern University). Richard D. Irwin, 1957. 357 pages, $7.00 text, $8.40 trade.

The Tree of Mathematics. Edited by James (Editor, Mathematics Magazine). Digest Press, 1957. 402 pages, $5.50. To be reviewed.

A series of articles from the *Mathematics Magazine.*

An Introduction to the Mechanics of Stellar Systems. By Kurth, translated by Kahn (University of Manchester). Pergamon Press, 1957. 175 pages, $9.00.

The Enjoyment of Mathematics. By Rademacher and Toeplitz (University of Pennsylvania). Princeton University Press, 1957. 240 pages, $4.50. Review: p. 603, 1957.

Mathematical Vocabulary (German-English). By Macintyre and Witte (Aberdeen University). Interscience, 1956. 106 pages, $1.75.

Through the Mathescope. By Ogilvy (Hamilton College). Oxford University Press, 1956. 162 pages, $4.00. Review: p. 592–3, 1956.

Prelude to Mathematics. By Sawyer (St. John's College, Cambridge). Penguin Books 1955. 224 pages, $0.85.

Monographs on Topics of Modern Mathematics. By Young. Dover, 1955. 416 pages, $2.00.

Relativity for the Layman. By Coleman (American International College, Springfield, Mass). William-Frederick Press, 1954. 224 pages, $2.75.

Mathematics and Plausible Reasoning: Volume I—Induction and Analogy in Mathematics. By Polya (Stanford University). Princeton University Press, 1954. 300 pages, $5.50 ($9.00 per set). Review: pp. 456–7, 1958.

Mathematics and Plausible Reasoning: Volume II—Patterns of Plausible Inference. By Polya (Stanford University). Princeton University Press, 1954. 200 pages, $4.50 ($9.00 per set). Review: pp. 456–7, 1958.

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

SUMMER SESSIONS

The following institutions announce advanced courses in mathematics for the summer of 1959.

Columbia University, July 6 to August 14: Mr. Thorpe, introduction to higher algebra; Mr. Glimm, differential equations; Mr. Clifton, probability; Professor Dorothy M. Stone, fundamental concepts of mathematics; Professor Taylor, general topology; Dr. Mendelson, theory of functions of a real variable; Professor A. H. Stone, theory of functions, third part; Dr. Taft, higher algebra, second part.

Columbia University, Teachers College, July 6 to August 14: Miss Allegri, teaching of geometry and teaching of junior high school mathematics; Dr. Rosskopf, teaching modern mathematics in the secondary school and foundations of geometry; Dr. Greitzer, applications of mathematics and professionalized subject matter in advanced secondary school mathematics.

Duke University, June 12 to July 17: Professor Roberts, advanced calculus; Professor Gergen, mathematical statistics.

Kent State University, June 22 to July 25; Professor Jenkins, theory of equations; Professor Kaiser, probability; Professor Stapleford, advanced methods of teaching high school mathematics. July 27 to August 29: Professor Iwanchuk, selected topics for classroom teachers, functions of a real variable; Professor Olson, differential geometry.

New York University, first term: Dr. Bazer, Dr. Schechter, advanced calculus (two sessions); Professor Isaacson, elementary numerical methods; Professor Shapiro, special functions. Second term: Dr. Kay, Mr. Ungar, advanced calculus (two sessions); Professor Karp, Laplace transform and Heaviside calculus.

Northwestern University, six week session June 23 to August 1, eight week session June 23 to August 15: advanced calculus, engineering mathematics 1-T; numerical methods in mathematics; introduction to the theory of numbers; the history of mathematics II; advanced geometry for teachers; foundations of calculus for teachers; complex variables for applications; topics in modern mathematics for teachers.

Syracuse University, June 29 to August 7: Professor Cole, analysis and applications; Professor Coté, statistics and probability; Professor Gilbert, modern algebra; Professor Hemmingsen, projective geometry; Staff, number theory, programming digital computers. Syracuse University will also offer two demonstration classes in new high school curricula: an eighth-grade class in the Beberman-Illinois-UICSM "First Course," and an eleventh-twelfth grade class in statistics, following the text prepared by the CEEB Commission on Mathematics. Anyone interested is invited to visit these courses at any time between June 30 and August 7.

University of Chicago, June 22 to August 29: The program stresses homological algebra. Advanced courses: Professor Hochschild, topics in Lie algebras; Professor MacLane, foundations of homological algebra; Professor Baily, analytic number theory; Professor Calderon, potential theory; Professor Halmos, ergodic theory; Professor Spanier, characteristic classes. Standard graduate courses: Professor Halmos, set theory and metric spaces; Professor Kaplansky, algebra IV; Dr. Swan, algebraic topology II.

University of Florida, June 16 to August 8: Professor Blake, Professor Meacham, introduction to mathematical thought; Professor South, mathematical statistics, advanced

topics in calculus; Professor Butson, Professor Gaddum, Professor Meacham, advanced mathematics for engineers and physicists; Professor Moore, introduction to topology; Professor Pirenian, vector analysis, theory of groups of finite order; Professor Smith, tensor analysis; Professor Morse, synthetic projective geometry, history of elementary mathematics; Professor Phipps, foundation of geometry; Professor Cowan, Fourier series; Professor Sobczyk, special topics in mathematics.

University of Maryland, June 22 to July 31: Professor Horváth, advanced calculus, higher geometry; Professor Rosen, vector analysis.

University of Michigan, June 22 to August 15: Professor Addison, introduction to the foundations of mathematics; Dr. Albrecht, differential equations, advanced mathematics for engineers; Professor Bartels, methods in high-speed computation (computer algorithms); Professor Carver, calculus of finite differences; Professor Clarke, mathematical theory of probability, theory of statistics I; Professor Coburn, vector analysis; Professor Craig, statistical analysis II; Professor Griffin, advanced calculus; Dr. Halpern, introduction to combinatorial topology; Professor Hay, intermediate course in differential equations; Dr. Hicks, differential geometry; Professor Jones, the teaching of elementary collegiate mathematics; Dr. Karrer, advanced mathematics for engineers, Fourier series and applications; Dr. Kincaid, statistical analysis I, theory of statistics II; Professor Kruskal, differential equations, introduction to matrices; Professor Leisenring, synthetic projective geometry; Professor Livingstone, theory of equations and determinants; Dr. Low, modern operational mathematics; Professor Lyndon, introduction to functions of a complex variable with applications, algebra; Professor Mayerson, differential equations; Professor McLaughlin, modern operational mathematics, topics in modern mathematics for teachers; Professor Reade, introduction to functions of a complex variable with applications, real analysis; Professor Ullman, differential equations, Fourier series and applications.

University of Minnesota, College of Science, Literature, and the Arts, June 15 to July 18: Professor Gil de Lamadrid, differential equations, theory of numbers; Professor Orey, critical reasoning in mathematical analysis. July 20 to August 22: Professor Engeler, advanced analytic geometry, advanced algebraic theory; Professor Storvick, calculus of variations.

University of Minnesota, Institute of Technology, June 15 to July 18: Professor Wilcox, vector analysis; Professor Munro, introduction to programming modern digital calculators, elementary numerical analysis in engineering; Professor Polansky, intermediate calculus. July 20 to August 21: Professor Wilcox, advanced calculus.

University of Nebraska: Professor Leavitt, theory of equations; Professor Harris, theory of games; Professor Miller, topics in algebra; Professor Guy, topics in analysis.

University of North Carolina, June 4 to July 14: Professor Cameron, fundamental concepts of mathematics with special reference to algebra; Professor Davis (University of Virginia), probability; Professor Garner, history of mathematics; Professor Hill, elementary mathematical statistics; Professor Linker, differential equations; Professor Mac Nerney, introduction to Hilbert spaces; Professor Pettis, topics in analysis; Mr. Wells, advanced calculus I, summability. July 15 to August 22: Professor Brauer, elementary theory of numbers I; Professor Hamstrom (Goucher College), elementary algebraic topology; Professor Hoyle, advanced calculus II; Professor Lasley, analytic geometry from a higher standpoint; Professor Mackie, theory of equations; Professor Mac Nerney, introduction to continued fractions; Professor Mann, introduction into numerical analysis.

University of Oklahoma, June 9 to August 6: Professor Bernhart, college geometry, vector analysis; Professor Pan, elementary differential equations; Professor LaFon, partial differential equations; Professor Brixey, theory of groups; Professor Giever, advanced partial differential equations.

University of Pittsburgh, June 8 to July 17, July 20 to August 28: Professors Christiano and Knipp, differential equations; Professor Benedicty, advanced calculus; Professor Taylor, functions of a complex variable; Professor Bryson, partial differential equations and Fourier series; Professor Bompiani, projective differential geometry; Professor Laush, infinite series; Professor Levine, topology. June 23 to August 15: Professor Elyash, differential equations; Professor Kovacs, mathematical theory of statistics; Professor Teats, history of mathematics; Professor Myers, recreational mathematics for teachers, theory of equations; Professor Leger, matrix theory. June 22 to August 14 (evenings): Professors Leger and Levine, differential equations; Professors Blumberg and Laird, mathematics of modern engineering; Professor Cooperman, advanced calculus; Professor Bryson, Laplace transform theory and applications.

University of Virginia, June 29 to August 22: Professor Ball, foundations of geometry; Professor Paige, foundations of algebra; Professor Malbon, differential equations and applied mathematics; Professor Paige, introductory analysis; Visiting Professor, advanced analysis.

University of Washington; Professor Walter, linear algebra; Professor Avann, introduction to modern algebra; Professor Brownell, topics in applied analysis; Professor Michael, advanced analytic and non-Euclidean geometry; Professor Pierce, foundations of mathematics; Professor Allendoerfer, special topics in modern mathematics for teachers.

University of Wisconsin, June 22 to August 15: Mr. Albright, theory and operation of computing machines; Professor Bicknell, applied differential equations; Visiting Lecturer Goblirsch, survey of the foundation of algebra; Visiting Lecturer Immel, applied mathematical analysis, introduction to mathematical probability; Dean Ingraham, topics in classical algebra; Dr. Losey, advanced analytic geometry; Visiting Lecturer Montgomery, advanced topics in algebraic topology; Dr. Posner, introduction to complex variables; Visiting Lecturer Sanderson, advanced calculus, elementary plane topology; Visiting Lecturer Schenkman, matrices and their applications, advanced topics in algebra.

West Virginia University, June 8 to July 17: Miss Hawkins, differential equations; Mrs. Easton, theory of equations; Professor Cunningham, group theory; Professor Posey, calculus of variations. July 20 to August 26: Mr. Gould, number theory; Professor Cochran, special topics; Professor Vest, advanced differential equations; Professor Peters, linear algebra.

OPERATIONS RESEARCH SOCIETY OF AMERICA

The Seventh Annual Meeting of the Operations Research Society of America will be held at the Shoreham Hotel, Washington, D. C., May 14–15, 1959.

PERSONAL ITEMS

Miss Mabel Williams of Tyler Junior College represented the Association at the inauguration of President R. W. Steen of Stephen F. Austin State College, Nacogdoches, Texas, on February 7, 1959.

University of North Carolina: Professor N. H. McCoy, Smith College, will be a Visiting Professor for the fall semester of 1959–1960; Professor T. H. Hildebrandt, University of Michigan, will be a Visiting Professor for the spring semester of 1959–1960.

Mathematics Research Center, U. S. Army, University of Wisconsin: Professor J. G. van der Corput, Mathematics Center, Amsterdam, Netherlands, Dr. H. M. Lieberstein, Ramo-Wooldridge Corporation, and Dr. S. M. Shah of India, have accepted appointments; Dr. Eberhard Hopf, Indiana University, has accepted a visiting appointment.

Professor Volodymyr Bohun-Chudyniv, Atlanta University, has been appointed Professor at Morgan State College.

Mr. K. C. Bullock, Oklahoma State University, has been appointed Instructor at

Murray State College.

Mr. W. H. Burgin, Jr., Princeton University, has been appointed Instructor at the Mercersburg Academy, Mercersburg, Pennsylvania.

Mr. R. L. Causey, Space Technology Laboratories, Los Angeles, California, has accepted a position as Senior Scientist with Lockheed Missiles Systems Division, Palo Alto, California.

Professor M. L. Curtis, University of Georgia, has received a Senior Postdoctoral Fellowship and will spend the year 1959–1960 in England.

Mr. C. M. Fast, General Electric Company, Syracuse, New York, has accepted a position as Mathematician with the Kirk Engineering Company, Philadelphia, Pennsylvania.

Mr. S. I. Gass, International Business Machines Corporation, has been appointed Chief, Operations Research Branch, Corporation for Economic and Industrial Research, Arlington, Virginia.

Dr. J. H. Griesmer, International Business Machines Corporation, Ossining, New York, has been appointed Staff Mathematician at the I.B.M. Research Center, Yorktown Heights, New York.

Professor G. B. Huff, University of Georgia, has resigned as Head of the Department of Mathematics to become Dean of the Graduate School.

Mr. M. R. Luttrell, Lincoln Laboratories, Massachusetts Institute of Technology, Lexington, Massachusetts, has been appointed Field Engineer, Sage Air Defense System of the Western Electric Company, Inc., New York.

Assistant Professor J. H. Manheim, Cooper Union, New York, has been appointed Assistant Professor at Montclair State College.

Assistant Professor Rev. P. M. Mino, College of Steubenville, has been appointed Assistant Professor at St. Francis College.

Assistant Professor V. J. Mizell, University of Tennessee, has been appointed Assistant Professor at Carnegie Institute of Technology.

Mr. P. M. Moskowitz, Metropolitan Life Insurance Company, New York City, is now a Senior Systems Analyst with the Radio Corporation of America, Princeton, New Jersey.

Mrs. Doretta M. Pasekoff, Ursinus College, has accepted a position as Mathematician in Univac Methods Department, Remington Rand Corporation, Pittsburgh, Pennsylvania.

Mr. B. L. Schwartz, Battelle Memorial Institute, has accepted a position as Mathematician with Technical Operations, Inc., Monterey, California.

Mr. D. J. Sine, West Virginia University, has accepted a position as Mathematician with Aero-Jet-General, Frederick, Maryland.

Sister Mary Vera, B.V.M., Mundelein College, has been appointed Chairman of the Mathematics Department at Clarke College.

Assistant Professor R. G. Stoneham, Santa Barbara College, has been appointed Teacher of Mathematics at The Walden School, New York.

Mr. D. V. Sward, Montana State University, has accepted the position of Associate Engineer with the Boeing Airplane Company, Seattle, Washington.

Assistant Professor Peter Yff, Fresno State College, has been appointed Associate Professor at the American University of Beirut, Beirut, Lebanon.

Professor Emeritus J. N. Michie, Texas Technological College, died on November 24, 1958. He was a member of the Association for 33 years.

Professor G. H. Pavlakos, Elmhurst College, died on November 10, 1958. He was a member of the Association since 1956.

Professor Emeritus B. F. Yanney, College of Wooster, died on August 10, 1958. He was almost 99 years old, and was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

THE FORTY-SECOND ANNUAL MEETING OF THE ASSOCIATION

The forty-second annual meeting of the Mathematical Association of America was held at the University of Pennsylvania, Philadelphia, Pennsylvania on Thursday and Friday, January 22 and 23, 1959, in conjunction with meetings of the American Mathematical Society, the Association for Symbolic Logic, and the Delaware Valley Section of the Society for Industrial and Applied Mathematics. There were registered 996 persons, including 621 members of the Association.

Sessions of the Association were held on Thursday morning and on Friday morning and afternoon in the Auditorium of the University Museum of the University of Pennsylvania. President G. B. Price presided at the sessions on Thursday and Friday mornings and at the Annual Business Meeting. Vice-President B. W. Jones presided at the Friday afternoon session. The Program Committee for the meeting consisted of J. G. Kemeny, Chairman; E. G. Begle, and M. A. Shader.

FIRST SESSION OF THE ASSOCIATION

"Professional Opportunities in Mathematics." Moderator: Dr. M. A. Shader, IBM Corporation; Panel Members: Professor Wallace Givens, Wayne State University, Dr. H. H. Goldstine, IBM Corporation, Dr. Arthur Grad, National Science Foundation, Dr. J. P. Nash, Lockheed Aircraft Corporation.

SECOND SESSION OF THE ASSOCIATION

"The Training of Secondary School Mathematics Teachers." Moderator: Professor J. G. Kemeny, Dartmouth College; Panel Members: Professor C. B. Allendoerfer, University of Washington, Professor H. F. Fehr, Teachers College, Columbia University, Professor E. P. Northrop, University of Chicago, Professor H. E. Vaughan, University of Illinois.

THIRD SESSION OF THE ASSOCIATION

Annual Business Meeting of the Association.

"High School Mathematics Courses." (Co-sponsored by the School Mathematics Study Group.) Moderator: Professor E. G. Begle, Yale University and School Mathematics Study Group; Panel Members: Dr. J. R. Mayor, American Association for the Advancement of Science, Professor E. E. Moise, University of Michigan, Dr. H. O. Pollak, Bell Telephone Laboratories, Professor G. B. Price, University of Kansas, Mr. R. E. K. Rourke, Kent School and Commission on Mathematics, Professor D. E. Richmond, Williams College.

MEETING OF THE BOARD OF GOVERNORS

The Board of Governors of the Association met on Thursday afternoon in the Jefferson Room of the Benjamin Franklin Hotel in Philadelphia with twenty-three members present. Among the items of business transacted were the following:

The Board approved the appointment by President Price of the following Nominating Committee for 1958: Mina Rees, Chairman, E. G. Begle, and A. E. Taylor.

The Board elected Professor Harley Flanders of the University of California as Second Vice-President for the two-year term 1959–1960.

Since the Board was not prepared to elect a Secretary, Professor H. M. Gehman was instructed to continue to act as Secretary of the Association.

The Board approved the following schedule of future meetings: University of Utah, August 31–September 3, 1959; Hotel Conrad Hilton, Chicago, January 28–30, 1960; Michigan State University, August 29–September 1, 1960; Washington, D. C., or New York City, January 1961; Oklahoma State University, August 28–31, 1961; Kansas City, Missouri, January 1962.

The Board voted to invite Professor William Feller of Princeton University to deliver the eighth Earle Raymond Hedrick Lectures at the 1959 Summer Meeting.

The Board authorized the publication as Carus Monograph No. 12 of "Statistical Independence in Probability, Analysis, and Number Theory," by Mark Kac.

The Board voted to accept with an expression of gratitude a grant from the Carnegie Corporation of New York of $75,000 over a three year period for the support of a Washington Office of the Association.

ANNUAL BUSINESS MEETING OF THE ASSOCIATION

The annual business meeting of the Association was held on Friday, January 23, 1959 in the Auditorium of the University Museum of the University of Pennsylvania, Philadelphia, Pennsylvania. President G. B. Price presided.

The ten amendments to the By-Laws which were printed in the November 1958 issue of the MONTHLY were unanimously adopted.

The balloting for officers in which 1046 votes were cast resulted in the election of Professor C. B. Allendoerfer of the University of Washington as President for the two-year term 1959–1960, and of Professors R. V. Churchill of the University of Michigan and Morris Kline of New York University as governors for the three-year term 1959–1961.

In spite of a more stringent policy on dropping from membership for non-payment of dues, membership in the Association was 7754 on January 16, 1959.

Reports were made by Dr. J. R. Mayor for the Committee on Secondary School Lecturers and by Professor L. W. Cohen for the Committee on Production of Films.

MEETINGS OF OTHER ORGANIZATIONS

The American Mathematical Society held sessions from Tuesday, January 20, through Thursday. The annual Gibbs lecture was delivered by Professor J. M. Burgers of the University of Maryland. Invited addresses were given by Professors G. D. Mostow and Felix Browder.

The Association for Symbolic Logic met on Thursday and the Delaware Valley Section of SIAM met on Wednesday evening for a lecture by Dr. John Mauchly and a social get-together.

ARRANGEMENTS, ENTERTAINMENT, AND RECREATION

The Committee on Arrangements for the meeting consisted of: Emil Grosswald, Chairman, P. A. Caris, J. H. Curtiss, Robert Ellis, H. M. Gehman, W. H. Gottschalk, R. D. Schafer, G. E. Schweigert, C. T. Yang.

Registration headquarters were in the foyer of Houston Hall of the University of Pennsylvania. The book exhibit and the employment register were located on the second floor of Houston Hall. Sleeping accommodations were available in the Benjamin Franklin Hotel and in other hotels. Meals were available at the hotels and at several University cafeterias.

An official reception by the University of Pennsylvania was held in the Rotunda of the University Museum on Wednesday afternoon.

A banquet was held on Thursday evening at the Benjamin Franklin Hotel. Professor Bernard Epstein served as Chairman. Professor J. S. Frame presented a resolution on behalf of the mathematical organizations expressing thanks to the administration of the

University of Pennsylvania, to the members of the Mathematics Department, and particularly to the members of the local committee on arrangements for their successful efforts on our behalf.

HARRY M. GEHMAN, *Secretary Treasurer*

NOVEMBER MEETING OF THE PHILADELPHIA SECTION

The Philadelphia Section of the Mathematical Association of America met at Lehigh University on Saturday, November 29, 1958, with Professor Charles Saalfrank presiding in lieu of the Section Chairman, Dr. I. E. Block. There were fifty-four present at this meeting, of whom thirty-nine are members of the Association.

For the year 1958–59 Professor Marguerite Lehr, Bryn Mawr College, was elected Chairman of the Section, Professor F. L. Dennis, Ursinus College, was elected Secretary-Treasurer, and Professor Bernard Epstein, University of Pennsylvania, was elected to the Executive Committee.

The following resolution was adopted by the Section:

Be it resolved that the Philadelphia Section of the Mathematical Association of America direct and empower the Executive Committee to take the necessary steps to institute a Mathematics Newsletter directed primarily toward the secondary schools of the area included in the Section or to take other action directed toward this goal if such is deemed desirable.

The entire meeting was devoted to the topic "Desirable Mathematical Training for the Mathematician Who Plans to Work in Industry." This was discussed in the morning program in three papers, as follows:

1. Dr. R. F. Drenick, Bell Telephone Laboratories. The mathematician's primary problem is communication, especially with nonmathematical colleagues. The gap between pure and applied mathematicians must be closed. Synthesis is the mathematician's primary job.

2. Dr. H. D. Mills, Princeton University. The mathematician must have a basic empathy for other people's problems, solid mathematical scholarship, inventive capacity, and self-esteem as a mathematician. Students should receive a concept of "wholeness" of mathematics, a survey based on the "storehouse of mathematics" so he will know what is in the literature, and case histories of mathematical discovery.

3. Dr. H. R. J. Grosch, International Business Machines Corporation. The influence of mathematicians in industry will diminish unless the curriculum is broadened; the present curriculum was based on training necessary for engineers. Work in information theory, logic, combinatorial analysis, probability and statistics is needed.

The afternoon program consisted of a panel discussion. Members of the panel were: Dean Mina Rees, Hunter College, Moderator; Professor Everett Pitcher, Lehigh University, and the morning speakers.

G. C. WEBBER, *Secretary*

THE DECEMBER MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The Fall Meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held on December 6, 1958, at George Washington University, Washington, D. C. Professor Joseph Milkman of the U. S. Naval Academy and Professor Herta T. Freitag of Hollins College presided. Mr. W. H. Norris, Section Chairman of the Mathematics Contest, announced the plans for the 1959 Contest.

The following papers were presented at the meeting:

1. *A mathematical model for transfer of training*, by Professor J. M. Long, College of William and Mary in Norfolk.

In this paper the author describes a mathematical model or analogy for the study of the transfer of training problem. The idea is to get a model which when analyzed will give results similar to that of a laboratory experiment. The "coefficient of transfer" is described and proposed as an appropriate quantitative measure of transfer. A simple geometrical model is the first model considered. This is expanded into an n-dimensional system with weighted parameters as the final model proposed.

2. *Algebraic compilers for scientific computers*, by Mr. R. C. Smith, Applied Science Representative, International Business Machines Corporation, introduced by the Secretary.

A progress report on the adaptation of computer language to mathematical language. A program in FORTRANSIT and the Soap Interpretive System for the solution of a mathematical problem on a computer.

3. *On the order of contact of two curves*, by Professor S. B. Jackson, University of Maryland.

Let two tangent curves have their points in 1-1 correspondence in the neighborhood of the contact point. The order of contact is defined by considering the distance d between corresponding points for a correspondence in which d is an infinitesimal of maximum possible order. A discussion is given of conditions under which a given correspondence is known to be optimal, and this result is applied to the case when corresponding points have equal arc lengths. It is shown that in general this is an optimal correspondence but an example is given to show this is not always the case.

4. *On the summability of a certain eigenfunction series*, by Professor Luna Mishoe, Morgan State College.

In this paper it is shown that if $u_n(x)$ are eigenfunctions or nonzero solutions of the equation $u''+q(x)+\lambda[P(x)u-u']=0$ such that $u(0)=u(1)=0$ and $P^2(x)+q(x)=0$ then the series $\sum_{-\infty}^{\infty} a_n u_n(x)$ (where a_n are the corresponding eigen-coefficients determined in the usual manner) behaves exactly as the Fourier Series of $f(x)$, with respect to summability.

5. *Practical adaptations of Ferrari's general quartic solution*, by Mr. C. R. White, Special Problems Section, Artillery Weapon Systems Branch, Aberdeen Proving Ground, Maryland.

The fourth degree equation is resolved into two quadratic factors by the method of Ferrari, in which the coefficients of these factors are themselves the real roots of two other quadratic equations and are also functions of K, a Ferrari real root of Euler's reducing cubic. The author's contribution, in addition to the discovery of the two associated quadratic equations is to provide direct methods for evaluating K. The formulas which result are then transformed for computing purposes on machines designed to extract square roots directly such as a Friden calculator, a digital computer, or White's two-dimensional slide rule. Furthermore, the formulas provide the basis for an improved technique of completely solving the general nth degree equation with real coefficients.

6. *Weak limit characterization of distributions*, by Professor T. P. Liverman, George Washington University.

Let K be the set of piecewise continuous functions on E_1 to E_1. A test function $\phi(x)$ is a function on E_1 to E_1 of class C^∞ which vanishes identically outside a finite interval. The A. L. Schwartz distribution T is an equivalence class of sequences $\{t_n(x)\}$ of K-functions which are weakly convergent, i.e., such that $I(\phi)=\lim_{n\to\infty}\int_{-\infty}^{\infty} t_n\phi(x)dx$ exists for every test function. With this approach, initiated by Mikusinski, G. Temple, and Lighthill, distributions, the natural tool in much applied

mathematics, should be accessible to undergraduates. This paper offers a simplification of the previously known theory; an elementary, though rigorous, justification of a commutative Heaviside calculus; and a proof, in elementary terms, of the fundamental theorem of distribution theory, *i.e.*, locally every distribution is a derivative of finite order of a continuous function.

7. *A new characterization of group*, by Professor Howard Campaigne, American University.

New investigations into semigroups conducted by G. B. Preston, V. V. Vagner, A. H. Clifford and others have led to the concept of "inverse" semigroups, in which for each element a there is a unique solution x to the simultaneous statements 1) $axa=a$ and $xax=x$. A solution to 1) implies a simultaneous solution of both, which suggests investigating semigroups which have unique solutions to 1). These are groups.

8. *Tetrahedra equivalent to cubes by dissection*, by Mr. Michael Goldberg, Bureau of Ordnance, U. S. Navy.

The list of special tetrahedra which can be dissected by plane cuts into a finite number of pieces to form cubes, published by J. J. M. Hill in 1896 and augmented by four new tetrahedra by J. P. Sydler in 1956, is now augmented by two new tetrahedra. Mr. Goldberg exhibited models of all the known dissectible tetrahedra and showed how the recent ones were derived. A tabulation of the lengths of the edges and dihedral angles was presented.

9. *Some published USSR research in mathematical economics*, by Dr. W. H. Marlow, Senior Staff Scientist, George Washington University. (By invitation.)

An exposition is given of some problems attacked by L. V. Kantorovich and various collaborators and by G. Sh. Rubinshtein in areas of organizing and planning production. These involve programming problems for which "Lagrange Multiplier" or "razreshaiushchiimnozhitel (resolving multiplier)" techniques have been devised by Kantorovich over the past twenty years. (Cf. *Doklady Akademii Nauk SSSR*, vol. 113 (1957), pp. 987–990; vol. 115 (1957, pp. 414–444; ibid., pp. 1058–1061; and errata: vol. 118 (1958), p. 1054.)

D. B. Lloyd, *Secretary*

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CALENDAR OF FUTURE MEETINGS

Fortieth Summer Meeting, University of Utah, Salt Lake City, Utah, August 31–September 3, 1959.

Forty-third Annual Meeting, Conrad Hilton Hotel, Chicago, Illinois, January 28–30, 1960.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, University of Pittsburgh, May 2, 1959.
ILLINOIS, Millikin University, Decatur, May 8–9, 1959.
INDIANA, Valparaiso University, May 2, 1959.
IOWA, Iowa Wesleyan University, Mount Pleasant, April 17, 1959.
KANSAS, Marymount College, Salina, April 11, 1959.
KENTUCKY, Centre College of Kentucky, Danville, April 25, 1959.
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Goucher College, Towson, Maryland, May 2, 1959.
METROPOLITAN NEW YORK, Polytechnic Institute of Brooklyn, April 18, 1959.
MICHIGAN
MINNESOTA, University of Minnesota, Minneapolis, April 25, 1959.
MISSOURI, Lindenwood College, St. Charles, April 25, 1959.
NEBRASKA, University of Nebraska, Lincoln, April 18, 1959.
NEW JERSEY, Princeton University, November 7, 1959.
NORTHEASTERN
NORTHERN CALIFORNIA
OHIO, Miami University, Oxford, May 9, 1959.
OKLAHOMA, Tulsa University, April 10–11, 1959.
PACIFIC NORTHWEST, University of Oregon, Eugene, June 19, 1959.
PHILADELPHIA, University of Delaware, Newark, November 28, 1959.
ROCKY MOUNTAIN, Utah State University, Logan, May 8–9, 1959.
SOUTHEASTERN
SOUTHERN CALIFORNIA
SOUTHWESTERN, Arizona State University, Tempe, April 10–11, 1959.
TEXAS, University of Texas, Austin, April 17–18, 1959.
UPPER NEW YORK STATE, Hartwick College, Oneonta, May 9, 1959.
WISCONSIN, Wisconsin State College, Platteville, May 2, 1959.

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CONTENTS

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The AMERICAN MATHEMATICAL MONTHLY

(Founded in 1894 by Benjamin F. Finkel)

Ralph D. James, *Editor*

EDITORIAL CORRESPONDENCE should be addressed to the Editor, R. D. James, Department of Mathematics, University of British Columbia, Vancouver 8, Canada. Articles should be typewritten and double-spaced on $8\frac{1}{2}$ by 11 paper. The greatest possible care should be taken in preparing the manuscript and authors should keep a complete copy.

ADVERTISING CORRESPONDENCE should be addressed to F. R. Olson, Mathematical Association of America, University of Buffalo, Buffalo 14, N. Y.

NOTICE OF CHANGE OF ADDRESS by members of the Association as well as correspondence regarding subscriptions to the Monthly should be sent to the Secretary-Treasurer, H. M. Gehman, Mathematical Association of America, University of Buffalo, Buffalo 14, N. Y.

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THE INTEGRABILITY OF CERTAIN FUNCTIONS AND RELATED SUMMABILITY METHODS*

I. J. SCHOENBERG, University of Pennsylvania

1. Introduction. This paper was especially written for the MONTHLY because it shows that a closer analysis of an old textbook example involves in a natural way a variety of subjects such as integrability, summability, and the asymptotic distribution of sequences. The most familiar example of a function $f(x)$ which is not Riemann integrable is due to Dirichlet ([2], p. 132): $f(x)$ is defined in the range $[0, 1]$ as being 0 if x is irrational and 1 if x is rational. In many textbooks (see *e.g.*, [1], p. 83) we find the function defined in $[0, 1]$ by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational,} \\ 1/q & \text{if } x = p/q, \qquad (p, q) = 1, \qquad 0 = 0/1, \end{cases}$$

where it is proposed as an exercise to show that $f(x)$ is discontinuous at all rationals and continuous at all irrationals. For indeed, if ξ is irrational and $p/q \to \xi$, then $q \to \infty$ so that

$$\lim f(p/q) = \lim 1/q = 0 = f(\xi).$$

Its discontinuity at rational points is clear. Thus the set of discontinuity points of $f(x)$ is a set of measure zero; the function being bounded we conclude by a classical criterion that $f(x)$ is Riemann integrable in the range $[0, 1]$ (see [4], Th. 1, p. 208).

We are here concerned with the following generalization of these functions: Let

$$\{\gamma_n\}, \qquad n = 1, 2, \cdots, \tag{1.1}$$

be a given sequence of reals and let $f(x)$ be defined in $[0, 1]$ by

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is irrational,} \\ \gamma_q & \text{if } x = p/q, \qquad (p, q) = 1. \end{cases} \tag{1.2}$$

When is this function Riemann integrable? We have just seen that it is integrable if $\gamma_n = 1/n$, but not integrable if $\gamma_n = 1$.

Let us now assume that

$$\lim \gamma_n = 0 \tag{1.3}$$

and let us show that $f(x)$ is then R-integrable. Indeed, the simple argument mentioned above applies also now. If ξ is irrational and $p/q \to \xi$, then $q \to \infty$, hence by (1.3)

$$\lim f(p/q) = \lim \gamma_q = 0 = f(\xi).$$

* The present work was partly sponsored by the Air Research and Development Command, USAF, at the University of Pennsylvania.

Hence (1.3) implies the continuity of $f(x)$ at all irrationals. Since $f(x)$ is evidently bounded, we again conclude as above that our function is integrable. However, the converse is also true as stated in the following

THEOREM 1. *The function $f(x)$ defined by* (1.1) *and* (1.2) *is Riemann integrable in the range* $[0, 1]$ *if and only if* (1.3) *holds.*

The sufficiency of (1.3) has already been shown. To prove the necessity let us assume that (1.3) does *not* hold and let us show that *$f(x)$ is discontinuous at every irrational point $x=\xi$.*

To prove this let n be given and let us consider the $\phi(n)$ integers $r_1, r_2, \cdots, r_\phi$ in natural order, which do not exceed n and are prime to n. The $\phi(n)$ reduced fractions

$$0 < \frac{r_1}{n} < \frac{r_2}{n} < \cdots < \frac{r_\phi}{n} < 1 \tag{1.4}$$

divide the interval $[0, 1]$ into $\phi+1$ subintervals. Let us denote by g_n the length of the largest of these subintervals. We shall now take for granted, proving it later, the interesting property

$$\lim_{n\to\infty} g_n = 0. \tag{1.5}$$

The discontinuity of $f(x)$ at irrational points is now easily established. For if ξ is irrational, $0<\xi<1$, then (1.5) shows that the inequalities

$$\frac{r_\nu}{n} < \xi < \frac{r_{\nu+1}}{n}$$

define uniquely the integer $\nu=\nu(n)$, provided that n is sufficiently large. Again (1.5) implies that

$$\lim_{n\to\infty} \frac{r_\nu}{n} = \xi. \tag{1.6}$$

Since $\gamma_n \not\to 0$ by assumption, we obtain

$$f(r_\nu/n) = \gamma_n \not\to 0 = f(\xi)$$

showing, in view of (1.6), that $f(x)$ is discontinuous at $x=\xi$.

The property (1.5), which we have just used, is a corollary of a stronger result due to G. Pólya (see [8] and [9], Problem 188, p. 75) to the effect that the set of fractions (1.4) is asymptotically equidistributed in the range $[0, 1]$. This means that if $g(x)$ is any Riemann integrable function in $[0, 1]$ then

$$\lim_{n\to\infty} \frac{g(r_1/n) + g(r_2/n) + \cdots + g(r_\phi/n)}{\phi(n)} = \int_0^1 g(x)dx. \tag{1.7}$$

That this implies the weaker statement (1.5) is seen as follows: Let us assume that (1.5) were false; this means that there exists a $\delta>0$ such that $g_n>\delta$ for arbitrarily large values of $n=n_\nu$ ($\nu=1, 2, \cdots$). An obvious application of the Bolzano-Weierstrass principle leads to the existence of an interval J, a subinterval of $[0, 1]$, such that J contains none of the fractions (1.4) if $n=n_\nu$. However, this clearly contradicts the relation (1.7) if

$$g(x) = \begin{cases} 1 & \text{if } x \in J, \\ 0 & \text{if } x \notin J. \end{cases}$$

This completes our proof of Theorem 1.

In the next section it is shown how the Riemann sums (2.4) of our function $f(x)$ lead to the regular summability method (2.5) which seems not to have been studied before. This summability method turns out to be peculiarly weak: If a sequence $\{\gamma_n\}$ is summable by it to the limit λ, then $\gamma_{n_\nu}\to\lambda$ over subsequences $\{n_\nu\}$ which can be readily described (Theorem 2 below). Known results on the arithmetical function $\phi(n)/n$ allow to establish contact with the notion of an asymptotic distribution function of a sequence of which equidistribution in $[0, 1]$ is a notable example. The case when a sequence $\{\gamma_n\}$ has an asymptotic distribution function having only one point of increase (with unit jump) is interpreted as a generalization of the notion of convergence called D-convergence. It is shown that sequences summable by the method (2.5) are also D-convergent (Theorem 3). An analytic criterion for D-convergence is established by elementary means (Theorem 4). A last brief section records an interesting remark by H. Rademacher exhibiting sequences $\{\gamma_n\}$ such that the function (1.2) is not only continuous but also differentiable at algebraic irrationalities.

2. The Riemann sums of $f(x)$ and a related summability method. A trivial generalization of Theorem 1 is as follows. Let the sequence (1.1) be given as well as a fixed real number λ. Let $f(x)$ be defined in $[0, 1]$ by

$$f(x) = \begin{cases} \lambda & \text{if } x \text{ is irrational} \\ \gamma_q & \text{if } x = p/q, \qquad (p, q) = 1. \end{cases} \tag{2.1}$$

Then $f(x)$ *is R-integrable if and only if*

$$\lim \gamma_n = \lambda. \tag{2.2}$$

Indeed, either repeat previous arguments or apply old result to $f(x)-\lambda$.

Let us now assume that (2.2) holds. Being R-integrable, $f(x)$ is also summable in the sense of Lebesgue, while

$$(R)\int_0^1 f(x)dx = (L)\int_0^1 f(x)dx = (L)\int_0^1 \lambda dx = \lambda,$$

since $f(x)=\lambda$ almost everywhere. The Riemann integral of $f(x)$ being λ we may

conclude that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\nu=1}^{n} f\left(\frac{\nu}{n}\right) = \lambda. \tag{2.3}$$

On the other hand, the Riemann sums

$$s_n = \frac{1}{n} \sum_{\nu=1}^{n} f\left(\frac{\nu}{n}\right), \qquad (n = 1, 2, \cdots) \tag{2.4}$$

are evidently finite linear combinations of the elements of the sequence $\{\gamma_n\}$. If we interpret the relations (2.4) as a transformation which transforms the sequence $\{\gamma_n\}$ into the sequence $\{s_n\}$, we may restate our result ($\gamma_n \to \lambda$ implies $s_n \to \lambda$) by saying that (2.4) is a *regular transformation* (*i.e.*, convergence and limit preserving; see [5], p. 43).

However, the matter just discussed belongs to the realm of algebraic analysis and may be easily settled directly without using integration theory. Let us express the sums (2.4) in terms of the γ_ν. For this purpose we classify the integers $\nu = 1, \cdots, n$ by the value of their g.c.d. with n. If d' is a divisor of n, then the ν with $(n, \nu) = d'$ are given by $\nu = d'l$, $n = d'd$, where l is such that $1 \leqq l \leqq d$, $(l, d) = 1$. By re-grouping the terms of the sum (2.4) according to this classification of the ν, we obtain, since $\nu/n = l/d$, that

$$\sum_{\nu=1}^{n} f\left(\frac{\nu}{n}\right) = \sum_{d|n} \sum_{\substack{(l,d)=1 \\ l \leq d}} f\left(\frac{l}{d}\right) = \sum_{d|n} \sum_{\substack{(l,d)=1 \\ l \leq d}} \gamma_d = \sum_{d|n} \phi(d)\gamma_d.$$

In the Dirichlet case when all the $\gamma_n = 1$, also $f(\nu/n) = 1$, and we obtain

$$n = \sum_{d|n} \phi(d),$$

which is a theorem of Gauss ([6], p. 30). The explicit expression of the Riemann sums (2.4) now becomes

$$s_n = \frac{\sum_{d|n} \phi(d)\gamma_d}{\sum_{d|n} \phi(d)} \qquad (n = 1, 2, \cdots). \tag{2.5}$$

The above conclusion to the effect that $\gamma_n \to \lambda$ implies $s_n \to \lambda$ now becomes again evident because the transformation (2.5) satisfies the conditions for a regular transformation given by the theorem of Toeplitz (see [5], Th. 2, p. 43). Indeed, notice that on writing (2.5) in the form

$$s_n = \sum_{\nu} a_{n\nu}\gamma_\nu,$$

we have $a_{n\nu} \geqq 0$, $\sum_\nu a_{n\nu} = 1$, $a_{n\nu} \to 0$ as $n \to \infty$ for each fixed ν.

Let us say for convenience that *the sequence* $\{\gamma_n\}$ *is* ϕ-*summable, or* ϕ-*con-*

vergent, to the ϕ-limit λ and write

$$\phi\text{-lim } \gamma_n = \lambda, \tag{2.6}$$

meaning thereby that the transformed sequence $\{s_n\}$ has the property

$$\lim s_n = \lambda. \tag{2.7}$$

We know by Theorem 1 that

$$\lim \gamma_n = \lambda \tag{2.8}$$

is necessary and sufficient for the R-integrability of the function (2.1). Also (2.3), or (2.7), is clearly a *necessary* condition. However, it is conceivable that the relation (2.7) might also be sufficient for the R-integrability of $f(x)$. Such would be the case if and only if the ϕ-convergence of a sequence $\{\gamma_n\}$ would imply its ordinary convergence. Such, however, is not the case as will be shown in the next section.

3. Properties of the transformation inverse to (2.5). Let us invert the transformation (2.5) or

$$ns_n = \sum_{d|n} \phi(d)\gamma_d.$$

By Moebius' inversion formula (see *e.g.* [6], p. 87) we obtain

$$\phi(n)\gamma_n = \sum_{d|n} \mu\left(\frac{n}{d}\right) ds_d, \tag{3.1}$$

or

$$\gamma_n = \frac{1}{\phi(n)} \sum_{d|n} \mu\left(\frac{n}{d}\right) ds_d \qquad (n = 1, 2, \cdots). \tag{3.2}$$

Also from (3.1), for all $\gamma_\nu = 1$ hence all $s_\nu = 1$, we obtain the classical relation

$$\phi(n) = \sum_{d|n} \mu\left(\frac{n}{d}\right) d. \tag{3.3}$$

Since $\phi(n) \to \infty$ as $n \to \infty$ (see [8], p. 6, formula (16) for a more precise result), we see that the inverse transformation (3.2), if written in the form

$$\gamma_n = \sum_m b_{nm} s_m \qquad (n = 1, 2 \cdots), \tag{3.4}$$

has the properties $\sum_m b_{nm} = 1$, $b_{nm} \to 0$ as $n \to \infty$ for each fixed m. The question of the possible regularity of (3.4) will now be decided by the behavior of the sums

$$B_n = \sum_m |b_{nm}| = \frac{1}{\phi(n)} \sum_{d|n} \left|\mu\left(\frac{n}{d}\right)\right| d = \frac{1}{\phi(n)} \sum_{d|n} |\mu(d)| \frac{n}{d} \cdot$$

If $n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ is the canonical decomposition of n into primes we obtain

$$B_n = \frac{n}{\phi(n)}\sum_{d|n} |\mu(d)|/d = \frac{n}{\phi(n)}\left\{1+\sum\frac{1}{p_1}+\sum\frac{1}{p_1p_2}+\cdots+\frac{1}{p_1p_2\cdots p_k}\right\}$$
$$= \frac{n}{\phi(n)}\prod_{i=1}^{k}\left(1+\frac{1}{p_i}\right) = \frac{n}{\phi(n)}\prod\left(1-\frac{1}{p_i^2}\right)\Big/\prod\left(1-\frac{1}{p_i}\right)$$

and finally

$$B_n = \left(\frac{n}{\phi(n)}\right)^2\cdot\prod_{i=1}^{k}\left(1-\frac{1}{p_i^2}\right).$$

Writing

$$\theta = \prod_{p}\left(1-\frac{1}{p^2}\right), \qquad 0<\theta<1,$$

where p runs over all primes, we obtain

$$\theta\left(\frac{n}{\phi(n)}\right)^2 < B_n \leqq \left(\frac{n}{\phi(n)}\right)^2. \tag{3.5}$$

Observing that the sequence

$$n/\phi(n) = 1\Big/\prod_{i=1}^{k}(1-p_i^{-1}) \qquad (n=2,3,\cdots)$$

is *not* bounded, we conclude from (3.5) that the sequence $\{B_n\}$ is likewise unbounded. By Toeplitz's theorem we conclude that *the transformation* (3.2) *is not regular.*

However, the inequalities (3.5) allow us to settle completely the question concerning the convergent *subsequences* of a ϕ-convergent sequence $\{\gamma_n\}$. We state the result as

THEOREM 2. *Let* $\{n_\nu\}$ *be a given sequence of increasing natural numbers. Then the assumption*

$$\phi\text{-}\lim \gamma_n = \lambda \tag{3.6}$$

implies that

$$\lim_{\nu\to\infty}\gamma_{n_\nu} = \lambda \tag{3.7}$$

if and only if the sequence $\{n_\nu\}$ *is such that*

$$\liminf_{\nu\to\infty}\frac{\phi(n_\nu)}{n_\nu} > 0. \tag{3.8}$$

Remarks. 1. The condition (3.8) is for instance satisfied if $\{n_\nu\}$ is the sequence of primes $\{p\}$; likewise if $\{n_\nu\} = \{2p\}$. For indeed, observe that

$$\frac{\phi(p)}{p} = 1 - \frac{1}{p} \to 1 > 0, \qquad \frac{\phi(2p)}{2p} = \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{p}\right) \to \frac{1}{2} > 0$$

as $p \to \infty$. Thus (3.6) implies the relations

$$\gamma_p \to \lambda, \qquad \gamma_{2p} \to \lambda \text{ as the prime } p \to \infty. \tag{3.9}$$

2. Theorem 2 is of the nature of a "limitation theorem" (see [**5**], p. 95). Thus the sequence 1, 0, 1, 0, 1, 0, $\cdots$, which is $(C, 1)$-convergent to the limit 1/2 is certainly *not* ϕ-convergent because for the present sequence

$$\gamma_p \to 1, \qquad \gamma_{2p} \to 0,$$

in contradiction to the necessary relations (3.9).

A proof of Theorem 2. Let us take from the system (3.2) or (3.4) only the relations corresponding to $n = n_\nu$:

$$\gamma_{n_\nu} = \sum_m b_{n_\nu, m} s_m, \qquad (\nu = 1, 2, \cdots). \tag{3.10}$$

The inequalities (3.5) which we write as

$$\left(\frac{\phi(n)}{n}\right)^2 \leqq \frac{1}{B_n} < \frac{1}{\theta}\left(\frac{\phi(n)}{n}\right)^2$$

show that the sequence $\{B_{n_\nu}\}$ is *bounded above*, or equivalently the sequence $\{1/B_{n_\nu}\}$ is bounded away from zero, if and only if the sequence $\{\phi(n_\nu)/n_\nu\}$ is *bounded away from zero.* Thus the transformation (3.10) is *regular* if and only if the condition (3.8) is verified. Therefore (3.8) is the necessary and sufficient condition for $s_n \to \lambda$ to imply $\gamma_{n_\nu} \to \lambda$ and our theorem is established.

4. A related notion of D-convergence of sequences. Let us again assume that $s_n \to \lambda$ which we express as above by writing

$$\phi\text{-lim } \gamma_n = \lambda. \tag{4.1}$$

Observe now that

$$\frac{\phi(n)}{n} = \prod_{p \mid n}\left(1 - \frac{1}{p}\right), \qquad (n = 1, 2, \cdots) \tag{4.2}$$

is a fraction which seems to roam freely in the range $[0, 1]$ as n ranges over the natural integers. By Theorem 2 the assumption (4.1) seems to imply that "essentially" $\gamma_n \to \lambda$, by which we mean that for any $\epsilon > 0$ the inequality $|\gamma_n - \lambda| < \epsilon$ holds for all n with the exception of some possibly infinite but "thin" sequence which depends on ϵ.

It is easy to make these vague ideas precise but requires some results con-

cerning the asymptotic behavior of the sequence (4.2), as $n\to\infty$, which are somewhat deeper than the tools which we have used so far. Let us start with the following definition (see *e.g.* [11], p. 316): If a sequence $S=\{n_\nu\}$ of increasing integers is given, we define

$$D^*(s) = \limsup_{n\to\infty} \frac{1}{n} \sum_{n_\nu \leqq n} 1, \qquad D_*(s) = \liminf_{n\to\infty} \frac{1}{n} \sum_{n_\nu \leqq n} 1,$$

and call them the *upper* and *lower* density of S, respectively. Clearly $0 \leqq D_*(S) \leqq D^*(S) \leqq 1$. If $D_*(S) = D^*(S)$ we say that the sequence S has a density whose value is

$$D(s) = \lim_{n\to\infty} \frac{1}{n} \sum_{n_\nu \leqq n} 1.$$

Evidently $D(S)=0$ if the sequence S is void or finite. For $n_\nu = 2\nu$ we find $D(S)=1/2$; for $n_\nu=\nu^2$, $D(S)=0$. More interesting examples for this concept are obtained as follows: Let

$$x_1, x_2, x_3, \cdots \tag{4.3}$$

be a given sequence of reals and let us denote by $\{n; x_n \leqq t\}$ the (possibly void) sequence of integers n for which $x_n \leqq t$, where t is given real. If we now assume that the sequence (4.3) is asymptotically equidistributed in $[0, 1]$ (see [9], Part II, Ch. 4, or [7], Ch. 6) we find without difficulty that

$$D\{n; x_n \leqq t\} = \begin{cases} 0 & \text{if} \quad t < 0, \\ t & \text{if} \quad 0 \leqq t \leqq 1, \\ 1 & \text{if} \quad t > 1. \end{cases} \tag{4.4}$$

Conversely, if (4.4) holds, we verify easily that the sequence (4.3) is equidistributed in $[0, 1]$. It is convenient to describe (4.4) by saying that the sequence (4.3) admits the *asymptotic distribution function*

$$\omega_1(t) = \begin{cases} 0 & \text{if} \quad t < 0, \\ t & \text{if} \quad 0 \leqq t \leqq 1, \\ 1 & \text{if} \quad t > 0. \end{cases} \tag{4.5}$$

The following generalization seems now natural. Let $\omega(t)$ be any given non-decreasing function defined for all real t and such that

$$\lim_{t\to-\infty} \omega(t) = 0, \qquad \lim_{t\to+\infty} \omega(t) = 1.$$

Functions $\omega(t)$ with these properties are called *distribution functions*. A sequence of reals (4.3) is said to admit the asymptotic distribution function $\omega(t)$ provided that the relation

$$D\{n;\, x_n \leqq t\} = \omega(t) \tag{4.6}$$

holds for every real t which is a point where $\omega(t)$ is continuous.

For example, the equidistribution in $[0, 1]$ of the sequence (4.3) is, by (4.4), a special case of the above definition for the special continuous distribution function (4.5). The particular sequence

$$x_n = \frac{\phi(n)}{n} \qquad (n = 1, 2, \cdots) \tag{4.7}$$

exhibits a somewhat different asymptotic behavior. We need the following

LEMMA 1. *The sequence* (4.7) *admits an asymptotic distribution function*

$$D\{n;\, \phi(n)/n \leqq t\} = \omega(t) \qquad (-\infty < t < \infty), \tag{4.8}$$

which is continuous for all real t.

A proof of this lemma can be found in the writer's dissertation [**10**], a simplified and more general version in [**11**]. Because $0<x_n<1$ for all n, it is clear that

$$\omega(t) = 0 \text{ if } t \leqq 0, \qquad \omega(t) = 1 \text{ if } t \geqq 1. \tag{4.9}$$

The nature of $\omega(t)$ in the range $[0, 1]$ does not concern us here; I may mention, however, that $\omega(t)$ increases strictly in the range $[0, 1]$ from $\omega(0)=0$ to $\omega(1)=1$ and that $\omega'(t)=0$ for all values of t with the exception of a set of measure zero, as was shown by P. Erdös [**3**].

An almost immediate consequence of Lemma 1 is the following

LEMMA 2. *If the sequence $S=\{n_\nu\}$ has the property*

$$\lim_{\nu\to\infty} \frac{\phi(n_\nu)}{n_\nu} = 0 \tag{4.10}$$

then

$$D(S) = 0. \tag{4.11}$$

Indeed, let $S_t=\{n;\, \phi(n)/n\leqq t\}$ be the sequence appearing in (4.8). Now (4.10) clearly implies that the elements of the sequence $S=\{n_\nu\}$ are contained in the sequence S_t provided that $t>0$ and that $\nu>\nu_0(t)$. But then evidently by (4.8),

$$D^*(S) \leqq D(S_t) = \omega(t).$$

Letting $t\to 0$ we have $\omega(t)\to\omega(0)=0$ which shows that $D^*(S)=0$ and hence (4.11) is established.

LEMMA 3. *If*

$$\phi\text{-lim } \gamma_n = \lambda \tag{4.12}$$

and $\epsilon > 0$ *then*

$$D\{n;\ |\gamma_n - \lambda| \geqq \epsilon\} = 0. \tag{4.13}$$

There is nothing to prove if the sequence defined by $|\gamma_n - \lambda| \geqq \epsilon$ is void or finite. Assuming this sequence to be infinite let us write

$$S(\epsilon) = \{n;\ |\gamma_n - \lambda| \geqq \epsilon\} = \{n_\nu\}_{\nu=1,2,\ldots}.$$

From the very definition of this sequence it is clear that $\gamma_{n_\nu} \not\to \lambda$. By Theorem 2 we conclude that

$$\liminf_{\nu\to\infty} \phi(n_\nu)/n_\nu = 0.$$

We claim that actually *the relation* (4.10) *holds*. For if not, then

$$\limsup_{\nu\to\infty} \phi(n_\nu)/n_\nu > 0$$

and we could therefore pick a subsequence $\{n_{\nu'}\}$, of $\{n_\nu\}$, for which

$$\liminf \phi(n_{\nu'})/n_{\nu'} > 0.$$

But now Theorem 2 would imply that $\gamma_{n_{\nu'}} \to \lambda$, which is absurd since

$$|\gamma_{n_{\nu'}} - \lambda| \geqq \epsilon \qquad \text{for } all\ \nu'.$$

Thus (4.10) indeed holds and Lemma 2 implies the conclusion (4.13).

Lemma 3 suggests the following generalization of the concept of a convergent sequence:

DEFINITION OF D-CONVERGENCE. *We say that a sequence* $\{\gamma_n\}$ *is D-convergent to the D-limit* λ *provided that for every* $\epsilon > 0$

$$D\{n;\ |\gamma_n - \lambda| \geqq \epsilon\} = 0, \tag{4.14}$$

in which case we write

$$D\text{-}\lim \gamma_n = \lambda. \tag{4.15}$$

Clearly $\lim \gamma_n = \lambda$ implies (4.15) since void or finite sequences have vanishing densities. In terms of this definition we may restate Lemma 3 as

THEOREM 3. *If* ϕ-lim $\gamma_n = \lambda$, *then also* D-lim $\gamma_n = \lambda$.

The converse is evidently false. Thus if

$$\gamma_n = \begin{cases} 1 & \text{if } n \text{ is a prime,} \\ 0 & \text{otherwise,} \end{cases}$$

then D-lim $\gamma_n = 0$ because the density of the sequence of primes is zero. On the other hand the relations

$$\gamma_p \to 1, \qquad \gamma_{2p} \to 0$$

hold and rule out the ϕ-convergence of $\{\gamma_n\}$ in view of the necessary conditions (3.9).

We conclude this section with a few obvious properties of D-convergence whose simple proofs may be omitted:

1. D-lim $x_n = \xi$, D-lim $x_n = \eta$ imply $\xi = \eta$.
2. D-lim $x_n = \xi$ implies D-lim $(cx_n) = c\xi$.
3. D-lim $x_n = \xi$, D-lim $y_n = \eta$ imply D-lim $(x_n + y_n) = \xi + \eta$.

5. A criterion for D-convergence. The thoughtful reader has surely noticed the close connection of D-convergence with the notion of an asymptotic distribution function $\omega(t)$ as described by the relation (4.6). Indeed, let us consider the special distribution function

$$\omega_0(t) = \begin{cases} 0 & \text{if} \quad t < 0, \\ 1 & \text{if} \quad t \geqq 1. \end{cases} \tag{5.1}$$

A moment's reflection will show that

$$D\text{-lim } x_n = \xi$$

if and only if the sequence $\{x_n\}$ *admits the asymptotic distribution function* $\omega_0(t-\xi)$.

Thus D-convergence, like equidistribution (see (4.4), (4.5)), is equivalent to the sequence admitting a certain special remarkable distribution function, in this case the function (5.1) or its translates $\omega_0(t-\xi)$.

The kinship of D-convergence to equidistribution raises the question whether there is also for D-convergence a criterion analogous to Weyl's criterion for equidistribution (see [7], p. 76 or [9], Problem 165, p. 70). The answer is affirmative and may be stated as

THEOREM 4. *For*

$$D\text{-lim } x_n = \xi \tag{5.2}$$

it is necessary and sufficient that

$$\lim_{n\to\infty} \frac{1}{n} (e^{itx_1} + e^{itx_2} + \cdots + e^{itx_n}) = e^{it\xi} \textit{ for every real } t. \tag{5.3}$$

Proof of necessity. This will readily follow from a few simple observations which we state as lemmas.

LEMMA 4. *If* (5.2) *holds and*

$$|x_n| < K \qquad \textit{for all } n, \tag{5.4}$$

then

$$(C,\,1)\text{-}\lim x_n = \xi. \tag{5.5}$$

Indeed, without loss of generality we may assume that $\xi=0$. This means that if $\epsilon>0$ and if we denote by N_n the number of ν among $\nu=1,\cdots,n$ for which $|x_\nu|\geqq\epsilon$, then

$$\lim_{n\to\infty}\frac{N_n}{n} = 0. \tag{5.6}$$

Now

$$\left|\frac{x_1+\cdots+x_n}{n}\right| \leqq \frac{N_nK+(n-N_n)\epsilon}{n} = \epsilon+(K-\epsilon)\frac{N_n}{n}$$

where, by (5.6), the right side is $<2\epsilon$ provided that n is sufficiently large. Thus (5.5) is established.

LEMMA 5. *If* (5.2) *holds and* $g(x)$, *defined for all real* x *is continuous at* $x=\xi$, *then*

$$D\text{-}\lim g(x_n) = g(\xi). \tag{5.7}$$

Indeed, to an $\epsilon>0$ corresponds a $\delta>0$ such that

$$|x-\xi| < \delta \text{ implies } |g(x)-g(\xi)| < \epsilon.$$

But then $|g(x)-g(\xi)|\geqq\epsilon$ implies $|x-\xi|\geqq\delta$ and in particular $|g(x_n)-g(\xi)|\geqq\epsilon$ implies $|x_n-\xi|\geqq\delta$. Thus $\{n;\,|g(x_n)-g(\xi)|\geqq\epsilon\}\subset\{n;\,|x_n-\xi|\geqq\delta\}$ and therefore

$$D^*\{n;\ |g(x_n)-g(\xi)| \geqq \epsilon\} \leqq D\{n;\ |x_n-\xi| \geqq \delta\} = 0$$

because of (5.2). This proves (5.7).

The *necessity* of (5.3) is now clear. Indeed, for a *fixed* value of t, e^{itx} is a *continuous* function of x. By Lemma 5 (5.2) implies

$$D\text{-}\lim_{n\to\infty} e^{itx_n} = e^{it\xi}. \tag{5.8}$$

Since $\{e^{itx_n}\}$ is a *bounded* sequence, (5.8) implies, by Lemma 4, that (5.3) holds.

Proof of sufficiency. This proof requires some integral calculus. Let us define a continuous function $M(x)$ by

$$M(x) = \begin{cases} 0 & \text{if } \quad x<-1, \\ 1+x & \text{if } -1\leqq x<0, \\ 1-x & \text{if } \quad 0\leqq x<1, \\ 0 & \text{if } \quad x\geqq 1. \end{cases} \tag{5.9}$$

We take for granted the fact that $M(x)$ allows the integral representation

$$M(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin t/2}{t/2}\right)^2 e^{itx}dt \qquad (-\infty < x < \infty), \tag{5.10}$$

as can be proved, for instance, by means of Cauchy's calculus of residues. We observe first that it suffices to show that (5.3) implies (5.2) for the case when $\xi=0$. For if this is already known, as (5.3) implies

$$\lim_{n\to\infty}\frac{1}{n}\left(e^{it(x_1-\xi)} + \cdots + e^{it(x_n-\xi)}\right) = 1,$$

we may conclude that D-lim $(x_n-\xi)=0$. But D-lim $\xi=\xi$. Adding the two results we obtain the desired conclusion (5.2).

Let us now assume

$$\lim_{n\to\infty}\frac{1}{n}\left(e^{itx_1} + \cdots + e^{itx_n}\right) = 1 \qquad \textit{for every real } t, \tag{5.11}$$

and let us prove that

$$D\{n; |x_n| \geqq \epsilon\} = 0 \qquad \textit{if } \epsilon > 0. \tag{5.12}$$

By an appropriate change of variables (5.10) furnishes

$$M(x/\epsilon) = \frac{\epsilon}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin \epsilon t/2}{\epsilon t/2}\right)^2 e^{itx}dt \tag{5.13}$$

whence

$$\frac{1}{n}\sum_{\nu=1}^{n} M(x_\nu/\epsilon) = \frac{\epsilon}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin \epsilon t/2}{\epsilon t/2}\right)^2 \left\{\frac{1}{n}\sum_{\nu=1}^{n} e^{ix_\nu t}\right\} dt.$$

Observe now that (5.13) is an *absolutely* convergent integral, *i.e.*, the relation is valid with the integral taken in the sense of Lebesgue. Moreover (5.11) holds while

$$\left|\frac{1}{n}\sum_{\nu=1}^{n} e^{ix_\nu t}\right| \leqq 1 \qquad \text{for all real } t, \text{ all } n.$$

By the bounded convergence theorem we obtain

$$\lim_{n\to\infty}\frac{1}{n}\sum_{\nu=1}^{n} M(x_\nu/\epsilon) = \frac{\epsilon}{2\pi}\int_{-\infty}^{\infty}\left(\frac{\sin \epsilon t/2}{\epsilon t/2}\right)^2 dt = M(0) = 1,$$

or

$$\lim \frac{1}{n}\sum_{\nu=1}^{n} M(x_\nu/\epsilon) = 1. \tag{5.14}$$

It is clear that a somewhat longer argument by elementary method may replace

our reference to Lebesgue integration theory. If we let N_n be the number of solutions of $|x_\nu| \geqq \epsilon$ among $\nu = 1, \cdots, n$, then clearly, by (5.9), we have

$$\sum_{\nu=1}^{n} M(x_\nu/\epsilon) \leqq n - N_n$$

or

$$\frac{N_n}{n} \leqq 1 - \frac{1}{n} \sum_{\nu=1}^{n} M(x_\nu/\epsilon).$$

But the right-hand side $\to 0$ as $n \to \infty$, in view of (5.14). Thus $N_n/n \to 0$ and (5.12) is established. This completes a proof of our theorem.

6. A remark by Hans Rademacher. I owe to Han's Rademacher the following interesting remark which is published here with his kind permission. Years ago he already considered the function $f(x)$ defined by our relation (1.2) and found the following

THEOREM 5 (Rademacher). *If the sequence* (1.1) *satisfies the condition*

$$\lim_{n \to \infty} n^k \gamma_n = 0 \qquad (k \text{ an integer} \geqq 2) \tag{6.1}$$

then the function $f(x)$ defined by (1.2) *is differentiable at $x = \xi$, where ξ is an algebraic irrationality of degree not exceeding k.*

We may indeed, assume that ξ is an algebraic number of degree k. A famous theorem of J. Liouville (see [7], pp. 90–91) insures then the existence of a constant $c = c(\xi) > 0$ such that

$$\left| \frac{p}{q} - \xi \right| > \frac{c}{q^k}$$

for any rational fraction p/q, $(q > 0)$. By (1.2) we now have

$$\left| \frac{f(p/q) - f(\xi)}{p/q - \xi} \right| \leqq \frac{1}{c} q^k \gamma_q.$$

Thus, if $p/q \to \xi$, hence $q \to \infty$, our assumption (6.1) implies that $f'(\xi) = 0$.

In particular, if for instance $\gamma_n = 2^{-n}$, then (6.1) holds for all k and $f(x)$ is differentiable at all algebraic irrationalities.

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HEURISTIC REASONING IN THE THEORY OF NUMBERS

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A deep but easily understandable problem about prime numbers is used in the following to illustrate the parallelism between the heuristic reasoning of the mathematician and the inductive reasoning of the physicist. The experts may judge whether the parallelism is more serious than the tone of presentation which is adapted to a wider audience.

1. "Till now the mathematicians tried in vain to discover some order in the sequence of the prime numbers and we have every reason to believe that there is some mystery which the human mind shall never penetrate. To convince oneself, one has only to glance at the tables of primes which some people took the trouble to compute beyond a hundred thousand, and one perceives that there is no order and no rule. This is so much more surprising as the arithmetic gives us definite rules with the help of which we can continue the sequence of the primes as far as we please, without noticing, however, the least trace of order."*

So wrote Euler about two centuries ago, yet the prime numbers may inspire the contemporary mathematician with the same feeling of mystery that Euler so vividly expressed. The primes remain puzzling in spite of many important discoveries made in the meantime. Let us look at some of these discoveries.

The intervals between successive primes are irregular, but these intervals seem to become larger "on the whole" (the primes seem to become scarcer) as we proceed in the sequence of numbers. Since Euler's time a definite law of this phenomenon was discovered (conjectured by Legendre and Gauss, investigated by Chebyshev and Riemann, finally proved by Hadamard and de la Vallée Poussin, proved recently in an essentially different "elementary" manner by Atle Selberg and Paul Erdös). We may formulate this law, the "prime number theorem," intuitively although not quite precisely, as follows: The probability

* See L. Euler, Opera Omnia, ser. 1, vol. 2, p. 241 or G. Pólya, Mathematics and Plausible Reasoning, Princeton, vol. 1, p. 91.

that a large integer x should be a prime, is $1/\log x$ (where $\log x$ is the natural logarithm of x).*

The following short table exhibits the first primes (with two exceptions) classified according to their last digit.

	11		31	41		61	71			101	
3	13	23		43	53		73	83		103	113
7	17		37	47		67			97	107	
	19	29			59		79	89		109	

If we set apart 2 and 5, the prime factors of 10, the last figure in the decimal symbol of a prime cannot be 0, 2, 4, 5, 6, or 8 (since neither 2 nor 5 should be a divisor) and must, therefore, be 1, 3, 7, or 9. Thus, with respect to ten (*modulo* 10) there are four kinds of primes which are listed in the four horizontal lines of the foregoing table, respectively. Since Euler's time, a general law has been discovered (most of the credit for its discovery is due to Dirichlet) which, applied to our particular case, asserts that there are infinitely many prime numbers of each kind and, what is more, that each kind is equally probable. Therefore, in an extensive table of prime numbers there must be roughly as many primes ending with 1 as primes ending with 3.

Euler mentions a table of primes that goes beyond 10^5. Since his time much more extensive tables have been computed, especially in the last decade with the help of machines. Data derived from these tables may suggest problems not yet considered by Euler.

2. The least possible distance between two consecutive primes is 2, if we set apart the unique case of the primes 2 and 3. Two primes having this minimum distance are called *twin primes*. Here is a list of the twin primes under 100:

3, 5 5, 7 11, 13 17, 19 29, 31 41, 43 59, 61 71, 73

We can generalize this situation and consider a prime p that is escorted at a given distance d by another prime $p'=p+d$. (This situation is uninteresting unless d is even; we do not care whether there are or are not other primes between p and p'.) Here is a list of all such pairs at the distance 6, in which the

* The irregular distribution of primes ("there is no order and no rule") strongly suggests the idea of probability and chance. Yet this is paradoxical: Whether any given integer is a prime or not, can be decided by the "definite rules" of arithmetic—where and how could chance enter the picture? The paradox can be somewhat explained (or deepened) by a physical analogy. The kinetic theory of matter considers the probability distribution of the velocities of the molecules in a gas. Yet this is paradoxical: The velocities resulting from the collision of two molecules can be exactly predicted from the data of the collision by the "definite rules" of classical deterministic mechanics —where and how could chance enter the picture? The determinateness of the simple single event and the probabilistic theory of the highly composite whole may seem to be equally compatible (or incompatible) in both cases.

first prime does not (but its escort may) exceed 100:

5, 11	7, 13	11, 17	13, 19	17, 23	23, 29	31, 37	37, 43
41, 47	47, 53	53, 59	61, 67	67, 73	73, 79	83, 89	97, 103

It is curious that the second kind of pairs is more numerous. We count 8 pairs of twin primes and exactly twice as many pairs of primes at the distance 6. Let us take now instead of 10^2 the considerably higher bound $3\cdot 10^7$. Under thirty million there are 152892 primes followed by another prime at the distance 2, but nearly twice as many, namely 304867 primes followed by another at the distance 6.

The numbers of these prime pairs have been obtained by Professor and Mrs. D. H. Lehmer with the use of appropriate computing apparatus; they computed, up to the same limit $3\cdot 10^7$, the number of primes escorted by another prime at the distance d for $d=2, 4, 6, 8, \cdots, 70$. I wish to thank them here for their kind permission to use their interesting material. I wish to use some of their results to offer the unprejudiced reader a particularly suitable opportunity for an inductive investigation in pure mathematics.

It will be convenient to introduce here some notation. Let $\pi_d(x)$ stand for the number of those prime numbers p that satisfy two conditions:

$$p \leqq x, \qquad p + d \text{ is a prime number.}$$

For instance,

$$\pi_2(100) = 8\ , \qquad \pi_2(30\ 000\ 000) = 152892,$$

$$\pi_6(100) = 16, \qquad \pi_6(30\ 000\ 000) = 304867.$$

I set

$$\pi_d(3\cdot 10^7)/\pi_2(3\cdot 10^7) = R_d.$$

For instance, $R_6 = 304867/152892 = 1.9940$, approximately. A small part of the material computed by Professor and Mrs. Lehmer is collected in Table I.

d	R_d	12	1.9985	24	1.9976	36	1.9997	48	1.9965	60	2.6632
2	1.0000	14	1.1985	26	1.0910	38	1.0566	50	1.3308	62	1.0341
4	0.9979	16	1.0001	28	1.1974	40	1.3330	52	1.0892	64	0.9999
6	1.9940	18	1.9982	30	2.6632	42	2.3987	54	1.9981	66	2.2186
8	0.9996	20	1.3311	32	0.9970	44	1.1097	56	1.1957	68	1.0663
10	1.3317	22	1.1088	34	1.0645	46	1.0467	58	1.0349	70	1.5977

TABLE I. VALUES OF R_d

3. Now, let us start our inductive research. At any moment at which the reader feels inspired, he should interrupt the reading and try to guess the result by himself.

The four kinds of prime numbers that we have considered in Section 1 (ending with 1, 3, 7 or 9 in the decimal notation, respectively) are known to be equally frequent. Are the 35 kinds of prime numbers with which Table I is concerned also equally frequent? If it were so, all the ratios R_d contained in Table I should be approximately equal to one. In fact, remarkably enough, a few entries in Table I are pretty close to the value 1, but the majority seem to deviate significantly from 1. The analogy with the previous case does not seem to go far. Yet, perhaps, the analogy holds at least in one respect: the ratio $\pi_d(x)/\pi_2(x)$ may converge towards some limit (not necessarily 1) when x tends to infinity, and the ratio $R_d=\pi_d(3\cdot 10^7)/\pi_2(3\cdot 10^7)$ entered into Table I may be an approximation to that limit.

We face here a situation somewhat analogous to the situation that the chemists faced around 1800 when they were about to discover the Law of Multiple Proportions. They had to perceive behind their experimental data distorted by unavoidable errors of observation the ratios of simple multiples of the atomic weights, and we have to perceive behind the approximate ratios R_d collected in Table I the true limiting ratios. To guess these limiting ratios is a challenging task.

We have already observed that some values of R_d are very close to 1; they correspond to $d=2$, 4, 8, 16, 64. (For $d=2$ the value is exactly 1, but this is trivial.) We can scarcely fail to notice here the powers of 2. By the way, these values of R_d so close to 1 are also the smallest values in the table. Are there other entries in the table so nearly equal to each other?

In trying to answer this question we may notice that the entries corresponding to

$$d = 6,\ 12,\ 24,\ 48$$

are approximately equal to each other, and so are those corresponding to

$$d = 10,\ 20,\ 40$$

or those corresponding to

$$d = 14,\ 28,\ 56.$$

In general, multiplication of d by 2 seems to leave the value of R_d almost unchanged.

What about multiplication by 3? It approximately doubles the value of R_d in certain transitions, as from

2 to 6,	4 to 12,	8 to 24,	16 to 48,
10 to 30,	20 to 60,	14 to 42,	22 to 66.

Yet it is not so in other cases, as

$$6 \text{ to } 18, \qquad 12 \text{ to } 36, \qquad 18 \text{ to } 54;$$

in these latter cases the multiplication of d by 3 leaves the value of R_d almost unchanged. How can you account for this different behavior?

And so on, from question to question, by observation and tentative generalization, carefully checking each guess, the reader may discover that many of the values R_d contained in Table I come very close to simple fractions; see Table II.

d	2, 4, 8, 16, 32, 64	6, 12, 18, 24, 36, 48, 54	10, 20, 40, 50	14, 28, 56	22, 44	30, 60	42	66	70
R_d (approx.)	1	$\frac{2}{1}$	$\frac{4}{3}$	$\frac{6}{5}$	$\frac{10}{9}$	$\frac{8}{3}$	$\frac{12}{5}$	$\frac{20}{9}$	$\frac{8}{5}$

TABLE II. SIMPLE APPROXIMATIONS TO SOME R_d

Table II strongly suggests that R_d *depends only on the decomposition of d into prime factors.* More precisely, just the presence of a prime factor in, or its absence from, the decomposition seems to be relevant; for instance, to all values of d of the form $2^\alpha 3^\beta$ with $\alpha, \beta = 1, 2, 3, \cdots$ there corresponds the same value of R_d (approximately).

Moreover, to each prime factor of d there seems to correspond a factor of R_d; to the (unavoidable) factor 2 of d, the (trivial) factor 1 of R_d; to the prime factors

$$3, \qquad 5, \qquad 7, \qquad 11$$

of d, the following factors of R_d:

$$\frac{2}{1}, \qquad \frac{4}{3}, \qquad \frac{6}{5}, \qquad \frac{10}{9},$$

respectively. Then, when d is a product of different primes (or powers of different primes) R_d seems to be the product of the corresponding factors.

4. All such observations point to the (conjectural) formula

$$\pi_d(x) \sim \pi_2(x) \prod_{p|d} \frac{p-1}{p-2}, \tag{1}$$

where the product $\prod_{p|d}$ is extended over all different *odd* prime factors p of the even number d.* The sign $\sim$ can be interpreted either vaguely or strictly. In a

* The usual abbreviation $a|b$ means "a divides b" or "a is a divisor of b." We shall need later also the abbreviation $a \nmid b$ which means "a is not a divisor of b."

vague interpretation $\sim$ means "approximately equal;" in the strict sense it means "the ratio of the two sides tends to 1 when x tends to ∞." The formula is merely a conjecture which we can conceive quite naively by examining Table I. In Table III, the observed values of R_d, taken from Table I and styled now R_d (obs.), are compared with the corresponding conjectural limiting values, styled R_d (theor.). This comparison yields strong inductive evidence for the conjecture which could be further strengthened by use of other data computed by Professor and Mrs. Lehmer.

d	R_d (obs.)	R_d (theor.)	24	1.9976	2.0000	48	1.9965	2.0000
2	1.0000	1.0000	26	1.0910	1.0909	50	1.3308	1.3333
4	0.9979	1.0000	28	1.1974	1.2000	52	1.0892	1.0909
6	1.9940	2.0000	30	2.6632	2.6667	54	1.9981	2.0000
8	0.9996	1.0000	32	0.9970	1.0000	56	1.1957	1.2000
10	1.3317	1.3333	34	1.0645	1.0667	58	1.0349	1.0370
12	1.9985	2.0000	36	1.9997	2.0000	60	2.6632	2.6667
14	1.1985	1.2000	38	1.0566	1.0588	62	1.0341	1.0345
16	1.0001	1.0000	40	1.3330	1.3333	64	0.9999	1.0000
18	1.9982	2.0000	42	2.3987	2.4000	66	2.2186	2.2222
20	1.3311	1.3333	44	1.1097	1.1111	68	1.0663	1.0667
22	1.1088	1.1111	46	1.0467	1.0476	70	1.5977	1.6000

TABLE III. VALUES OF R_d, OBSERVED AND "THEORETICAL"

5. We have before us a precise, general, but enigmatic formula derived from, and quite well verified by, observations. Of course, we wish to understand it, we wish to explain it. When we are looking at it, our situation is similar to that of Newton looking at the laws of Kepler or to that of Niels Bohr looking at Balmer's formula. The word "similar" must be correctly understood. Similar figures may be very different in magnitude, but they show the same proportions, and so do in a sense the three situations we have just compared.

We wish to explain that conjectural formula about prime numbers. Both the irregular distribution of the primes and the structure of the conjectural formula strongly suggest an explanation by probability. I wish to present such an explanation. We shall arrive at it in two steps (of which the second is much more dangerous).

PROBLEM I. *Let p denote a given prime number, d a given integer, and x a large integer chosen at random. Find the probability that neither x nor $x+d$ is divisible by p.*

The reader may visualize the integers as successive intervals of equal length along an infinite straight line, some sort of super-roulette. The interval is red or green, according as the integer is, or is not, divisible by p; among any p consecutive intervals there is always just one that is red. A ball is rolled along the line and steps in the interval x.

We have to distinguish two cases.*

First case: $p|d$. In this case $x+d$ falls on a multiple of p (a red space) if, and only if, x itself falls on such a multiple. Therefore, out of any p consecutive numbers (spaces), $p-1$ are favorable (green) and so the required probability is $(p-1)/p$.

Second case: $p\nmid d$. Even if x does not fall on a multiple of p, $x+d$ may. Therefore, out of any p consecutive numbers just $p-2$ are favorable. The required probability is $(p-2)/p$.

PROBLEM II. *Let d denote a given even integer, and x a large integer chosen at random. Find the probability P_d that both x and $x+d$ are prime numbers.*

In order that both x and $x+d$ should be prime numbers, a sequence of conditions must be satisfied:

First, neither x nor $x+d$ is divisible by 2;

then, neither x or $x+d$ is divisible by 3;

then, neither x nor $x+d$ is divisible by 5;

and so on. The general form of this condition is: neither x nor $x+d$ is divisible by p where p is a prime number.

We have computed above the probability for the fulfillment of any single one of these conditions. Now we have to compute the probability that all these conditions are fulfilled at the same time, all these events are realized simultaneously.

Two difficulties arise here: Are these events independent? How far should we go with p? In fact, these two difficulties may be connected, but at this stage of the game it will be better not to examine them too thoroughly; let us now proceed quickly and see whether anything worthwhile turns up.

Are the events independent? We do not know, but let us assume it. Also the physicist is inclined to assume the independence of the probabilities he deals with—not because he knows that they are independent, but interdependent probabilities are so much more difficult to handle—and so let us assume independence in our case too, although we have no better reasons than the physicist.

Having made this assumption all we have to do is to multiply probabilities computed above. We distinguish three cases:

* For the symbols $|$ and $\nmid$, see footnote p. 379.

$p=2$ (which is a divisor of the even number d);

p is odd and is a divisor of d;

p is odd and is not a divisor of d.

Accordingly, the required probability P_d is a product of three kinds of factors:

$$P_d = \frac{1}{2} \prod_{p \mid d} \frac{p-1}{p} \prod_{p \nmid d} \frac{p-2}{p}. \tag{2}$$

In this formula (2) (and in the following formulas (3), (4)) the letter p stands for an *odd* prime number.

How far should we go with p? Of course, on the right hand side of formula (2) we extend the first product over all odd prime factors of the given number d. In the second product, we take all the odd primes not dividing d up to a certain large upper bound, depending on the considered large number x—but let us *postpone* the decision, how far to go, how large that upper bound should precisely be.

We can transform formula (2) as follows:

$$P_d = \prod_{p \mid d} \frac{p-1}{p-2} \cdot \frac{1}{2} \prod_{p} \frac{p-2}{p}; \tag{3}$$

the second product on the right hand side of (3) is extended over *all* odd primes p under a certain (large, but not yet definitely characterized) upper bound. The first product is extended over the odd prime divisors of d; if d happens to be 2 (or a power of 2) there are no odd prime divisors, that first product is empty, and has to be replaced by 1. Therefore

$$P_d = \prod_{p \mid d} \frac{p-1}{p-2} \cdot P_2. \tag{4}$$

Yet the ratio of the probabilities P_d/P_2 should be approximately the same as the ratio of the observed numbers $\pi_d(x)/\pi_2(x)$—and so the formula (4) just derived justifies the conjectural formula (1)—complete success!

6. Unfortunately, our reasoning is vulnerable and the success is illusory. We left a gap in our derivation (we did not decide how far to go with p) and if we try to fill this gap, we run into trouble. The trouble becomes manifest if we try to apply our reasoning to the simplest analogous problem, the result of which is well known.

PROBLEM III. *Find the probability that x, a large integer chosen at random, is a prime number.*

By reasoning as we did in solving Problem I and assuming the independence of the probabilities involved as we did in solving Problem II we obtain the answer $\prod(p-1)/p$; the product is extended to all primes p not surpassing a cer-

tain bound—but what should be the bound? The number x is certainly prime if it is not divisible by any prime $p<x$. This leads to the evaluation of the desired probability

$$\prod_{p<x} \frac{p-1}{p} \sim \frac{\mu}{\log x} \tag{5}$$

where $\mu=0.561459 \cdots = e^{-c}$ and $c=0.577215 \cdots$ is the familiar constant of Mascheroni and Euler; the asymptotic evaluation in (5) (on the right hand side of the sign $\sim$) which is valid for $x \to \infty$, is due to Mertens.*

Now, the value (5) is too small. The probability in question is known to be $1/\log x$; this is just the prime number theorem. And we can "explain" somehow why the result is wrong: If the integer x is not divisible by any prime p which does not exceed $x^{1/2}$, x itself must be a prime—and so divisibility by primes exceeding $x^{1/2}$ is, in fact, *not* independent of the smaller primes.

Let us try to modify (5) by considering only primes p not exceeding $x^{1/2}$. This leads us to

$$\prod_{p \leqq x^{1/2}} \frac{p-1}{p} \sim \frac{\mu}{\log (x^{1/2})} = \frac{1.122 \cdots}{\log x} \tag{6}$$

(we used Mertens' result (5)) and this value is too large.

Let us, however, imitate the physicists who, without hesitation, modify their theories to fit the observed facts. And so let us do a thing between (5) and (6) and extend the product to all *primes not exceeding* x^μ. We obtain so

$$\prod_{p<x^\mu} \frac{p-1}{p} \sim \frac{1}{\log x}, \tag{7}$$

the right result.

I do not pretend to understand why the introduction of the upper bound x^μ *should* yield the right result. For that matter, when the quanta were introduced, no physicist pretended to understand why energy should be obtainable (as salt or sugar is in the self-service store) only in uniform little packages, in multipla of a certain unit. Yet the criterion of a physical theory is its applicability. Let us apply the (unintelligible) trick that gave us the right expression for the prime number theorem to our formula (3). Extending the second product to *odd* primes p inferior to x^μ, we are led to

$$\begin{aligned} P_d &= \prod_{p \mid d} \frac{p-1}{p-2} \cdot \frac{1}{2} \prod_{p<x^\mu} \frac{p-2}{p} \\ &\sim \prod_{p \mid d} \frac{p-1}{p-2} \cdot 2 \prod_{p<x^\mu} \frac{(p-2)p}{(p-1)^2} \frac{1}{(\log x)^2}; \end{aligned} \tag{8}$$

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* *Cf.* G. H. Hardy and E. M. Wright, An Introduction to the Theory of Numbers, Oxford, 1938, p. 349, Th. 430.

we have used Mertens' result (5). It is easily seen that (8) is equivalent to

$$P_d \sim 2C_2 \prod_{p|d} \frac{p-1}{p-2} \frac{1}{(\log x)^2}, \tag{9}$$

where C_2 stands for the convergent infinite product

$$\prod \left(1 - \frac{1}{(p-1)^2}\right)$$

extended to all odd primes $p=3, 5, 7, 11, \cdots$. The asymptotic formula (9) is due to Hardy and Littlewood, yet even their argument, which is incomparably deeper and more difficult than the one presented here, does not prove (9); it just confers on (9) another kind of plausible evidence. Yet all available numerical data also seem to support (9).

Let us recall that we have attained (9) by combining two analogies, one of which was extremely "natural" and the other (the "trick of the magic μ") extremely "artificial." And let us try to draw the moral: mathematicians and physicists think alike; they are led, and sometimes misled, by the same patterns of plausible reasoning.*

A CHAIN OF CYCLIC GROUPS

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1. Introduction. Consider the chain of groups $\mathfrak{G}_0, \mathfrak{G}_1, \cdots, \mathfrak{G}_i, \cdots$ where $\mathfrak{G}_0$ is a cyclic group of order m, and $\mathfrak{G}_i$ is the automorphism group of $\mathfrak{G}_{i-1}$, $i=1, 2, \cdots$. We ask when the chain consists entirely of cyclic groups. Obviously, when $m=1$ this will be so, and we suppose henceforth that $m>1$.

When $\mathfrak{G}_0$ is cyclic of order m with generator a it is well known that its automorphism group, $\mathfrak{G}_1$, is of order $t=\phi(m)$ and that $\mathfrak{G}_1$ is isomorphic to the multiplicative group modulo m of integers less than and relatively prime to m.

* See G. H. Hardy and J. E. Littlewood, Some problems of "Partitio numerorium": On the expression of a number as a sum of primes, Acta Math., vol. 44, 1922, pp. 1–70, especially Conjecture B on p. 42. The more general conjecture on p. 61 (Theorem X 1) is also obtainable by the foregoing reasoning. See also the literature quoted (and criticized) on pp. 32–34, especially the writings of Sylvester, concerning the use of probabilities in questions of similar nature. The crux of the matter may be so expressed: When we consider a fixed number of primes, the "probabilities" introduced can be regarded as "independent," but they cannot be so regarded when the number of primes considered increases in an arbitrary manner. (*Added in proof.* Profesor E. M. Wright drew my attention to a paper by the late Lord Cherwell in the Quart. J. Math., vol. 17, 1946, pp. 46–62, which has a certain contact with the present paper, and to a paper by Lord Cherwell and himself which is scheduled to appear in a coming volume of the Quarterly.)

2. $\mathfrak{G}_0$, $\mathfrak{G}_1$, $\mathfrak{G}_2$, and $\mathfrak{G}_3$. We write

$$\mathfrak{G}_1 = [1, a_2, \cdots, a_t],$$

where $1, a_1, a_2, \cdots, a_t$ are the $\phi(m)$ integers less than and relatively prime to m. Then, since $\mathfrak{G}_1$ must be cyclic, there exists an a_i such that $\phi(m)$ is the smallest exponent s with the property that $a_i^s \equiv 1 \pmod{m}$. That is, one of the a_i must be a primitive root of m. But this will occur* if and only if $m = p_0^{n_0}$, $m = 2p_0^{n_0}$, $m=2$, or $m=4$. ($p_0, p_1, p_2, \cdots$ will always denote an odd prime.) Under these circumstances, $\mathfrak{G}_2$ will be isomorphic to the multiplicative group mod $\phi(m)$ of order $\phi^2(m)$ of integers less than and relatively prime to $\phi(m)$.

If $m \neq 2$ or 4 and $\mathfrak{G}_2$ is also cyclic, we must have either

$$(1)\ \phi(m) = 2; \quad (2)\ \phi(m) = 4; \quad (3)\ \phi(m) = p_1^{n_1}; \quad \text{or} \quad (4)\ \phi(m) = 2p_1^{n_1}.$$

Case 1. $\phi(m) = 2$. Then $\phi(m) = p_0^{n_0-1}(p_0-1) = 2$, $p_0 = 3$, $n_0 = 1$, $m = 3$ or $2\cdot 3 = 6$.

Case 2. $\phi(m) = 4$. Then $\phi(m) = p_0^{n_0-1}(p_0-1) = 4$ so that $p_0 = 5$, $n_0 = 1$, and $m = 5$ or $2\cdot 5 = 10$.

Case 3. $\phi(m) = p_1^{n_1}$. Then $p_0^{n_0-1}(p_0-1) = p_1^{n_1}$, which is impossible.

Case 4. $\phi(m) = 2p_1^{n_1}$. Then $p_0^{n_0-1}(p_0-1) = 2p_1^{n_1}$.
(a) If $p_0 = p_1$, $p_0 = 3$, $n_1 = n_0 - 1$, $m = 3^{n_0}$ or $2\cdot 3^{n_0}$.
(b) If $p_0 \neq p_1$, $n_0 = 1$, $m = p_0 = 2p_1^{n_1}+1$ or $m = 2p_0$.

We continue, then, with Case 4b, where

$$m = p_0 = 2p_1^{n_1} + 1 \text{ or } m = 2p_0. \quad \text{Then} \quad \phi(m) = p_0 - 1 = 2p_1^{n_1}$$

and demand further that

$$\phi^2(m) = \phi(2p_1^{n_1}) = p_1^{n_1-1}(p_1 - 1)$$

be equal to either

$$(1)\ 2; \quad (2)\ 4; \quad (3)\ p_2^{n_2}; \quad \text{or} \quad (4)\ 2p_2^{n_2}.$$

Case 1. $p_1 = 3$, $n_1 = 1$, $m = 2\cdot 3+1 = 7$ or $2\cdot 7 = 14$.

Case 2. $p_1 = 5$, $n_1 = 1$, $m = 2\cdot 5+1 = 11$ or $2\cdot 11 = 22$.

Case 3. Impossible.

Case 4. (a) $p_1 = p_2$, $p_1 = 3$, $n_2 = n_1 - 1$, $m = p_0 = 2\cdot 3^{n_1}+1$ or $m = 2p_0$ where, then, $2\cdot 3^{n_1}+1$ must be a prime.

(b) $p_1 \neq p_2$, $n_1 = 1$, $p_1 = 2p_2^{n_2}+1$, $m = p_0 = 2p_1^{n_1}+1 = 2(2p_2^{n_2}+1)+1 = 4p_2^{n_2}+3$ or $m = 2p_0$.

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* See, for example, L. E. Dickson, Introduction to the Theory of Numbers, Chicago, 1929, p. 20.

3. The induction. We now perform an induction and investigate the passage from $\phi^{k-1}(m)$ to $\phi^k(m)$ where we suppose that $p_{k-1}\neq 3$ or 5 so that the process has not terminated. Thus, we assume that $p_{k-2}=2p_{k-1}^{n_{k-1}}+1$,

$$\phi^{k}(m) = p_{k-1}^{n_{k-1}}(p_{k-1} - 1) \quad \text{and} \quad n_{k-2} = 1$$

and demand that this be equal to

$$(1)\ 2; \qquad (2)\ 4; \qquad (3)\ p_k^{n_k}; \quad \text{or} \quad (4)\ 2p_k^{n_k}.$$

Case 1. $p_{k-1}=3$, $n_{k-1}=1$.

Case 2. $p_k-1=5$, $n_{k-1}=1$.

Case 3. Impossible.

Case 4. (a) $p_{k-1}=3$.
(b) $n_{k-1}=1$, $p_{k-1}=2p_k^{n_k}+1$.

Suppose first that, at some stage, we end with $p_{k-1}=5$. Then $p_{k-2}=2\cdot 5+1=11$, $p_{k-3}=2\cdot 11+1=23$, $p_{k-4}=2\cdot 23+1=47$, $p_{k-5}=2\cdot 47+1=95$. Since 95 is not a prime, $k=1$, 2, 3, or 4. Hence

$$p_0 = 5,\ 11,\ 23, \text{ or } 47 \quad \text{and} \quad m = 5,\ 10,\ 11,\ 22,\ 23,\ 46,\ 47, \text{ or } 94.$$

Now, suppose that, at some stage, $p_{k-1}=3$. Then $p_{k-2}=2\cdot 3^{n_{k-1}}+1$ and $p_{k-3}=2p_{k-2}^{n_{k-2}}+1=2(2\cdot 3^{n_{k-1}}+1)+1=4\cdot 3^{n_{k-1}}+3$ is not a prime. Hence $k=1$ or 2, $p_0=3^{n_0}$ or $2\cdot 3^{n_1}+1$ and $m=3^n$, $2\cdot 3^n$, $2\cdot 3^n+1$, or $2(2\cdot 3^n+1)$ where $2\cdot 3^n+1$ is a prime.

Finally, we note that, if we do not come to 3 or 5, the process must continue indefinitely. But since $p_0, p_1, \cdots$, *etc.*, form a decreasing sequence of primes, this is impossible.

4. Conclusions. The results obtained above give us the

THEOREM. *The chain of automorphism groups $\mathfrak{G}_0, \mathfrak{G}_1, \mathfrak{G}_2, \cdots$, consists entirely of cyclic groups if and only if $\mathfrak{G}_0$ is cyclic of order m where $m=1$, 2, 4, 5, 10, 11, 22, 23, 46, 47, 94, 3^n, $2\cdot 3^n$, $2\cdot 3^n+1$, or $2(2\cdot 3^n+1)$, where $2\cdot 3^n+1$ is a prime.*

Because the only Abelian groups with Abelian automorphism groups are the cyclic groups* the theorem above may be restated with the requirement that $\mathfrak{G}_0, \mathfrak{G}_1, \cdots$ be Abelian.

* G. A. Miller, On the groups which have the same group of isomorphisms, Trans. Amer. Math. Soc., vol. 1, 1900, pp. 395–401. It is interesting to note that this fact does not seem to be at all well known. Thus Kurosh, in his recent book on group theory, simply mentions that an Abelian group *need* not have an Abelian automorphism group and cites one example.

A TOURNAMENT PROBLEM

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Introduction. In his book,† Steinhaus discusses the problem of ranking n objects according to some transitive characteristic, by means of successive pairwise comparisons. In this paper we shall adopt the terminology of a tennis tournament by n players. The problem may be briefly stated: "What is the smallest number of matches which will always suffice to rank all n players?"

Steinhaus proposes an inductive method whereby, the first k players having been ranked, the $(k+1)$-st player is matched against the median player in the first k, and by a "halving" process is finally ranked into this chain. Then the $(k+2)$-nd player is ranked into the new chain of $k+1$ players in the same manner.

Using this process, a player can be ranked into a chain of k others in $S(k) = 1+[\log_2 k]$ matches. Steinhaus thus shows that $M(n)$ matches always suffice for n players where

$$M(n) = 1 + nS(n) - 2^{S(n)}.$$

He then states, "It has not been proved that there is no shorter proceeding possible, but we rather think it to be true."

The purpose of this note is to present an improved procedure, compare it with Steinhaus' $M(n)$ as an upper bound and with a lower bound $L(n)$ derived from information theory, and to discuss the asymptotic behavior of these three functions for large n.

A lower bound, $L(n)$, is easily seen to be $L(n)=1+[\log_2 (n!)]$, since each pairing can do no more than divide the remaining possibilities into two complementary sets; the results of the comparison then selects one or the other of these. Observing $n!$ possibilities initially, with halving the best we can do at each stage, we are led directly to the above formula for $L(n)$.

The improved procedure. This may be explained inductively in three steps (illustrated for the case $n=19$ by Fig. 1). Suppose $n=2r$ or $2r+1$.

1. Pair off $2r$ of the players and let the pairs play in the first round leaving one man out if n is odd.

2. By a continuation of the present method applied to r players, give a complete ranking of these r first round winners.

3. The third step is best explained by a diagram.

At this point in the ranking we have a hierarchy of the form illustrated in Figure 1 for $n=19$. First round winners $J, I, \cdots, B$ are ranked in that order with J the best player. A is the first round loser to B, and other first round losers

* Now at General Analysis Corporation.

† H. Steinhaus, Mathematical Snapshots, New York, 1950, pp. 37–40.

are indicated directly below their respective victors. The odd man drawing a first round bye is considered a loser and put in the position at the extreme left of the diagram.

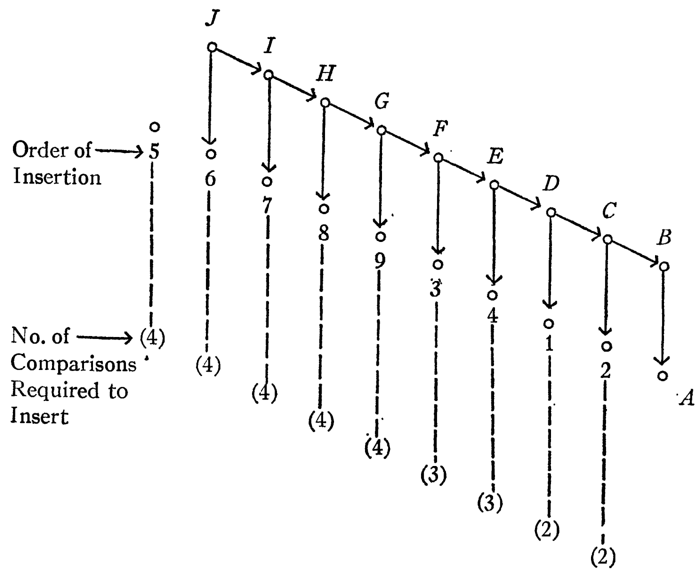

FIG. 1

The phrase "main chain" initially will refer to the chain $JIH \cdots CBA$, and the procedure will be to insert the numbered points in the main chain in the order indicated. The procedure is based on the fact that the insertion of a single point into a chain by the Steinhaus method is most efficient if the number of points on the chain is of the form 2^K-1.

Hence we start by inserting point 1 in the chain ABC. After this has been done, the "main chain" under point 2 consists of AB and possibly point 1; this insertion can also be performed with two comparisons.

We now turn to chains of length $2^3-1=7$, and observe that point 3 is as high as we can go, the main chain under 3 being composed of $ABCDE$ and 1 and 2, and so forth.

This represents a ranking technique, requiring $U(n)$ comparisons, where $U(n)$ is given recursively as follows.

$$U(1) = 0, \qquad U(2) = 1,$$

$$U(2k) = k + U(k) + \sum_{i=2}^{k} T(i),^*$$

$$U(2k+1) = k + U(k) + \sum_{i=2}^{k+1} T(i),$$

* The $T(i)$ are exactly the parenthetical numbers along the bottom of Figure 1.

where

$$T(i) = 2 \quad \text{for} \quad 1 < i \leqq 3,$$
$$T(i) = 3 \quad \text{for} \quad 3 < i \leqq 5,$$
$$T(i) = 4 \quad \text{for} \quad 5 < i \leqq 11,$$
$$\cdots\cdots\cdots\cdots$$
$$T(i) = j \quad \text{for} \quad t_{j-1} < i \leqq t_j,$$

and $t_j = \{2^{j+1} + (-1)^j\}/3$.

We shall now give a table of values of $M(n)$, $U(n)$, and $L(n)$ for some selected values of n, thereafter making some empirical observations.

n	1	2	3	4	5	6	7	8	9	10	11	12	13
$M(n)$	0	1	3	5	8	11	14	17	21	25	29	33	37
$U(n)$	0	1	3	5	7	10	13	16	19	22	26	30	34
$L(n)$	0	1*	3	5	7	10	13	16	19	22	26	29	33

FIG. 2

It may be observed that $U(n) = L(n)$ for $n = 20$ and 21 also; thus the proposed procedure is optimal for those values of n in addition to the values $n \leqq 11$. We conjecture that the value $L(12) = 29$ can not be achieved, but it seems to be difficult of proof. We further conjecture that $U(n)$ is best possible, for all n, but have no mathematical grounds on which to attack a similar conjecture regarding $L(n)$.

Asymptotic formulae. We state the following formulae for the asymptotic behavior of $U(n)$, $L(n)$, and $M(n)$. The proofs are sufficiently cumbersome to be omitted. (The formulae for $U(n)$ and $M(n)$ are based on the subsequence of n which appear to give local minima for $U(n) - L(n)$, *i.e.*, $n = [2^k/3]$ for some k. These being the "best" values, the comparison may be somewhat invidious.) The decimal values which appear as coefficients for n are, of course, approximations.

$$M(n) \sim n \log_2 n - .915n + O(\log_2 n),$$
$$U(n) \sim n \log_2 n - 1.415n + O(\log_2 n),$$
$$L(n) \sim n \log_2 n - 1.443n + O(\log_2 n).$$

On the other hand, for the subsequence of $n = 2^k$ for some k, which are the "best" values for $M(n)$, the coefficients of n in the expressions for $M(n)$ and $U(n)$ are -1 and -1.333, respectively.

* A more correct statement for $L(n)$ should be "the least integer $K \geqq \log_2 n!$." Our formula is equivalent except for $n = 2$.

DESCARTES AND THE GEOMETRIZATION OF ALGEBRA

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The great accomplishment of Descartes in mathematics invariably is described as the arithmetization of geometry [1]. Is it not true that his celebrated *Géométrie* of 1637 is far more concerned with algebra than with geometry [2]? Have we not read [3] that he "freed himself completely from the superstition of homogeneity?" Did not Descartes express the fundamental principle of analytic geometry?

> For the solution of any one of these problems of loci is nothing more than the finding of a point for whose complete determination one condition is wanting. . . . In every such case an equation can be obtained containing two unknown quantities [4].

At least one writer [5] consequently has jumped to the conclusion that "The real objects with which he dealt were numbers." It is the purpose of this paper to point out that all such assertions are but one side of a two-faced coin, for the truth is that Descartes had no intention of arithmetizing geometry. In fact, the purpose of *La géométrie* might with equal validity be described as the translation of algebraic operations into the language of geometry. The very first section of the work is entitled "How the calculations of arithmetic are related to the operations of geometry," and the second describes "How multiplication, division, and the extraction of square roots are performed geometrically." Also included in Book I of *La géométrie* are detailed instructions on the solution of quadratic equations, not in the algebraic sense of the ancient Babylonians or the medieval Arabs or the modern cossists, but geometrically, as in classical Greece. Book II of *La géométrie*, "On the nature of curved lines," comes close to the subject which now is known as analytic geometry, but Book III reverts to the theme of making algebra intelligible through geometric constructions of the roots of polynomial equations.

La géométrie was published as an appendix to the celebrated *Discours de la méthode*, and in the main treatise Descartes seems to have been partial neither to geometry nor to algebra. The former he accused of relying too heavily on diagrams which fatigue the imagination unnecessarily, and the latter he criticized as a confused and obscure art which embarrasses the mind [6]. The aim of his method, then, was two-fold: (1) Through algebraic procedure to free geometry from the use of diagrams, and (2) to give meaning to the operations of algebra through geometric interpretation. His work was indeed a linking of the two fields of geometry and algebra, but the association was bipolar and not prejudiced in favor of either direction.

The significance of Descartes' accomplishment becomes clearer when one recalls that the circumspect mathematicians of ancient Greece, shocked by the discovery of the incommensurable, had divided their field into two distinct halves—geometry (the study of continuous magnitude) and arithmetic (the

study of discrete number). An equation could form part of either half, but not both. If it arose in geometry, the unknown magnitude necessarily was to be exhibited as a line, an area, or a volume; in arithmetic the answer had to be either a whole number or a ratio of whole numbers. Quadratic equations, for example, appear both in Euclid's *Elements* and in Diophantus' *Arithmetic*, but the forms of the answers are quite different in the two cases. For cubic equations the geometric point of view predominated in Greece after it was found that such equations could be solved graphically through intersecting conic sections. The arithmetical equation $x^3=2$, for example, could not be solved, but the geometrical equation $x^3=2a^3$ was solved by finding the coordinate lines corresponding to the points of intersection of the parabolas $x^2=ay$ and $y^2=2ax$.

The decline of Greece and the rise of the Hindu and Arabic cultures led to the uncritical acceptance of irrational numbers, and the equation $x^3=2$ became arithmetically solvable. Until the early sixteenth century, however, it was generally believed that most cubic equations were arithmetically unsolvable, even in terms of radicals. The discovery, by del Ferro, Tartaglia, and Cardan, that all cubic equations are solvable numerically was a triumph which led to the development of algorithmic algebra without recourse to geometry, a direction which was encouraged by the discovery that imaginary cube roots could combine to give real numbers. Viète, the greatest algebraist of the sixteenth century, handled his equations so deftly by algebraic rules, despite his observance of the principle of homogeneity, that Descartes later criticized his work as marking too great a separation of algebra from geometry [**7**]. The obscurities of the new algebra repelled Descartes, and he set out to show that this subject, like geometry, is simply a description of magnitude. That is, Descartes in a sense was returning in thought to the ancient geometrical algebra, while at the same time he encouraged the development of symbolic forms of expression. In so doing Descartes initiated a great reform in mathematics, but he scarcely anticipated the form that this was to take in the nineteenth-century arithmetization of the subject.

Descartes was convinced that all mathematical sciences proceed from the same basic principles, and he decided to use the best of each branch. He began, in the *Discours*, by supposing all magnitudes, without exception, to be represented by lines, for he could think of nothing simpler which could be represented more distinctly to his imagination and his reason. But in order to hold them in mind and to understand them collectively, he found it necessary to express them in ciphers. In this way he felt that he would borrow the best of both geometric analysis and algebra, correcting the faults of each [**8**].

Descartes recognized that there are distinct advantages—in generality and in facility of handling—in thinking of every magnitude, whether an area, a volume, or any other quantity, as a line. It enables one to transcend three dimensions, and it obviates the necessity for a strict homogeneity in expressions to be combined. Descartes showed that one now could take the cube root of quantities such as a^2b^2-b, for one can think of the first term as divided once by

the unit line segment and the latter as multiplied twice by this unit. The cube root then will take the form which Descartes preferred; it will be a line segment [9]. Two things in particular should be noted here. In the first place, Descartes did not exactly abolish the principle of homogeneity; he simply substituted a mental geometric homogeneity for an apparent algebraic inhomogeneity. Secondly, and more importantly, it will be remarked that Descartes' basic magnitudes are not numbers at all; they are line segments, in conformity with the emphasis which Descartes placed on the idea of extension in his philosophical system. Descartes' approach would geometrize even arithmetic itself, oblivious of the qualms felt by Greek thinkers (qualms stilled much later by the Cantor-Dedekind axiom), and at least one seventeenth-century commentator [10] subsumed many of the Cartesian constructions under the caption "Arithmétique par Géométrie."

Descartes was not thoroughgoing in his geometrization of numbers, despite his realization that one of the chief advantages of algebra was its generality. Statements found in many a reputable history of mathematics notwithstanding, he did not systematically explain the graphical status of negative numbers [11], and imaginary numbers were completely beyond the geometric pale. In a vague sort of way Descartes realized that when a line turned out to be negative it was to be drawn in a sense contrary to that which had been taken as positive, but so little did he appreciate the significance of this that in 1638 he sketched his folium as a leaf only, restricting himself to positive values of the coordinates [12]. Occasionally he made use of negative ordinates, but not of negative abscissas.

The cavalier attitude of Descartes toward negative coordinates undoubtedly was a consequence of the goal of *La géométrie*. The object of the work was not the sketching of curves, as is now often assumed. There is in the whole of *La géométrie* not a single new curve plotted directly from its equation. Descartes' purpose, as brought out in the last book of the treatise, is the geometric construction of the roots of polynomial equations, using intersecting curves of lowest possible degree. This is, of course, reminiscent again of Greek geometric algebra, where the search for a geometric solution of $x^3 = 2a^3$ had led to the discovery of the conic sections. That so much of *La géométrie* is concerned with the theory of equations—the number of possible positive roots, increasing and decreasing the roots of an equation, finding rational roots, and depressing the degree when a root is known—does not indicate a preference for algebra over geometry. It stems from the need to know, if one is to use the simplest possible curves for the geometric solution, whether or not a given equation is reducible [13]. While Descartes did indeed convert geometric problems into the language of algebra, this was not an end in itself; it was but an intermediate step in determining the most appropriate geometrical construction. Consequently his theory of equations, being an algorithm of lines rather than of numbers, is not more appropriately described as an arithmetization of geometry than as a geometrization of algebra.

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NOMOGRAPHIC SOLUTIONS FOR POSITION RELATIONS BETWEEN A CLOSE EARTH SATELLITE AND ITS OBSERVER

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Introduction. The routine ephemeris predictions for an artificial satellite can be expected to give for any instant only the latitude and longitude of the subsatellite point, together with the height of the satellite above the earth's surface. The problem would then remain to determine, for an individual observer not directly under the satellite, its apparent azimuth, and altitude or elevation above the horizon; also its distance or slant range.

Although the formulas for such relations are quite simple and easy to apply for those versed in computation, for repeated use requiring only moderate accuracy, and especially for nontechnical observers not possessing mathematical tables, simple charts or nomograms for quick graphical solutions are desirable.

Solution for the ground range and azimuth. The ordinary astronomical triangle of spherical trigonometry yields the following equation for the arc distance or ground range d° of the subsatellite point as related to the observer's

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latitude ϕ_0, the satellite latitude ϕ, and the longitude difference $\Delta\lambda$:

$$\cos d^\circ = \sin \phi_0 \sin \phi + \cos \phi_0 \cos \phi \cos \Delta\lambda. \tag{1}$$

The law of sines gives the satellite's apparent azimuth A:

$$\sin A = \sin \Delta\lambda \cos \phi \csc d^\circ. \tag{2}$$

Numerical solutions for these equations are tabulated in *Hydrographic Office Publication No. 214*. A universal graphical solution has been published in the form of a single nomogram by Hughes [1]. It consists of two parallel scales and a network of curved line scales, one curve for every 5° of observer's latitude. A more workable chart would be obtained by omitting all but two adjacent curved line scales, printing up all such combinations and choosing for each observer the chart appropriate to his latitude interval. Another simplification would be to print a separate chart for the satellite latitude—observer latitude—longitude difference—ground range relation, and another for the satellite latitude—longitude difference—ground range—azimuth relation. In the published tables and charts for which notations fit the astronomical triangle, replace *Hour Angle* by *Longitude Difference*, *Declination* by *Satellite Latitude*, and *Altitude* by 90° minus *Range*.

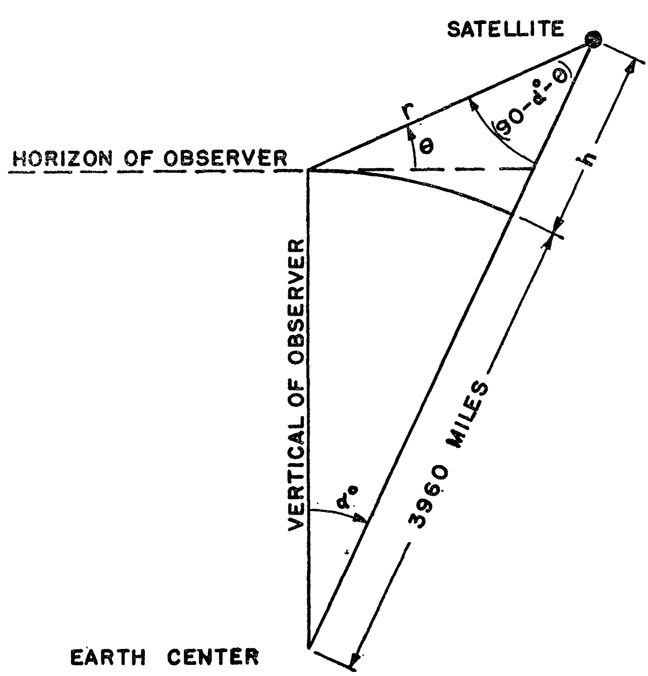

FIG. 1

Solution for the apparent altitude or elevation. The elevation θ is related to the ground range d° and the satellite height h in miles (Fig. 1) by the equation

$$\tan\theta = \cot d^\circ - \left(\frac{3960}{3960+h}\right)\csc d^\circ. \tag{3}$$

This equation, like (1), is of the form [2]

$$f_1(u) = f_4(w) - f_2(v)f_3(w) \tag{4}$$

depictable on a nomogram consisting of two parallel scales and a curved scale. Two parallel vertical scales, linear in $\tan\theta$ and $3960/(3960+h)$ (but labelled only in θ and h) are constructed with an arbitrary metric separation of K. Then, if m_θ and m_h are the metric lengths of units of $\tan\theta$ and $3960/(3960+h)$, respectively, also if the X-axis is horizontal through the zero points of these functions, with origin at the left scale, the metric coordinates of points on a curved d° scale are given by

$$X_{d^\circ} = \frac{Km_\theta \csc d^\circ}{m_\theta \csc d^\circ + m_h}, \qquad Y_{d^\circ} = \frac{m_\theta m_h \cot d^\circ}{m_\theta \csc d^\circ + m_h}. \tag{5}$$

A straightedge laid over the chart will then always intersect readings of θ, d°, and h which satisfy (3).

Solution for the distance or slant range. The slant range r in miles is given by

$$r = (3960 \sin d^\circ)[\sec(d^\circ+\theta)] \tag{6}$$

which may be represented by a so-called Z-chart, an alignment chart constructed as follows [**3**]. Parallel vertical scales of arbitrary separation, linear in r and $3960 \sin d^\circ$ (but labelled only in r and d°), have the first with zero point at the bottom and the second with zero point at the top, and a diagonal of metric length K connecting these zero points. Then, if m_r and m_{d° are the metric lengths of units of r and $3960 \sin d^\circ$, respectively, the distance along the diagonal from its upper right end to a point corresponding to $d^\circ+\theta$ is given by

$$X_{d^\circ+\theta} = \frac{K}{(m_r/m_{d^\circ})\sec(d^\circ+\theta)+1}. \tag{7}$$

A straightedge will then always intersect values of r, $d^\circ+\theta$, and d° which satisfy (6).

Summary. At each station for observing a close satellite there will be required only four simple nomograms and a straightedge to obtain from a routine published satellite ephemeris the satellite's azimuth, elevation, and slant range at any instant. The first two nomograms will give the azimuth and ground range, respectively, of the subsatellite point from the known geographical positions of observer and satellite, the third nomogram will give the apparent elevation of the satellite from its known ground range and linear height above the

earth, and the fourth nomogram will give the distance or slant range of the satellite from its known ground range and altitude. No computation or tables are required for such results, which will have accuracies of the order of 1° and 5 to 10 miles, depending only on the precision of scale readings at the intersections with an overlaid straightedge.

The author is indebted to Drs. John P. Hagen and Joseph W. Siry for discussion of this problem, and to Mrs. Mary P. Hann and Miss Ann Eckels for computing and drafting the nomogram scales.

Example. Suppose an observer, located at latitude 35° N and longitude 85° W, is looking for a satellite at latitude 30° N, longitude 70° W, and height 300 miles. It is required to find the satellite's azimuth, elevation, and slant range.

1. On the NOMOGRAM FOR SATELLITE GROUND RANGE, LATITUDES OF OBSERVER AND SATELLITE, AND LONGITUDE DIFFERENCE, lay a straightedge between the 15° point on the scale for Satellite-Observer Difference in Longitude and the intersection of the area marked "North and North" of curves marked "Satellite Latitude 30°" and "Observer's Latitude 35°." (For convenience, note that the base of the latter curve on the "Ground Range" scale is at $90°-35°=55°$.) The straightedge will then intersect the Ground Range scale at a point which may be estimated as about 13°. This ground range is in degrees of arc on a great circle; for present purposes there is no need to convert it to statute or nautical miles.

2. In examples such as the present one, where the difference in longitude is less than 20°, ground range may be more conveniently and accurately determined on the APPROXIMATE NOMOGRAM FOR GROUND RANGE, AND LATITUDE AND LONGITUDE DIFFERENCES BETWEEN OBSERVER AND SATELLITE. On this lay one straightedge between the 15° point on the $\Delta\lambda$ scale and $\phi_0=35°$ on the diagonal scale above it. From the point where this intersects the horizontal T-scale, extend a second straightedge to the 5° point on the $\Delta\lambda$ scale at the top. The second straightedge is then seen to intersect the $d°$ (ground range) scale at 13.6°.

3. On the NOMOGRAM FOR SATELLITE AZIMUTH, GROUND RANGE, AND LATITUDES OF OBSERVER AND SATELLITE, extend a straightedge between the 30° point on the Satellite Latitude scale and the intersection of curves marked "Observer's Latitude 35°" and "Ground Range—Observer North" corresponding to 13.6°. (For convenience note that the latter curve would intersect the Observer's Latitude scale at $90°-13.6°=76°$.) The straightedge will then intersect the Azimuth from North scale at 107°.

4. On the NOMOGRAM FOR HEIGHT, GROUND RANGE, AND ELEVATION ANGLE, lay a straightedge between the 300-mile point on the h (height) scale at the extreme right and the point on the $d°$ (ground range) curve to its left corresponding to $d°=13.6°$. This straight edge will then intersect the θ (elevation) scale at 10.2°, the apparent altitude of the satellite above the horizon.

5. On the NOMOGRAM FOR GROUND RANGE, SLANT RANGE, AND ELEVATION ANGLE, lay a straightedge between the 13.6° point on the $d°$(ground range) scale at the right and the point on the diagonal scale corresponding to $d°+\theta=13.6°+10.2°=23.8°$. This straightedge will then intersect the r (slant range) scale at 1018 statute miles.

Larger scale charts are available on request to Code 4143, Beltsville Space Center, National Aeronautics and Space Administration, Washington 25, D. C.

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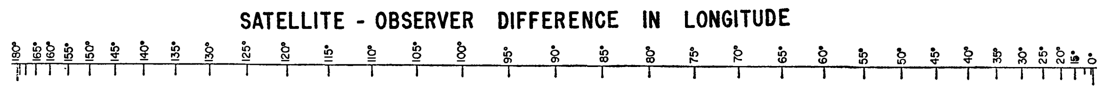
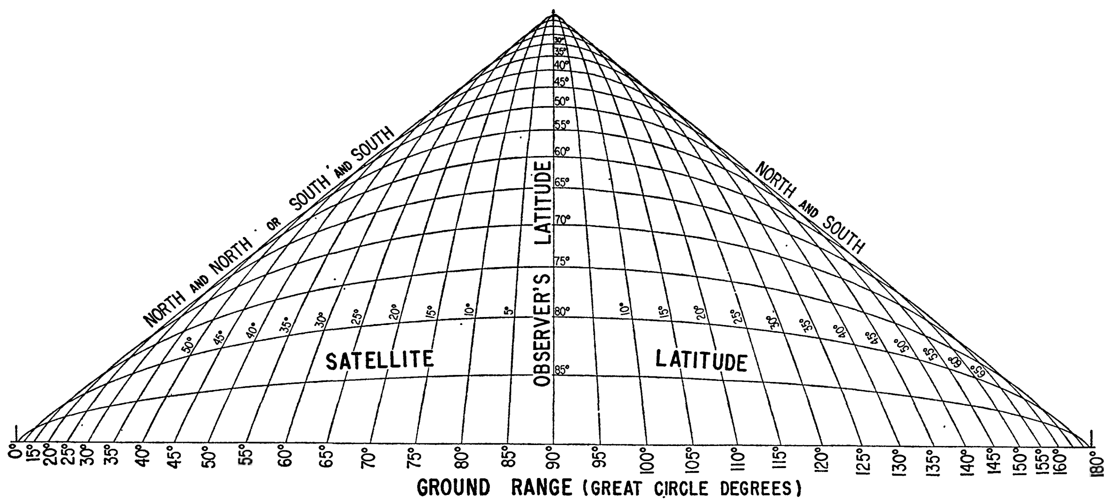
2. A. S. Levens, Nomography, New York, 1948, p. 99.

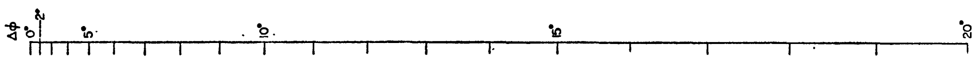
3. ———, Nomography, New York, 1948, p. 44.



NOMOGRAM FOR SATELLITE GROUND RANGE, LATITUDES OF OBSERVER AND SATELLITE AND LONGITUDE DIFFERENCE

(Straight Edge Connects Corresponding Values Using the Intersection of Satellite and Observer Latitudes)





APPROXIMATE NOMOGRAM FOR GROUND RANGE, AND LATITUDE AND LONGITUDE DIFFERENCES BETWEEN OBSERVER AND SATELLITE

(Two Straight Edges Intersecting on T-Scale Connect Corresponding Values)

$(d^{\circ})^2 = (\Delta\phi)^2 + (\Delta\lambda)^2 \cos^2 \phi_0$

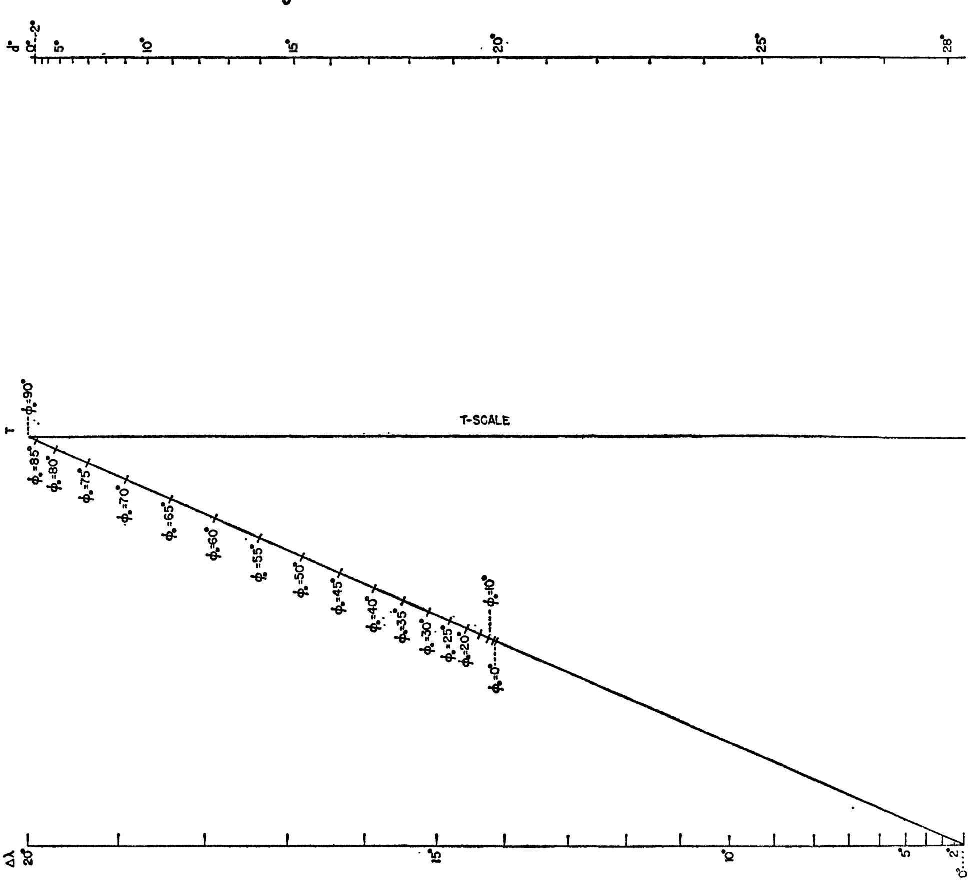
useful approximation when $\Delta\lambda < 20^{\circ}$ and $\phi_0 < 60^{\circ}$

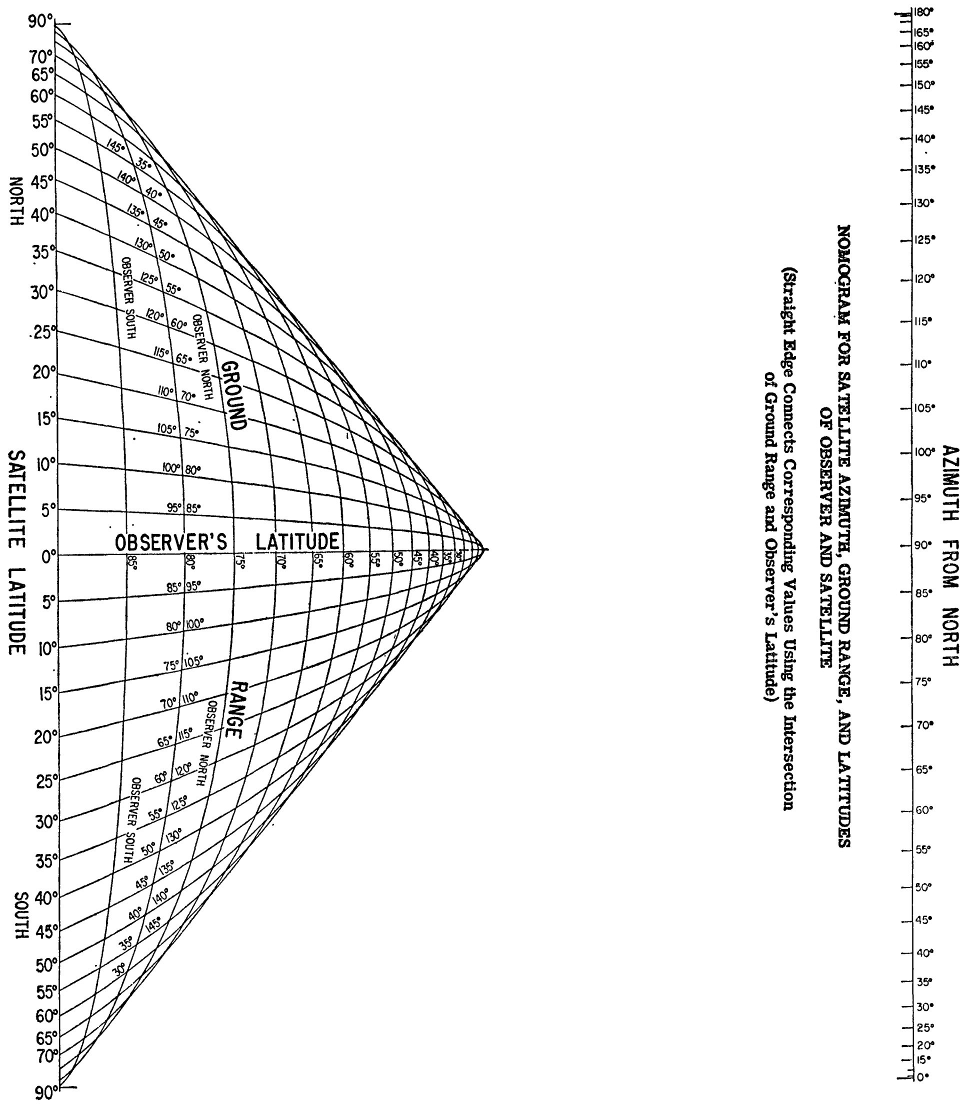
d° = ground range

$\Delta\phi$ = difference in latitude between observer and satellite

$\Delta\lambda$ = difference in longitude between observer and satellite

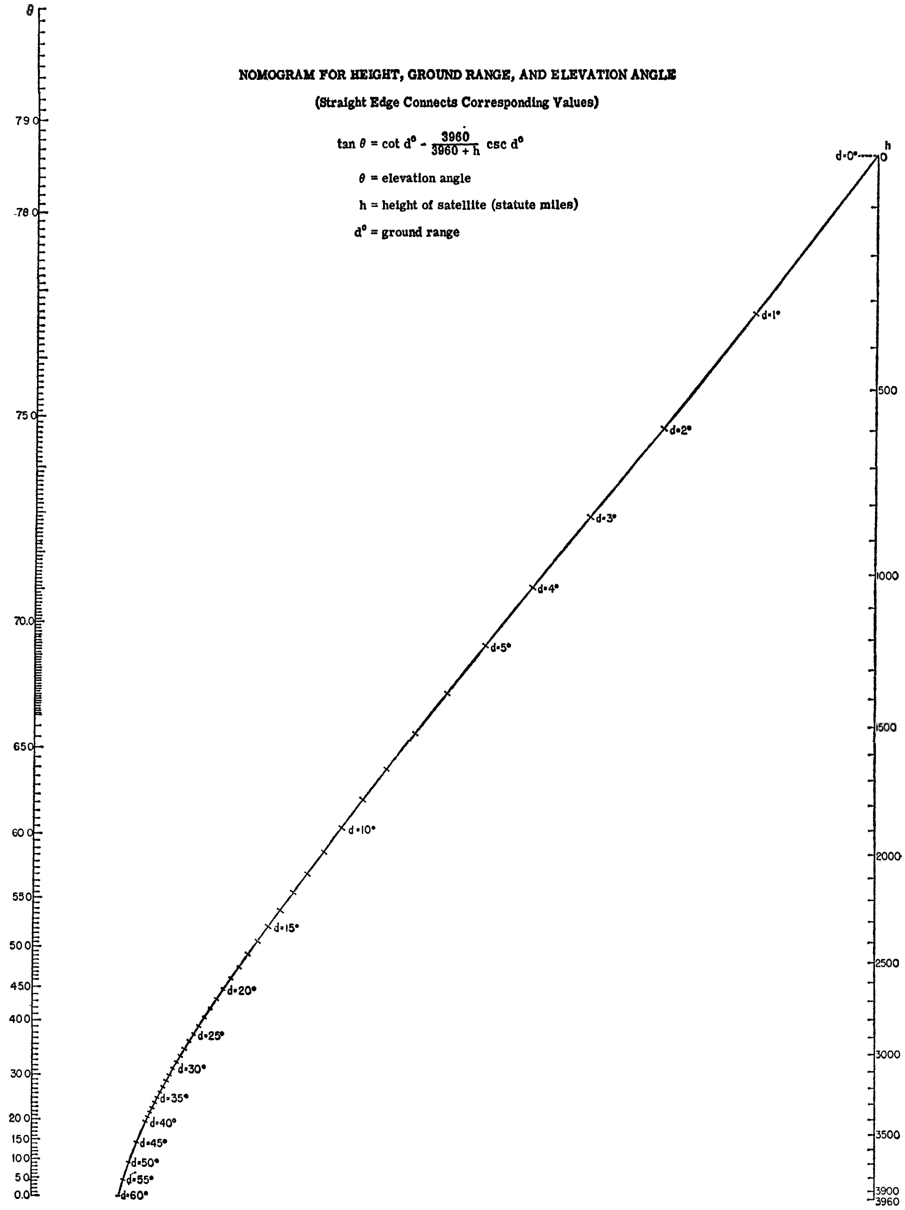
ϕ_0 = latitude of observer





NOMOGRAM FOR SATELLITE AZIMUTH, GROUND RANGE, AND LATITUDES OF OBSERVER AND SATELLITE

(Straight Edge Connects Corresponding Values Using the Intersection of Ground Range and Observer's Latitude)


NOMOGRAM FOR HEIGHT, GROUND RANGE, AND ELEVATION ANGLE
(Straight Edge Connects Corresponding Values)
$\tan\theta = \cot d^\circ - \frac{3960}{3960+h}\csc d^\circ$
θ = elevation angle
h = height of satellite (statute miles)
d° = ground range
θ
79.0
78.0
75.0
70.0
65.0
60.0
55.0
50.0
45.0
40.0
30.0
20.0
15.0
10.0
5.0
0.0
h
d=0°
0
500
1000
1500
2000
2500
3000
3500
3900
3960
d=1°
d=2°
d=3°
d=4°
d=5°
d=10°
d=15°
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d=35°
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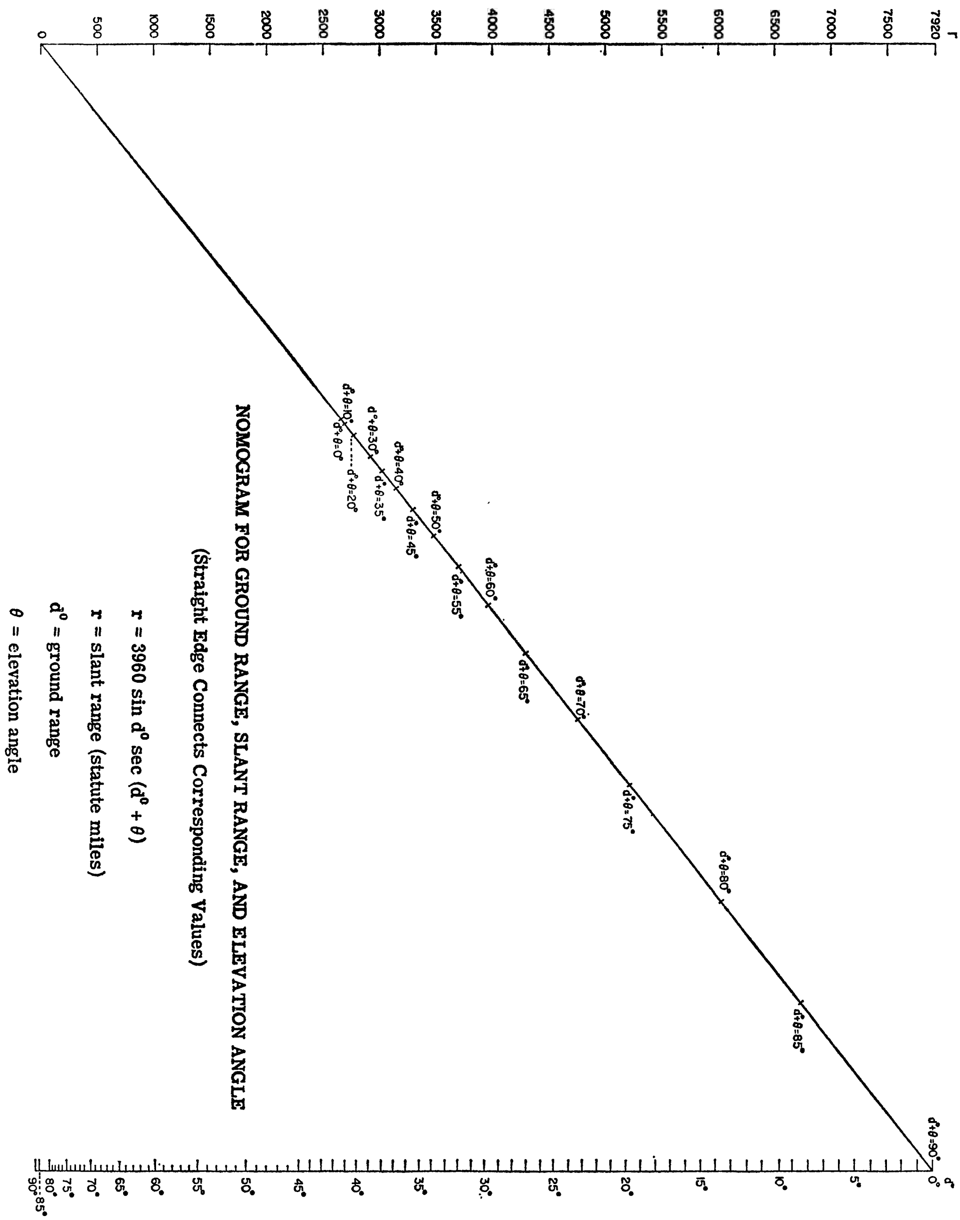
d=45°
d=50°
d=55°
d=60°



NOMOGRAM FOR GROUND RANGE, SLANT RANGE, AND ELEVATION ANGLE

(Straight Edge Connects Corresponding Values)

$r = 3960 \sin d^0 \sec (d^0 + \theta)$

r = slant range (statute miles)

d^0 = ground range

θ = elevation angle

MATHEMATICAL NOTES

EDITED BY ROY DUBISCH, Fresno State College

Material for this department should be sent to Roy Dubisch, Department of Mathematics, Fresno State College, Fresno 26, California.

A RODRIGUES' FORMULA

FRANK J. PALAS, Southern Methodist University and University of Oklahoma

In this note we shall discuss a set of polynomials which is defined by a generating function and is shown to satisfy a Rodrigues' formula and several recurrence relations. A portion of this work generalizes the results of a paper by Humbert [1] concerning a set of polynomials of even degree.

DEFINITION. *A k-set of polynomials $\{P_{kn}(x)\}$ is a sequence of polynomials such that $P_{kn}(x)$ is of exactly degree kn, $n=0, 1, 2, \cdots$; k a natural number.*

The k-set with which we shall be concerned is generated by

$$g(x, t) = (1 - t)^{-1} \exp [x^k u(t)] = \sum_{n=0}^{\infty} T_{kn}(x)t^n,$$

where $u(t)=1-(1-t)^{-k}$. The k-set $\{T_{kn}(x)\}$ is determined by a recurrence relation which we shall proceed to develop. We see that

$$\frac{\partial}{\partial x} g(x, t) = g(x, t)[kx^{k-1}u(t)].$$

This may be written as

$$(1 - t)^k \frac{\partial}{\partial x} g(x, t) = kx^{k-1}[(1 - t)^k - 1]g(x, t).$$

Writing each side of this equation as a power series in t and equating corresponding powers of t, we find that

$$\sum_{i=0}^{k} \binom{k}{i} (-1)^i T'_{k(n-i)}(x) = kx^{k-1} \sum_{i=1}^{k} \binom{k}{i} (-1)^i T_{k(n-i)}(x).$$

This equation yields the recurrence relation

$$T'_{kn}(x) = \sum_{i=1}^{k} (-1)^i \binom{k}{i} [kx^{k-1}T_{k(n-i)}(x) - T'_{k(n-i)}(x)]. \tag{1}$$

This recurrence relation may be used to show that the k-set $\{T_{kn}(x)\}$ satisfies a Rodriques' formula. However another method is more convenient.

Maclaurin's theorem allows us to write

$$g(x, t) = \sum_{n=0}^{\infty} \frac{1}{n!}\left[\frac{\partial^n}{\partial t^n} g(x, 0)\right] t^n = \sum_{n=0}^{\infty} T_{kn}(x) t^n,$$

so that

$$\begin{aligned} T_{kn}(x) &= \frac{1}{n!} \frac{\partial^n}{\partial t^n} g(x, 0) \\ &= \frac{\exp (x^k)}{n!} \frac{\partial^n}{\partial t^n} \left[(1 - t)^{-1} \exp \left[-x/(1 - t)\right]^k\right]_{t=0} \qquad (2) \\ &= \frac{\exp (x^k)}{n!} \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} x^{kr}(kr + 1)_n, \end{aligned}$$

where $(kr+1)_n=(kr+1)(kr+2) \cdots (kr+n)$. On the other hand we find

$$D^n[x^n \exp (-x^k)] = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} x^{kr}(kr + 1)_n.$$

Comparing this with the previous result we are able to state

THEOREM 1. *The k-set $\{T_{kn}(x)\}$ defined by the generating function*

$$g(x, t) = (1 - t)^{-1} \exp (x^k[1 - (1 - t)^{-k}])$$

satisfies the Rodrigues' formula $T_{kn}(x) = \exp (x^k)D^n[x^n \exp (-x^k)]/n!$.

The explicit formula

$$T_{kn}(x) = \frac{1}{n!} \sum_{p=0}^{n} \frac{x^{kp}}{p!} \sum_{r=0}^{p} (-1)^r \binom{p}{r} (kr + 1)_n$$

is a direct consequence of equation (2).

As an application of Theorem 1 we show that the k-set $\{T_{kn}(x)\}$ satisfies a property related to orthogonality.

THEOREM 2. *The moments of $\{T_{kn}(x)\}$ with respect to the weight function $w(x) = \exp [-x^k]$ have values*

$$M(s, n) = \begin{cases} 0, & s = 0, 1, \cdots, n - 1; \\ (-1)^n \binom{s}{n} \dfrac{n!}{k} \Gamma\left(\dfrac{s + 1}{k}\right), & s = n, n + 1, \cdots, \end{cases}$$

where $M(s, n) = \int_0^\infty w(x)x^s T_{kn}(x)dx$.

Proof. By means of the Rodrigues' formula we have

$$M(s, n) = \int_0^\infty x^s D^n[x^n w(x)]dx$$

$$= \sum_{i=0}^{r-1} (-1)^i i! x^{s-i} D^{n-1-i}[x^n w(x)] + (-1)^r \binom{s}{r} r! \int_0^\infty x^{s-r} D^{n-r}[x^n w(x)] dx.$$

This expression is the result of the rth integration by parts.

If $s=0, 1, \cdots, n-1$ then for $r=s$ we obtain $M(s, n)=Q(x)w(x)]_0^\infty=0$, since $Q(x)$ is a polynomial with the property $Q(0)=0$.

If $s=n, n+1, \cdots$ then for $r=n$ we have

$$M(s, n) = (-1)^n \binom{s}{n} n! \int_0^\infty x^s w(x) dx = (-1)^n \binom{s}{n} \frac{n!}{k} \Gamma\left(\frac{s+1}{k}\right).$$

For the case $k=1$ the polynomial set is the Laguerre type of order zero and the property of Theorem 2 is equivalent to orthogonality.

The Rodrigues' formula may also be used to develop a particularly simple recurrence relation. We rewrite the formula as

$$(r+1)! \exp(-x^k) T_{k(r+1)}(x) = D^{r+1}[x \cdot x^r \exp(-x^k)]$$

and apply the Leibniz formula. After some rearrangement we obtain the recurrence relation

$$(3) \qquad (r+1) T_{k(r+1)}(x) = x T'_{kr}(x) + (r + 1 - kx^k) T_{kr}(x).$$

As a closing remark we mention that the k-set $\{T_{kn}(x)\}$ satisfies a differential equation of order $k+1$. We make use of the relation (1) rewritten as

$$(4) \qquad T'_{k(n+k)}(x) = \sum_{i=1}^{k} (-1)^i \binom{k}{i} [kx^{k-1} T_{k(n+k-i)}(x) - T'_{k(n+k-i)}(x)]$$

and

$$\begin{aligned} (r+1) T^{(m)}_{k(r+1)}(x) &= x T^{(m+1)}_{kr}(x) + (r + m + 1 - kx^k) T^{(m)}_{kr}(x) \\ (5) \qquad &\quad - k \sum_{i=1}^{m} \binom{m}{i} \binom{k}{i} i! x^{k-i} T^{(m-i)}_{kr}(x), \qquad m \leqq k, \end{aligned}$$

which is an immediate consequence of relation (3). An iterative procedure with the relations (3) and (5) yields

$$(6) \qquad (r+1)_m T_{k(r+m)}(x) = x^m T^{(m)}_{kr}(x) + \sum_{i=0}^{m-1} p_{im}(x) T^{(i)}_{kr}(x),$$

where $p_{im}(x)$ is a polynomial of degree $ik-i+m$. When relation (6) and its derivative are inserted into (4) it follows that $T_{kn}(x)$ satisfies a homogeneous linear differential equation of order $k+1$ with polynomial coefficients.

Reference

1. P. Humbert, Sur certains polynômes orthogonaux, C. R. Acad. Sci. Paris, vol. 176, 1923, pp. 1282–1284.

AN ELEMENTARY THEOREM ON GRAPHS

FRANK HARARY, University of Michigan, Princeton University and Institute for Advanced Study

The following theorem, while very easy to state and prove, does not appear to be in the literature. (See König [1] as a general reference on graph theory.) Let G be a connected graph and r be the number of cut points of G. A *block* of a graph is a maximal connected subgraph containing no cut points of itself. Let N be the number of blocks of G. Let n_i be the number of components of the subgraph of G obtained on removing the ith point, and let the points of G be numbered so that the cut points are the first r points. Then,

$$N - \sum_{i=1}^{r} n_i + r = 1. \tag{1}$$

Before proving this result, we verify it for trees, *i.e.*, connected graphs with no cycles. For any graph G, let p and q be the number of points and the number of lines respectively. If G is connected and m is the number of independent cycles of G, then the well-known Euler characteristic can be stated in the form

$$p - q + m = 1. \tag{2}$$

For trees, this specializes at once to

$$p - q = 1. \tag{3}$$

It is convenient in verifying equation (1) for trees to first restate it in the following form:

$$N - \sum_{i=1}^{p} n_i + p = 1. \tag{1'}$$

It is of course obvious that (1′) is equivalent to (1) since the quantity $p-r$ has been both added and subtracted to the left side of equation (1) to obtain (1′). We note that $n_i = 1$ whenever $i > r$.

Now for trees, n_i is the degree of the ith point and $N = q$ since each line is a block. But the sum of the degrees of all the points of any graph is $2q$. Therefore for trees, equation (1′) immediately becomes equation (3).

We now prove (1) for arbitrary connected graphs by induction on r. If $r = 0$, then $N = 1$ and equation (1) is trivial. For connected graphs in which $r > 0$, an *endblock* is a block of G which contains exactly one cut point of G. Just as every tree has at least one endpoint, so does every connected graph with more than one block contain at least one endblock. As our inductive hypothesis, we assume that the theorem is true for all connected graphs containing exactly r cut points and consider a connected graph G having exactly $r+1$ cut points. Clearly, when $r > 1$ there exists a cut point c of G such that exactly one block at c is not an endblock. Let H be the graph obtained from G by removing all the end-

blocks of G which contain c. Then H has exactly r cut points and is connected. Therefore, by the inductive hypothesis, equation (1) holds for H. Thus if $N(H)$ is the number of blocks of H, then $N(H)-\sum_{i=1}^{r} n_i+r=1$. But obviously $N(G)-N(H)=n_{r+1}-1$. Therefore equation (1) holds for G and the proof is completed. We illustrate with Figure 1.

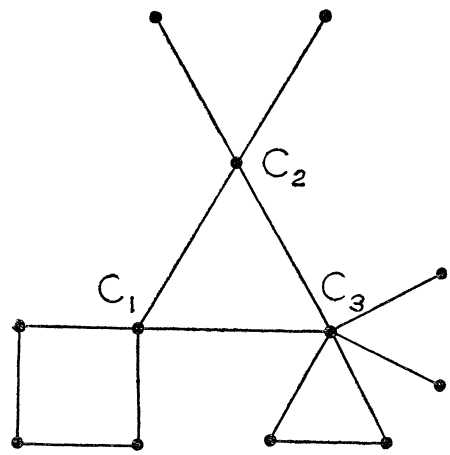


FIG. 1

Here $r=3$, $N=7$, $n_1=2$, $n_2=3$, $n_3=4$, and equation (1) is verified.

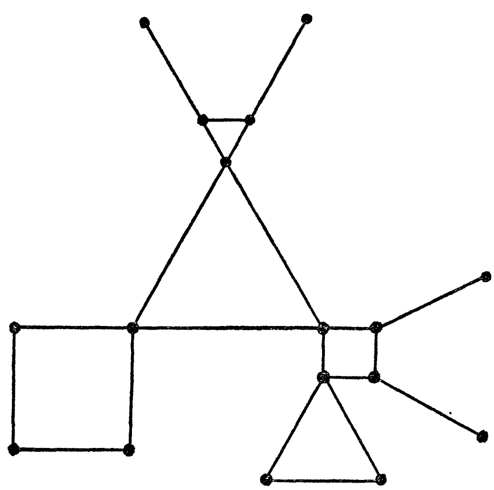


FIG. 2

There is a simple alternative proof of equation (1) which we now outline; it resembles one of the steps in a proof of the five color theorem. First, note that equation (1) is easily verified for connected graphs in which each cut point lies

on exactly two blocks. Then to prove equation (1) for any connected graph G, replace each cut point lying on more than two blocks by a cycle each of whose points is on exactly one of the blocks on which this point lies. We illustrate by showing in Figure 2 the graph obtained by modifying Figure 1 in this way. It is clear that such a modification of a connected graph does not affect the left-hand member of equation (1).

Reference

1. D. König, Theorie der endlichen und unendlichen Graphen. Leipzig, 1936; reprinted New York, 1950.

SOME FORMULAS PERTAINING TO A TETRAHEDRON*

VICTOR THÉBAULT, Tennie, Sarthe, France

Bretschneider gave, without proof, the relations

$$
\begin{aligned}
a^2 + a'^2 + 2aa' \cot a \cot a' &= b^2 + b'^2 + 2bb' \cot b \cot b' \\
&= c^2 + c'^2 + 2cc' \cot c \cot c'
\end{aligned} \tag{1}
$$

among the lengths of the edges and the cotangents of the dihedral angles of a tetrahedron [1]. J. Hecquet rediscovered them and established them by elementary geometry [2]. R. Deaux and V. Thébault gave further details [3]. It might be of interest to deduce equations (1) directly from some formulas obtained by G. Dostor [4] and T. C. Lewis [5], as this gives an interpretation of the constant value of the quantities (1).

Let a, a', b, b', c, c' denote the lengths of the edges BC, DA, CA, DB, AB, DC, as well as the corresponding dihedral angles, of a tetrahedron $T \equiv ABCD$, and let (A), (B), (C), (D), V, and R denote the areas of the faces BCD, CDA, DAB, ABC of T, the volume of T, and the radius of the sphere $ABCD$. Let us set

$$a^2 + a'^2 = \alpha, \qquad b^2 + b'^2 = \beta, \qquad c^2 + c'^2 = \gamma.$$

According to Dostor [6], the volume of a tetrahedron is equivalent to two thirds of the product of two faces by the sine of the included dihedral angle divided by the length of the edge of this dihedral angle. Thus

$$V = 2(A)(D)(\sin a)/3a = 2(B)(C)(\sin a')/3a' = \cdots,$$

whence

$$
\begin{aligned}
V^2 &= 4(A)(B)(C)(D)(\sin a)(\sin a')/9aa' \\
&= 4(A)(B)(C)(D)(\sin b)(\sin b')/9bb' = \cdots .
\end{aligned} \tag{2}
$$

Furthermore, T. C. Lewis gives the formulas [5]

* Translated from the French by W. E. Byrne, Virginia Military Institute.

$$
(3)\qquad
\begin{aligned}
256(A)(B)(C)(D)\cos a\cos a' &= -288\alpha V^2 + H,\\
256(A)(B)(C)(D)\cos b\cos b' &= -288\beta V^2 + H,\\
256(A)(B)(C)(D)\cos c\cos c' &= -288\gamma V^2 + H,
\end{aligned}
$$

where

$$
\begin{aligned}
H = {} & 144(\alpha+\beta+\gamma-4R^2)V^2\\
& + a^2a'^2(\alpha-\beta)(\alpha-\gamma)\\
& + b^2b'^2(\beta-\alpha)(\beta-\gamma)\\
& + c^2c'^2(\gamma-\alpha)(\gamma-\beta).
\end{aligned}
$$

After dividing any one of equations (3) by $288V^2$ and making use of (2) we obtain for the common value of the quantities (1),

$$
(4)\qquad H/288V^2 = (\alpha+\beta+\gamma-4R^2)/2 + \sum a^2a'^2(\alpha-\beta)(\alpha-\gamma)/288V^2.
$$

If T is isosceles, relations (1) and (4) yield

$$
\begin{aligned}
&2a^2/\sin^2 a = 2b^2/\sin^2 b = 2c^2/\sin^2 c\\
&= \frac{3(a^2+b^2+c^2)}{4} + \frac{a^4(a^2-b^2)(a^2-c^2)+b^4(b^2-c^2)(b^2-a^2)+c^4(c^2-a^2)(c^2-b^2)}{(b^2+c^2-a^2)(c^2+a^2-b^2)(a^2+b^2-c^2)}.
\end{aligned}
$$

Because of (2), relations (1) also reduce to

$$
\alpha + k\cos^2 a = \beta + k\cos^2 b = \gamma + k\cos^2 c, \qquad k = 2S^4/9V^2,
$$

where S is the area of a face of T. It can also be shown that k is the ratio of the area of any face of T to the sine of the supplementary trihedral angle of the trihedral angle opposite this face.

If T is orthocentric, then $\alpha=\beta=\gamma$, and relations (1) reduce to

$$
\cos a\cos a' = \cos b\cos b' = \cos c\cos c',
$$

which characterizes this particular type of tetrahedron.

References

1. Archiv de Grunert, 1841.
2. Mathesis, 1955, p. 414; 1956, p. 487. Question 3713.
3. *Ibid.*
4. Archiv der Mathematik und Physik, t. 87, 1875, pp. 113–190.
5. The dihedral angles of a tetrahedron, Messenger of Mathematics, vol. 50, 1921, pp. 190–192.
6. Archiv der Mathematik und Physik, t. 58, p. 150.

CLASSROOM NOTES

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PARAMETER VARIATION AND THE SOLUTION OF BERNOULLI'S EQUATION

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Mr. A. N. Aheart in his note (this MONTHLY, vol. 58, 1951, pp. 696–697) has shown that the solution of the Bernoulli differential equation:

$$\frac{dy}{dx} + Py = Qy^n, \qquad n \neq 0, n \neq 1,$$

where P and Q are either constants or functions of x only, can be found directly without reducing it to the linear equation $(n=0)$ by a proper substitution.

This note is to show that his method is similar to the method of variation of parameters used in determining the solution of the linear equation ([1], p. 20). In the following method, we can allow n to have the value zero, but not 1, in which case the variables can be separated and the equation integrated.

Consider the Bernoulli equation

$$\frac{dy}{dx} + Py = Qy^n, \qquad n \neq 1, \tag{1}$$

where P and Q are either constants or functions of x only.

The equation (1) with right-hand member equal to 0, gives after separating the variables and integrating:

$$y = ze^{-\int P dx}. \tag{2}$$

(If we set $u = e^{-\int P dx}$, (2) is the solution assumed by Mr. A. N. Aheart.)

Considering z as a parameter and differentiating with respect to x:

$$\frac{dy}{dx} = e^{-\int P dx}\left(\frac{dz}{dx} - Pz\right). \tag{3}$$

Substitution in (1) from (2) and (3) gives:

$$\frac{dz}{dx} = z^n Q e^{(1-n)\int P dx}.$$

Separating the variables and integrating:

$$\frac{z^{(1-n)}}{(1-n)} = C + \int Q e^{(1-n)\int P dx} dx,$$

where C is an arbitrary constant of integration.

Substitution in (2) gives the solution of (1) as:

$$y^{(1-n)}e^{(1-n)\int P dx} = (1-n)C + (1-n)\int Qe^{(1-n)\int P dx}dx.$$

[*Remark.* For $n=0$, this gives the solution of the linear equation $dy/dx+Py=Q$.]

Reference

1. A. R. Forsyth, A Treatise on Differential Equations, London, 1929.

THE INSOLVABILITY OF THE QUINTIC RE-EXAMINED

MORTON J. HELLMAN, Rutgers University

In the April 1958 issue of this MONTHLY a unifying technique for the solution of the quadratic, cubic, and quartic was presented using the relations between the roots and coefficients. It is interesting to pursue the editor's suggestion to see where this method breaks down in the case of the quintic. That is the purpose of this note.

It is sufficient to consider the reduced quintic

$$x^5 + ax^3 + bx^2 + cx + d = 0. \tag{1}$$

Denote the five roots by r_1, r_2, r_3, r_4, r_5 and set

$$\begin{aligned} &r_1 + r_2 + r_3 = s, && r_1r_2 + r_1r_3 + r_2r_3 = t, \\ &r_1r_2r_3 = v, && r_4 + r_5 = -s. \\ &r_4r_5 = w, \end{aligned}$$

Then the remaining four equations connecting the roots and coefficients are

$$-s^2 + t + w = a, \tag{2}$$

$$-ts + sw + v = -b, \tag{3}$$

$$tw - vs = c, \tag{4}$$

$$vw = -d. \tag{5}$$

From (5) $v=-d/w$ and from (4) $t=(c+vs)/w=(wc-ds)/w^2$ and substituting these relations in (2) and (3) we obtain, after simplification,

$$-w^2s^2 + wc - sd + w^3 = aw^2,$$

$$s^2d - wsc + sw^3 - dw = -bw^2$$

which upon elimination of either w or s yields an equation of higher degree than the fifth. Hence, as expected, this technique fails in the case of the quintic. It is easy to show, as one might expect, that equations (2) to (5) are equivalent to the equations which result when one attempts to factor (1) into cubic and quadratic factors.

COMPLEX ROOTS OF REAL POLYNOMIALS

J. P. BALLANTINE, University of Washington

1. Introduction. When a complex root, of the form $x+iy$, of a real polynomial, $P(z)$, is found by Newton's method, it is necessary to compute both $P(x+iy)$ and $P'(x+iy)$ for several trial values of $x+iy$. Substitution by the usual methods is laborious, since each complex multiplication entails four real multiplications.

The amount of calculation at each step is reduced by the Lin-Bairstow method* by avoiding almost all work with complex numbers. Instead of closing down on a complex root, the method approximates a real quadratic factor, which then can be solved for its complex roots.

The substitution of $x+iy$ into $P(z)$ and $P'(z)$ can be carried out in a manner, as we shall show, which avoids almost all work with complex numbers. With this method of substitution, Newton's method entails about the same amount of work as the Lin-Bairstow Method.

2. Quadratic synthetic division. Use is made of quadratic synthetic division of $P(z)$ by z^2-pz-q. The procedure is similar to ordinary synthetic division of $P(x)$ by $x-r$. A parameter, λ, has been introduced the purpose of which will appear later.

$$(1)\qquad \begin{array}{cccccc} a_0 & a_1 & a_2 & a_3 \cdots & a_{n-1} & a_n(p+q \\ & pb_0 & pb_1 & pb_2 \cdots & pb_{n-2} & \lambda pb_{n-1} \\ & & qb_0 & qb_1 \cdots & qb_{n-3} & qb_{n-2} \\ \hline b_0 & b_1 & b_2 & b_3 \cdots & b_{n-1} & b_n \end{array}$$

The value of each b_j under the line is found as the sum of the numbers in the same column above the line. That is:

$$(2)\qquad b_j = a_j + pb_{j-1} + qb_{j-2}, \qquad 0 \leqq j < n,$$

$$(3)\qquad b_n = a_n + \lambda pb_{n-1} + qb_{n-2},$$

where it is assumed that all a_j and b_j are zero for j negative.

The quotient, $Q(z)$, and remainder, $R(z)$, of the division shown in (1) may be read as follows:

$$(4)\qquad Q(z) = b_0z^{n-2} + b_1z^{n-3} + \cdots + b_{n-2},$$

$$(5)\qquad R(z) = b_{n-1}(z - \lambda p) + b_n.$$

The choice of λ is arbitrary. $R(z)$ does not depend on λ, since the λ appearing in (5) cancels with the λ in b_n. The usual choice is to take $\lambda=0$, and so omit all terms mentioned in this note which involve λ. The more useful choice, $\lambda=\frac{1}{2}$, will be explained later.

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* Kaiser S. Kunz, Numerical Analysis, New York, 1957, pp. 34–37; W. E. Milne, Numerical Analysis, Princeton, 1949, pp. 53–57.

3. The substitution. First form a real quadratic having the factor $z-(x+iy)$, namely,

$$\begin{aligned} z^2 - pz - q &= (z - x - iy)(z - x + iy) \\ &= z^2 - 2xz + x^2 + y^2, \end{aligned}$$

whence

$$p = 2x, \qquad q = -x^2 - y^2. \tag{6}$$

Now, divide $P(z)$ by z^2-pz-q, leaving a quotient $Q(z)$ and a remainder $R(z)$.

$$\begin{aligned} P(z) &= Q(z)(z^2 - pz - q) + R(z), \\ P(x + iy) &= 0 + R(x + iy), \\ &= b_{n-1}(x + iy - \lambda p) + b_n, \text{ by (5)}, \\ &= b_n + b_{n-1}(iy), \end{aligned} \tag{7}$$

by (6), provided $\lambda = \frac{1}{2}$.

Formula (7) gives the value of $P(x+iy)$, the two terms being respectively the real and imaginary parts. It helps memorize (7) to note that the imaginary part of $P(x+iy)$ must contain the factor y, since its sign must depend on the sign of y.

4. Arrangement of the computation. When the coefficients are in decimal form, the rows of (1) are more conveniently displayed as columns. The values of a_j go in the first column, with a_0 at the top. Next proceed with the quadratic synthetic division, noting that the multiplications by p and q, and the additions are done on the desk calculator, so that the second and third rows of (1) do not have to be written. The values of b_j are put in a second column. (See (9)). The value of $P(x+iy)$ is read from the two final values of b_j, using (7).

To find $P'(x+iy)$, fill in a column showing the exponents of the successive terms. The coefficients of $P'(z)$ are found as the product of each a_j by the exponent. This multiplication can be done mentally, as when $a_j=30$ and the exponent is 5. Or, the machine will handle the multiplication, as when $a_j=2.179$ and the exponent is 7. In fact, on machines equipped with "automatic multiplication," it is as easy to enter (7)(2.179) into the machine as to enter 2.179.

So $P'(x+iy)$ is computed from $P'(z)$ by the process explained for computing $P(x+iy)$ from $P(z)$. The resulting coefficients are called b_j', and are entered in a column. Note that $P'(z)$ is of degree $n-1$.

By Newton's Formula,

$$\Delta z = -\frac{P(x + iy)}{P'(x + iy)} = -\frac{b_n + iyb_{n-1}}{b'_{n-1} + iyb'_{n-2}}\,. \tag{8}$$

Example 1. Given that

$$z^6 - 2z^5 + 2z^4 + 2z^3 + 7z^2 - 5z + 9 = 0$$

has a root near $0.42+0.87i$, carry out one Newton correction.

Solution. The computation in tabular form is displayed in (9). $x=0.42$, $y=0.87$, $p=0.84$, $q=-0.9333$, and $n=6$.

(9)

	a	b	b'
6	1	1.	6.
5	-2	-1.16	-4.96
4	$+2$	$+0.0923$	-1.7662
3	$+2$	$+3.16016$	$+9.14556$
2	$+7$	$+9.568391$	$+23.330665 = b_4'$
1	-5	$+0.088071 = b_5$	$-3.736672 = b_5'$
	$+9$	$+0.106810 = b_6$	

$$\begin{aligned}\Delta z &= -\frac{0.106810 + 0.088071(0.87)i}{-3.736672 + 23.330665(0.87i)}\\ &= -0.002714 + 0.005760i\\ z &= +0.42 \qquad + 0.87i\\ \hline &\ \ +0.417286 + 0.875760i = \text{corrected } z.\end{aligned}$$

5. A generalization. The suggested method of substitution applies in the more general case of the equation

$$P(z) + F(z) = 0, \tag{10}$$

where $P(z)$ is a real polynomial and $F(z)$ is any analytic function. In applying Newton's Method, the substitutions $P(x+iy)$ and $P'(x+iy)$ can be made as in Example 1, while some other method may be used to find $F(x+iy)$ and $F'(x+iy)$.

Example 2. Find a root of (10), where

$$\begin{aligned}P(x) &= z^6 + 3z^5 + 8z^4 + 12z^3 + 20z^2 - 35z + 180,\\ F(z) &= 10iz.\end{aligned}$$

Solution. After a few trials and corrections, there seems to be a root near $1.2+1.2i$. One correction, starting with this approximation is shown, in (11). $x=1.2$, $y=1.2$, $p=2.4$, $q=-2.88$.

(11)

	a	b	b'
6	1	1	6.
5	3	5.4	29.4
4	8	18.08	85.28
3	12	39.84	156.00
2	20	63.5456	168.7936
1	-35	2.77024	-281.72768
	180	0.31296	

	Function	Derivative
$P(z)$	$0.313 + 3.324i$	$-281.7 + 202.6i$
$F(z)$	$-12.000 + 12.000i$	$10.0i$
$P(z) + F(z)$	$-11.687 + 15.324i$	$-281.7 + 212.6i$

$$\Delta z = -\frac{-11.687 + 15.324i}{-281.7 + 212.6i}$$

$$\Delta z = -0.0526 + 0.0147i$$

$$x + iy = +1.2 \quad + 1.2i$$

$$\overline{+1.1474 + 1.2147i = \text{corrected } z}\ .$$

ON A PROPERTY OF HARMONIC FUNCTIONS

H. B. MANN, Ohio State University

It is well known that an harmonic function has derivatives of all orders. Hence if (x, y) is a point in the interior of a region where $f(x, y)$ is harmonic and if $D_x f = f_x$, $D_y f = f_y$ the form

$$(1) \qquad (D_x h + D_y k)^n f(x, y) = \phi_n(h, k)$$

is well defined. Here h and k denote arbitrary real numbers.

THEOREM. *If $f(x, y)$ is an harmonic function in a neighborhood of the point (x, y) then $\phi_n(h, k)$ is either identically* 0 *or it is an indefinite form.*

Proof. We have, since $(D_{xx}+D_{yy})f(x, y)=0$,

$$(2) \qquad (D_x h + D_y k)^2 f = D_x(D_x h' + D_y k')f,$$

where $h' = h^2 - k^2$, $k' = 2hk$. The pair h', k' can take all pairs of values since the hyperbolas $h^2 - k^2 = h'$, $2hk = k'$ intersect for any values h' and k'.

We now prove the theorem by induction. The form (2) is obviously indefinite or identically 0. If n is odd then (1) is clearly indefinite or identically 0. If $n = 2n'$ then by (2)

$$(D_x h + D_y k)^{2n'} f = D_x^{n'} (D_x h' + D_y k')^{n'} f,$$

where h' and k' can take all values. Since $D_x^{n'} f$ is also an harmonic function the theorem follows by induction.

The theorem improves on the well-known result that an harmonic function cannot take a minimum or maximum in the interior of its domain of definition. The proof moreover is shorter and more direct than the known proofs of this property of harmonic functions.

MATHEMATICAL EDUCATION NOTES

Edited by John A. Brown, University of Delaware, and John R. Mayor, AAAS and University of Maryland

Contributions for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington 5, D. C.

THE ROLE OF MATHEMATICS IN AN INTEGRATED 9TH GRADE SCIENCE-MATHEMATICS COURSE

Alexander Calandra, Washington University

The contemporary interest in the reexamination of the teaching of mathematics at the elementary and high school levels has, for the most part, developed into an appraisal of the significance of modern mathematics for the curriculum at these levels. While the relevance of this appraisal has considerable merit, it has tended to obscure the heuristic value of the relationship of mathematics to the sciences, particularly physics.

The traditional curriculum does not do any better in this regard, for most of the mathematics needed in the conventional physics course is developed in the 9th grade while the physics course is usually taken in the 12th. This raises the question of the advisability of developing a substantial amount of elementary algebra which is taught at the 9th grade level, with the aid of a parallel systematic treatment of elementary physics.

As far as the author has been able to find out, this approach has not been previously attempted in any school, although a few related projects have been undertaken. The one system in operation after a number of years is the Baldwin School Project, Bryn Mawr, Pa., which combines the teaching of mathematics and science at the 7th grade level. In this project, however, the physics which is taught as part of the science is not a systematic treatment but consists of specially selected topics.

A more sophisticated attempt, at the 10th grade level, combining algebra and physics was introduced at the Putney School in Vermont a number of years ago, but the effort was not continued. In the summer of 1957, the author introduced an integrated course in physics and algebra at the 9th grade level at Bayless High School. The objectives of this course, as far as the mathematics were concerned, were as follows:

(1) To provide the student in elementary algebra with a systematic body of scientific material which could be explored with the mathematical skills learned in algebra.
(2) To decrease the time between the learning of mathematical skills and their application.
(3) To stimulate the re-thinking of the teaching of mathematics and science at the high school level.

Procedure. This integrated algebra-physics course at the 9th grade level was given five days a week for two hours a day for one school year. The students were given credit for one year of science and one year of algebra. The subject matter for the first semester was primarily geometric optics, a topic which the

writer had found to be very useful in introducing physics to students with weak backgrounds in mathematics. Although the mathematical content was not greatly different from that in the conventional algebra course, it was extensively structured to provide a pattern which would make sense both in mathematics and physics.

A major modification in the approach to algebra consisted of the immediate introduction of ratio and proportion and a delay in the treatment of signed numbers. The thinking behind this reorganization was that the topic of ratio and proportion is extensively used in geometric optics and is easy to teach, while the topic of signed numbers is a fairly sophisticated concept which can be advantageously delayed until the students have had considerably more facility in the handling of literal numbers.

Of the unconventional topics introduced, it was found necessary to include a small amount of very simple training in geometry and trigonometry. In geometry, this was accomplished by the use of a few special theorems that gave the student some insight into similar triangles. In trigonometry the only function used was the sine, which was thoroughly explored in terms of the unit circle concept. The emphasis on the sine came about as the result of the discussion of Snell's Law dealing with the Refraction of Light and the Diffraction Law in connection with a simple procedure for measuring the wave length of light. Although the subject matter of the first semester was primarily geometric optics, there was a substantial amount of other science introduced at such points where the mathematical treatment made them relevant. Thus, the Inverse Square Law was introduced in optics and was immediately applied to problems in gravitation, acoustics, radioactivity, electricity and magnetism. To the extent that this was done, the physics pattern was structured by the mathematics rather than the other way around. The mathematics of the second semester was developed around the concept of functions with a very elementary but integrated treatment of graphs, tables and equations. The linear equation $y=mx+b$ was explored intensively with a large variety of examples but with special emphasis on problems in kinematics. The second semester spent a considerable amount of time on mechanics including problems in rotation involving calculations with moments of inertia which gave extensive practice with quadratic equations. After mechanics, the course was concluded with electricity and magnetism. Electricity was an excellent medium for drill with simultaneous equations in connection with Kirchoff's Laws, and this was developed in considerable detail to advantage.

After a year's trial at Bayless High School, where the course was run by one instructor in both areas of science and mathematics, the course was expanded to other institutions in a variety of different circumstances. Sufficient experience is now available to give the following conclusions with considerable confidence.

(1) The course can be successfully given in the ninth grade by a teacher of average competence, with an average (but not overly heterogeneous) student group provided lesson plans are made available in considerable detail.

(2) The course can be given at the eighth grade level with an above-average group of students with a skilled and enthusiastic teacher. Students in these groups must also be studying algebra.

(3) It is not necessary that the same teacher be involved in both the mathematics and science. The science teacher can work cooperatively with the mathematics teacher, informing the latter of the background needed at different times. Although some of the administrators had misgivings about this practice, it has worked out exceedingly well.

(4) There is no deficiency in the resulting mathematical training of the students in the program. In one of the first trials of this program in the St. Louis Public School system, the median grade of one group of students (above average) in this program on the Lankton Algebra Test corresponded to the 94th national percentile.

(5) Although the course has been well received by students of average ability, it does particularly well with students of above-average ability.

Future Plans. The first class in this integrated course enrolled 22 students in 1957–58. In 1958–59 the venture was expanded to 200 students in 7 classes. The best estimates that can be made of enrollment in 1959–60 is about 5000 students in 250 classes. The actual figure will depend largely on the continued assistance of the United States Office of Education and possibly additional assistance from foundations.

The course is not an isolated science course, but part of a pattern of integrated science and mathematics throughout the secondary school system. It is part of a plan in which this course is followed by a sophomore chemistry course which emphasizes mathematical skills. This in turn is followed by a junior biology course which takes advantage of physics and chemistry previously learned and uses mathematical techniques in the field of statistics and probability in connection with the study of heredity and genetics. In the 4th year the science course planned is a second year of physics whose level of sophistication is comparable to the present typical college freshman physics course, and contains the kind of modern physics which is being presented in the Continental Classroom course in physics on television.

WISCONSIN PHYSICS FILM EVALUATION PROJECT

Progress Report

W. A. Wittich, The University of Wisconsin

The following is a brief summary of what appears in the January progress report.

"In September 1956 Dr. Harvey White began to teach the first one-year television course in high school physics ever to be offered as a continuing day-to-day experience in master teaching through television means. The original 30-minute telecasts were made over educational television station, WQED, in

Pittsburgh. One year later the University of Wisconsin undertook to evaluate the effectiveness of the Harvey White telefilm materials as they might be used in on-going, representative high school physics study situations in Wisconsin communities.

"The study undertook to discover (1) the effectiveness of the telefilms in typical classroom situations and (2) the effectiveness of telefilms in locally supervised correspondence study situations."

The January report is concerned only with the classroom phase of the study since the correspondence study phase has not been completed and is going on at the time of the writing of this summary.

The progress report states that:

"Television films can be described as a hybrid. They are made under rigid 30-minute television format conditions and therefore include many of the repetitions or unobtrusive errors or sometimes lengthy explanations which are characteristic of television *per se*. However, a limited amount of post judgment and improvement in quality is possible through later editing. This has been the case in the Harvey White physics films where the two media, sound motion picture film and television, have been brought into inter-play to create a new kind of educational material, the 16 mm. sound television film or T-film."

The report goes on to state that in September 1957, 83 schools began their participation in the T-film-using and control groups during the 90-day first semester of 1957–58 experimental period. All school groups, T-film-using as well as control, proceeded to study units on mechanics, properties of matter and heat during this 90-day semester.

The report describes how the development of the supplementary study materials, manuals for use by the instructors, tests for use in evaluation, and student supplementary study materials, were made through the cooperative efforts of Wisconsin State College Physics Department staff members. It describes the physics project as a high level, far-reaching, coordinated attempt by many of the educational teacher-training agencies in Wisconsin.

The report includes a series of questions and answers concerned with the key findings among traditional classroom groups which used the telefilms as experimental materials. Such questions as these were included:

Question: How did the use of the 30-minute film per day as measured by the Wisconsin final physics test influence the learning of students enrolled in high school physics classes?

Answer: The control and T-film using groups showed no significant difference in the amount of physics learned that was presented in both T-film and textbook.

The T-film-using group showed a significant level of achievement above that accomplished by control groups in the case of the physics being included in the Harvey White T-films alone.

Question: What influence did the use of a 30-minute Harvey White T-film have on the teacher's knowledge of physics?

Answer: The teachers who saw and used the film learned a significant amount of physics information above that learned by control teachers.

Question: How did the T-film using group compare with the control group in terms of retention of physics information?

Answer: The control group retained more Semester 1 physics information than the T-film-using group after a period of three months. Measurement was made on the basis of physics information found both in textbooks as well as in films.

It is believed by the committee that the Wisconsin Physics Film Project will have far-reaching implications and influences in the conduct of classroom teaching of physics. Information to be found in T-films only was tested for.

Interested persons may secure copies of the phase I report by addressing W. A. Wittich, 113 Education Building, University of Wisconsin, Madison. The committee which guides the Wisconsin project includes Professor Milton O. Pella, Department of Education, Professor Charles Wedemeyer, Director of Correspondence Study, Extension Division, and Professor W. A. Wittich, Department of Education.

THE UNIVERSITY OF ILLINOIS ARITHMETIC PROJECT

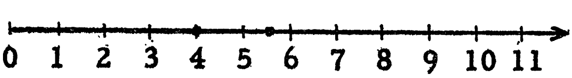
DAVID A. PAGE, University of Illinois

The University of Illinois Arithmetic Project was created in the fall of 1958. It is directed by Professor David A. Page. The project interprets the term "arithmetic" broadly and is actually concerned with many kinds of possible content in elementary mathematics classes. In November the Carnegie Corporation of New York announced a grant of $307,000 to the project for five years of work. Over the five year period the staff of the project plans to create an entirely new elementary mathematics course including textbooks, guides for teachers, and suitable teaching aids. Also they will be concerned with helping teachers learn to teach the new course.

An example of a classroom topic that has already been used in the experimental classes in Champaign-Urbana will give some idea of the kinds of new material which the project will be developing. The topic is "number line games and stand-still points." Quasi-elementary school language is used in order to substantiate the possibility of communicating with 4th and 5th grade students.

Number line games and stand-still points.

Here is a *number line:*



Think of it as going on to the right forever. On a number line every number (of the kind we work with) has its place. On the number line above

the places for 4 and $5\frac{1}{2}$ are shown by dots.

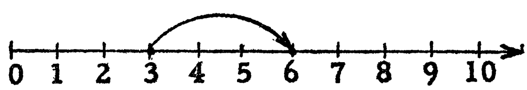
We will now learn about number line games. There are different number line games. In order to play one you must have a rule. Here is a rule for one number line game.

$$\square \longrightarrow \square \times 2$$

We will read this rule by saying, "Box goes to box times 2." This rule tells you how to make a *move* on the number line. First you must have a starting point. Suppose you start at 3. Put a 3 into each box of the rule. This gives you

$$\boxed{3} \longrightarrow \boxed{3} \times 2$$

The rule now tells you, "3 goes to 3 times 2." That is, it tells you that if you start at 3 and make one move, you go to 6. We can show this jump from 3 to 6 on a number line this way:



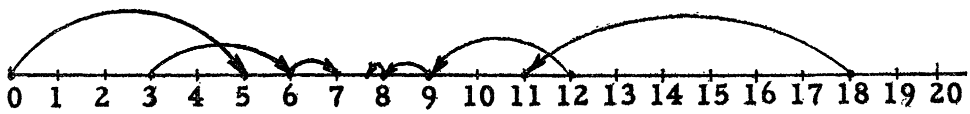
Where would you jump to if you started at 5 and made one move? Where would you jump to if you started at 2 and made 3 moves? Where would you have to start in order to get to 10 in 2 moves?

At this stage students can be given an enormous amount of computational practice appropriate to their grade level by varying the complexity of the rule and the "difficulty" of the starting point. Instead of indicating the wide variety of topics which can be taken up next, a specific one, of stand-still points, will be considered.

Take another number line game with this rule

$$(*) \qquad \square \longrightarrow \frac{\square}{3} + 5$$

We read this rule, "Box goes to box divided by 3, plus 5." If we choose several starting points and show the jump that this rule tells us to make for each of the starting points on a number line, it looks like this:



(Remember that these students have no algebraic technique for solving such a problem.) Here, in some abbreviation, are various conjectures and observations that students made in their search for the stand-still point:

1) In one part of the number line all of the jumps are to the right and in another part all the jumps are to the left. If I could find the place where the jumps stop going to the right and start going to the left I would have the stand-still point.

2) When you start at some points on the line you make long jumps. When you start at other points on the line you make short jumps. If you are at the stand-still point you don't jump at all. So if I keep heading toward shorter and shorter jumps I will be headed toward the stand-still point.

3) Look at the picture. One jump takes me from 3 to 6. Another jump takes me from 12 to 9. Since these jumps have the same length but are in opposite directions, it seems that the stand-still point should be halfway between the two starting points. It should be $7\frac{1}{2}$.

4) It seems to me that wherever I start on the number line my jumps carry me toward smaller jumps and toward the jumps in the opposite direction. So no matter where I start jumping on the number line, I always get closer to the standstill point. Therefore, if I pick any old point as a starting point and keep making one jump after another, I am bound to get to the stand-still point. (In practice, if the stand-still point is an "easy" one, students applying this "just keep jumping" method make a few successive jumps and then guess the stand-still point correctly. Cases have come up, however, where students recognized that although they kept getting closer and closer to the stand-still point, they would never be able to reach it.)

5) Look at the rule(*) above. At the far right you add 5. In order to stand still, you must lose 5. The only place you can lose 5 is when you divide by 3. So you are looking for a number such that when you divide it by 3, the result is down 5 from what you start with. Try something, say 15. Fifteen divided by 3 is 5 and so you have gone from 15 down to 5 or down 10. Too much. Try 9. Nine divided by 3 is 3 and you have gone from 9 to 3 or down 6. Still too much but getting closer. Try 6. Too little. Keep trying. Try $7\frac{1}{2}$. Seven and one-half divided by 3 is $2\frac{1}{2}$ and going from $7\frac{1}{2}$ to $2\frac{1}{2}$ is going down by 5. I must have the stand-still point.

6) My way is like the one before but faster. Whenever your rule is to divide by 3 and add something, take that something and divide it by 2. Take what you get and add it to the something and you have the stand-still point.

7) Another student made a two-dimensional graph with one axis an axis of starting points and the other axis the axis of landing points (he did not use this language!). Thus one point on his graph corresponded to a complete jump for the game. From this graph he was able to read off rather easily the location of stand-still points.

The reader can correctly infer from the responses indicated above that these students were of high ability. The project is far from ashamed when it discovers material which challenges and interests bright students. However, in the long run, the project is concerned at least with students of average ability and above and perhaps with students of below average ability. Able students were needed in order to find so many approaches to the job of finding stand-still points which can in turn be presented in a more gradual way to less able students.

AAAS TWO-YEAR STUDY ON THE USE OF SCIENCE COUNSELORS

J. A. Brown, University of Delaware

The AAAS Study on the Use of Science Counselors, developed as a result of the awareness of the present and anticipated future shortage of well-qualified science and mathematics teachers, is based on the recognition of the valuable service that can be provided to teachers by competent supervisors or counselors. Four centers for study were designated to represent the west, the south, the midwest, and the east. These centers were the Universities of Nebraska, Oregon, Texas, and Pennsylvania State University. In each university the study had the full support of the science, mathematics, and education departments. The universities were responsible for establishing satisfactory relationships with the cooperating schools and the state departments of education. Two experienced teachers, each capable of functioning as a counselor in science and mathematics teaching, were employed by the universities with funds made available to AAAS by a grant from the Carnegie Corporation of New York.

The direct-contact work of the counselors was in five areas: subject-matter counseling, materials and methods counseling, curriculum counseling, career-guidance counseling, and teacher professionalization.

During the two years of the study, the counselors visited some 150 science and mathematics teachers in secondary schools in these regions. These schools varied in size and were dispersed geographically in the states. Counselors visited schools, sometimes as a team and sometimes as individuals. They sought to bring new ideas, current thinking, and recent developments in science and mathematics teaching directly to the teachers. Group sessions for science and mathematics teachers were held for teachers from time to time, both on the university campuses and in cooperating schools.

The evaluation of the study under the supervision of John W. Gustad, Professor of Psychology of the University of Maryland, was first reported to the annual AAAS meeting in December. The evaluation involved questionnaires

and interviews. A total of 700 questionnaires were sent to participating teachers, with a 91 per cent return. Data were obtained from counselors, and interviews were conducted with a total of 60 individuals, including teachers, administrators, university staff members, and representatives of state departments of education. Gustad reported that virtually all expressed appreciation for the program and a desire to see it continued; and, in addition, all expressed a desire for substantially more contacts with counselors under a new program. Teachers as a group are moderately well satisfied with the support they receive from administrators and also with administrators' understanding of the problems of science teaching. Some indicated that the Study activity assisted on this point. There was general agreement among all groups that the study had stimulated the interest of the teacher in his own field, had provided him with new ideas about teaching methods, created new interest in science on the part of the library staff, assisted the guidance counselor, and stimulated greater student interest in science. Questionnaire and interview responses emphasized the desire of the teachers for more training in science subject matter and the opinion that participation in professional societies in science was more valuable to them than participation in professional education organizations.

A final report of the Study may be obtained upon request addressed to the Director of Education, AAAS, 1515 Massachusetts Ave. N.W., Washington 5, D. C.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1366. *Proposed by V. E. Hoggatt, Jr., San Jose State College*

Show that if a, b, c form a triangle, then $\sqrt{a}$, $\sqrt{b}$, $\sqrt{c}$ form a triangle.

E 1367. *Proposed by H. Lindgren, Patent Office, Canberra, Australia*

Prove that the numerator of the sum $1-1/2+1/3- \cdots$ to $[2p/3]$ terms is divisible by p when p is a prime greater than 3.

E 1368. *Proposed by M. S. Klamkin, AVCO Research and Development*

Show that if all roots of $ax^4-bx^3+cx^2-x+1=0$ are positive, then $c \geqq 80a+b$.

E 1369. *Proposed by R. E. Shafer, University of California*

If $y=(1+x^{1/2})^{2n+2}$, show that $y^{(n)}=4(n+1)(n+1)!$ at $x=1$.

E 1370. *Proposed by A. J. Goldman, National Bureau of Standards, Washington, D. C.*

(a) Evaluate $\int_0^\infty (1-e^{-x})^n x^{-n} dx$ for integral $n>1$.
(b) Evaluate $\int_0^\infty (1-e^{-x})^2 x^{-2} \cos^2 kx\, dx$ for arbitrary $k>0$.

SOLUTIONS

A Theorem of Darboux Type

E 1336 [1958, 708]. *Proposed by Albert Wilansky, Lehigh University*

Let $f(0)>0$, $f(1)<0$. Prove that $f(x)=0$ for some x under the assumption that there exists a continuous function g such that $f+g$ is nondecreasing.

Solution by the Proposer. Let E be the set of x such that $f(x)\geqq 0$. Let $s=$ l.u.b. E. Let $h=f+g$. Then, since h is nondecreasing, for any $x\in E$, $h(s)\geqq h(x)\geqq g(x)$. Hence, since g is continuous, $h(s)\geqq g(s)$ and so $f(s)\geqq 0$.

Next, $g(1)>h(1)\geqq h(s)\geqq g(s)$. Since g is continuous, there exists $t\geqq s$ such that $g(t)=h(s)$. Then $h(t)\geqq h(s)=g(t)$, so that $f(t)\geqq 0$. By definition of s, $t=s$ so that $g(s)=h(s)$ and $f(s)=0$.

Also solved by H. F. Bechtell, J. L. Brown, Jr., R. F. Brown and Joel Levy (jointly), C. H. Cunkle, S. J. Einhorn, Michael Goldberg, L. C. Graue, R. E. Greenwood, L. E. Hanson, Lawrence House, Joe Lipman, D. C. B. Marsh, James Misho, L. E. Pursell, Hans Schneider, Robert Spira, G. B. Thomas, Jr., and Dale Woods. Late solutions by J. W. Baldwin, D. A. Freedman, A. R. Hyde, and Jack Silver.

Generalization of a Russian Olympiad Problem

E 1337 [1958, 708]. *Proposed by the Student-Faculty Colloquium, Carleton College*

What is the largest value of n, in terms of m, for which the following sentence is true? If from among the first m natural numbers any n are selected, among the remaining $m-n$ at least one will be a divisor of another.

Solution by D. D. Strebe, University of South Carolina. Let $n=[m/2]-1$. The remaining $m-n$ numbers contain the integer 1 or a pair of the form $(k, 2k)$, satisfying the divisibility requirement. To show that this is the largest value of n, select a subset consisting of the first $n+1$ natural numbers. In the remaining set of $m-(n+1)$ natural numbers, each number is greater than one half of any other number in the set; thus there are no divisors.

Also solved by R. G. Albert, Merrill Barnebey, Lawrence Botsford, R. F. Brown and Joel Levy (jointly), S. J. Einhorn, E. L. Ellis, E. A. Fay and R. S. Gardner and T. L. Reynolds (jointly), R. E. Greenwood, J. H. Hodges, Joe Lipman, D. C. B. Marsh, J. B. Muskat, J. L. Selfridge, Charles Wexler, and the proposers. Late solutions by J. W. Baldwin, D. A. Breault, Helen M. Marston, and W. A. Veech.

Editorial Note. See problems 3739 [1937, 120] and 3820 [1939, 240].

A Property of Two Projectively-related Point Rows

E 1338 [1958, 708]. *Proposed by V. F. Ivanoff, San Carlos, California*

Let P and P' be a pair of corresponding points of two projectively-related point rows. If the rows are inverted about P and P', respectively, show that the resulting point rows are similar.

Solution by L. D. Goldstone, New York State Public Works Laboratory, Albany, N. Y. Let Q, R, S be any three points on the point row containing P, and let Q', R', S' be the projectively-related points on the point row containing P'. Let P_i, Q_i, R_i, S_i denote the inverses of P, Q, R, S, and let P_i', Q_i', R_i', S_i' denote the inverses of P', Q', R', S'. Since cross-ratio is preserved under inversion we have

$$(P_iQ_iR_iS_i) = (PQRS) = (P'Q'R'S') = (P_i'Q_i'R_i'S_i').$$

But, since P and P' were the centers of inversion, P_i and P_i' are both ideal points. It now follows that $Q_iS_i/Q_iR_i = Q_i'S_i'/Q_i'R_i'$, and the theorem is established.

Also solved by Joe Lipman, D. C. B. Marsh, and the proposer.

"Un Calcul Simple"

E 1339 [1958, 708]. *Proposed by A. J. Goldman, National Bureau of Standards, Washington, D. C.*

Let $f(t)=t-t^3/6+(t^4/24)\sin(1/t)$ for $t>0$. Prove that for $x>0$, $z>0$, $x+z<1$, the relation $f(x+z)\leqq f(x)+f(z)$ holds.

Solution by D. C. B. Marsh, Colorado School of Mines. Define

$$g(t) = f(t)/t = 1 - t^2/6 + (t^3/24)\sin(1/t)$$

and note that for $0<t<1$

$$g'(t) = -(t/24)[8 - 3t\sin(1/t) + \cos(1/t)] < 0.$$

Therefore $g(t)$ is decreasing in this interval, and for $x>0$, $z>0$, $x+z<1$, we have $g(x+z)<g(x)$ and $g(x+z)<g(z)$, which imply that

$$xg(x+z) + zg(x+z) < xg(x) + zg(z)$$

or $f(x+z)<f(x)+f(z)$.

Also solved by A. F. Kaupe, Jr., and the proposer. Late solution by C. H. Cunkle.

The proposer pointed out that this problem is equivalent to "un calcul simple" omitted from M. J. Haantjes' paper (see p. 91) *Sur la géométrie infinitésimal des espaces métriques* (Colloque de Géométrie Différentielle, Louvain, 1951).

A Type of Periodicity for Fibonacci Numbers

E 1340 [1958, 708]. *Proposed by Peter Beisswanger, Tübingen, Germany*

Let b_n be the number given by the last $k+2$ digits of a_n, where a_n is the nth

term of the Fibonacci sequence $a_0=0$, $a_1=1$, $a_{n+1}=a_{n-1}+a_n$. Show that for each $k\geqq 1$, the sequence $\{b_n\}$ is periodic with $150\cdot 10^k$ as a period.

I. *Solution by D. C. B. Marsh, Colorado School of Mines.* The b_n are congruent to the a_n (mod 10^{k+2}). Since the a_n have periods of $3\cdot 2^{m-1}$ (mod 2^m) and $4\cdot 5^m$ (mod 5^m), the b_n have period of l.c.m. $(3\cdot 2^{k+1}, 4\cdot 5^{k+2})=150\cdot 10^k$.

II. *Solution by Peter Greiner, University of British Columbia.* By mathematical induction one may establish the following two lemmas:

LEMMA 1. $(1+\sqrt{5})^n=2^{n-1}(a+b\sqrt{5})$, where a and b are integers with $a+b$ even.

LEMMA 2. $(1+\sqrt{5})^{150\cdot 10^k}=2^{150\cdot 10^k}[(10^{k+2}e+1)+10^{k+2}d\sqrt{5}]$, where d and e are integers and $k\geqq 1$.

One may now easily show that

$$a_n = [(1+\sqrt{5})^n-(1-\sqrt{5})^n]/2^n\sqrt{5} = b,$$

$$a_{n+150\cdot 10^k} = 10^{k+2}(ad+bc)+b,$$

whence

$$a_{n+150\cdot 10^k} - a_n = 10^{k+2}(ad+bc).$$

Also solved by Vern Hoggatt, Joe Lipman, Walter Penney, and the proposer.

A solution of the problem may be found in Vern Hoggatt, *A type of periodicity for Fibonacci numbers*, Mathematics Magazine, Jan.–Feb. 1955, pp. 139–142. The problem arose in a research by Hoggatt concerning the existence or nonexistence of three Fibonacci numbers which are squares of the sides of a right triangle. This latter problem is still unsolved.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4845. *Proposed by Fulton Koehler, University of Minnesota*

Let α be a real square matrix $[a_{ij}]$ and let $D(\alpha)=\sum_{i\geqq j}a_{ij}$. Find the maximum of $D(\alpha)$ for the group of orthogonal matrices of given order n; and show that, as $n\rightarrow\infty$, the maximum is asymptotic to $(n/\pi)\log n$.

4846. *Proposed by Olga Taussky, California Institute of Technology*

Show that a matrix which is similar to a real diagonal matrix is the product of two hermitian matrices one of which is positive definite. (The converse is known.)

4847. *Proposed by P. T. Bateman, University of Illinois*

Suppose f is a nonconstant polynomial function over a field K. Show that the additive group of K is generated by the values taken on by f provided either (a) K is of characteristic zero or (b) K has prime characteristic p and f has degree less than p. Also show by an example that some restriction such as (a) or (b) is needed.

4848. *Proposed by M. S. Klamkin, AVCO Research, Wilmington, Massachusetts*

Without performing any integration determine the ratio

$$\int_0^1 \frac{dt}{\sqrt{1-t^4}} \quad : \quad \int_0^1 \frac{dt}{\sqrt{1+t^4}}.$$

4849. *Proposed by L. A. Rubel, Institute for Advanced Study*

Suppose that $\phi(t)$ is a bounded measurable real function defined for $t>0$, and let $f(x)=(\phi*\phi)(x)=\int_0^x\phi(t)\phi(x-t)dt$. Must $f(x)$ be nonnegative for all sufficiently small positive x?

4850. *Proposed by Ward Cheney and Allen Goldstein, Convair-Astronautics, San Diego, California*

Let $\mathcal{A}$ and $\mathcal{B}$ denote two closed convex sets in Hilbert space. For a point x, let Ax and Bx denote respectively the points of $\mathcal{A}$ and $\mathcal{B}$ closest to x. Prove that A satisfies the Lipschitz condition $\|Ax-Ay\| \leqq \|x-y\|$. Furthermore, if x is a fixed point of AB, then $\|x-Bx\| = \text{dist}\ (\mathcal{A}, \mathcal{B})$.

SOLUTIONS

An Invalid Conjecture

4673 [1956, 125]. *Proposed by Paul Erdös, Technion, Haifa, Israel*

Let $a_1<a_2< \cdots$; $b_1<b_2< \cdots$ be two sequences of integers. Prove that from the sequence a_i+b_j one can always select an infinite subsequence such that no one divides another.

Note by the proposer. The proposition is not true as a counterexample shows. Trostrum, in *Mathematica*, June 1958, constructed two sequences a_i, b_i, $1\leqq i<\infty$ such that amongst the integers a_i+b_j there do not exist infinitely many, no one of which divides another.

Continuous Function with Rational Values Almost Everywhere

4799 [1958, 530]. *Proposed by K. L. Chung, Syracuse University*

Find a function continuous in $(0, 1]$ without any interval of constancy which takes on rational values almost everywhere.

Solution by Robert Spira, Berkeley, California. Define $f_{10}(x)$ as $1/2-|x-1/2|$. At the midpoint of each slanted portion of the graph insert an interval of constancy of length 3/8 and having the same midpoint. Reconnect the ends of the slanted portion to the ends of the interval of constancy and delete the old slanted portion. The result is the graph of $f_{11}(x)$. Now replace the center part of each horizontal portion with an inverted V, of base 1/16 and height 1/32. This is $f_{12}(x)$. There are now four disjoint horizontal sections; at the center of each put an inverted V with base 1/64 and height 1/128; the total of the four bases is 1/16. This corresponds to $f_{13}(x)$. Continue on in this manner; in general, the graph of $f_{1n}(x)$ will have 2^n horizontal sections, and the total sum of the bases of the inverted V's which have been added since the first step will be $(1/8+1/16+\cdots+1/2^{n+1})$.

The $f_{1n}(x)$ converge uniformly to a continuous function $f_1(x)$ which takes on the value 1/4 on a nondense set of measure 1/2.

For $f_{20}(x)$ take $f_1(x)$. On each slanted portion, insert at the midpoint an interval of constancy with length 3/4 of the difference of the abscissas of the endpoints of the slanted portion. Connect the endpoints of each interval of constancy to its respective endpoints of the slanted portion. This is the graph of $f_{21}(x)$. Altogether, the intervals of constancy of $f_{21}(x)$ add up to 3/8.

Each interval of constancy of $f_{21}(x)$ has length $(3/2)\cdot(1/2^m)$. In any such interval, in a manner analogous to the previous construction, a nondense set may be formed as the limit of $f_{2n}(x)$; where the total measure of the added inverted V's is $(1/2)\cdot(1/2^m)$. Thus $f_2(x)=\lim f_{2n}(x)$ has rational values on a set of measure 3/4.

Going on in this fashion, there is determined a sequence of continuous functions $f_1(x), f_2(x), \cdots$. These functions converge uniformly to some $f(x)$, which is continuous and takes on rational values on a set of measure 1, and has no interval of constancy.

Also solved by Jan Lipinski and by S. P. Lloyd.

Representation as Sums of Squares

4801 [1958, 530]. *Proposed by S. W. Golomb, Pasadena, California*

Let $R(n)$ denote the minimum number of terms required in representing n as a sum of squares. If n is restricted to perfect numbers, show that

$$R(n)=\begin{cases}2 \text{ for odd } n, & \text{if any;}\\ 3 \text{ for } n=6; & \\ 4 \text{ for even } n, & n\neq 6.\end{cases}$$

Solution by P. T. Bateman, University of Illinois. First, suppose n is an odd perfect number. Then (Euler) $n=p^r m^2$, where p is prime, m is not a multiple of p, and $p\equiv r\equiv 1 \pmod 4$. Thus n is not a square but there exist integers a, b (Euler) such that $p=a^2+b^2$. So $R(n)=2$. In fact

$$n = \{p^{(r-1)/2}ma\}^2 + \{p^{(r-1)/2}mb\}^2.$$

Clearly $R(6)=3$.

Finally, suppose n is an even perfect number other than 6. Then (Euler) $n=2^{p-1}(2^p-1)$, where p is an odd prime. Thus $n=4^{(p-1)/2}(2^{p-3}\cdot 8-1)$, which is of the form $4^a(8b+7)$, and hence (Fermat) n is not expressible as a sum of three squares. Finally (Lagrange) $R(n)=4$.

For the theorems referred to, one may consult Landau, *Elementary Number Theory*, New York, 1958; theorems 32, 165, 169, 186, and ex. 4, p. 235.

Also solved by Anders Bager, J. L. Brenner, L. Carlitz, Emil Grosswald, J. H. Hodges, M. I. Knopp, J. B. Muskat, Benjamin Sapolsky, Robert Spira, M. Sugunamma, and the proposer.

Lower Bound for the Maximum Element in a Matrix

4802 [1958, 530]. *Proposed by Ky Fan, Oak Ridge National Laboratory*

Let $A=(a_{ij})$ be a real symmetric matrix of order n $(\geqq 3)$ and of rank $\leqq 3$. If A is positive, semi-definite and if $a_{ii}=1$ $(1\leqq i\leqq n)$, then

$$\operatorname*{Max}_{i\neq j} a_{ij} \geqq \tfrac{1}{2}\operatorname{cosec}^2 \omega_n - 1,$$

where $\omega_n=n\pi/6(n-2)$.

Solution by the proposer. The positive semi-definite matrix A can be expressed as $A=B^2$, where B is a real symmetric matrix of order n. Let b_i denote the ith row vector of B. Then the inner product $(b_i, b_j)=a_{ij}$ $(i, j=1, 2, \cdots, n)$. Since $a_{ii}=1$, we have $\|b_i\|=1$. Since A is of rank $\leqq 3$, the maximum number of linearly independent vectors among $b_1, b_2, \cdots, b_n$ is $\leqq 3$. Hence the points $b_1, b_2, \cdots, b_n$ lie on the unit sphere S^2 of a 3-dimensional linear subspace of the Euclidean n-space. Now, by a theorem of L. Fejes Tóth and H. Hadwiger, among any n $(\geqq 3)$ points on the unit sphere S^2, there is at least one pair of distinct points with distance $\leqq(4-\operatorname{cosec}^2\omega_n)^{1/2}$. [See L. Fejes Tóth, *Lagerungen in der Ebene auf der Kugel und in Raum*, Springer, 1953, p. 115; also W. Habicht and B. L. van der Waerden, *Lagerung von Punkten auf der Kugel*, Math. Annalen, 123 (1951), pp. 223–234.] It follows that

$$\operatorname*{Min}_{i\neq j} \|b_i - b_j\|^2 \leqq 4 - \operatorname{cosec}^2 \omega_n.$$

As $\|b_i\|^2=1$, this means $\operatorname{Max}_{i\neq j} a_{ij}=\operatorname{Max}_{i\neq j} (b_i, b_j)\geqq\frac{1}{2}\operatorname{cosec}^2\omega_n-1$.

A direct proof without using the theorem of Fejes Tóth and Hadwiger is greatly desired. It is also hoped that a generalization by considering arbitrary upper bound r (instead of 3) for the rank of A will be discussed.

Series with Negative Coefficients

4803 [1958, 530]. *Proposed by D. J. Newman, AVCO Research, Wilmington, Mass.*

Let $\sum a_n x^n$ be analytic at 0 and suppose $a_n > 0$, $a_{n+1}a_{n-1} > a_n^2$. Prove that the expansion of $1/\sum a_n x^n$ has all negative coefficients (except for the constant term).

Solution by Leonard Carlitz, Duke University. This result is due to Kaluza, *Mathematische Zeitschrift*, vol. 28, 1928, pp. 161–170, in particular, Theorem 3. The simple proof follows. We may assume $a_0 = 1$. Put

$$1 \Big/ \sum_{n=0}^{\infty} a_n x^n = 1 - \sum_{n=1}^{\infty} c_n x^n,$$

so that

$$0 = a_n - \sum_{r=1}^{n} c_r a_{n-r} \qquad (n \geqq 1), \tag{1}$$

$$c_{n+1} = a_{n+1} - \sum_{r=1}^{n} c_r a_{n-r+1}. \tag{2}$$

If we multiply (1) by $-a_{n+1}$ and (2) by a_n, and then add, we get

$$a_n c_{n+1} = \sum_{r=1}^{n} c_r (a_{n+1}a_{n-r} - a_n a_{n-r+1}).$$

Now the hypothesis $a_{n+1}a_{n-1} > a_n^2$ implies $a_{n+1}a_{n-r} > a_n a_{n-r+1}$ $(1 \leqq r \leqq n)$. Hence, by induction every $c_n > 0$ and the theorem follows.

Other references are: Szegö, *Math. Z.*, vol. 25, p. 177; Hardy, *Divergent Series*, Th. 22, pp. 68–69.

Also solved by A. C. Aitken, P. T. Bateman, O. Buchta, W. S. Lawton, D. C. Russell, W. F. Trench, and the proposer.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

College Geometry. By L. H. Miller. Appleton-Century-Crofts, New York, 1957. x+201 pp. $4.50.

This is not "just another college geometry." The author has incorporated much of the spirit and some of the content of projective geometry. It is an

elementary text presupposing high school plane geometry, elementary trigonometry, and familiarity with the concept of a limit.

Chapter 1 (27 pp.) is concerned with the nature of proof and a review of elementary theorems from plane geometry. In Chapter 2 (19 pp.) directed line segments and angles are used to introduce the ideal point on a line (the point which divides a line segment in the ratio -1), cyclic quadrilaterals, and the Theorems of Menelaus and Ceva.

Chapter 3 (11 pp.) is an introduction to constructions with ruler and compass. Chapter 4 (11 pp.) is based on six theorems (including the Circle of Apollonius) involving geometric loci. Chapter 5 (18 pp.) is based on transformations—line reflections (orthogonal), point reflection, translation, rotation, homothetic transformation (treated in terms of a change of scale with a coefficient of expansion). Similar and homothetic figures are included along with homothetic constructions.

Chapter 6 (22 pp.) involves inverse points and curves. The construction of a circle tangent to three given circles, orthogonal circles, the invariance of angles under inversion, and Ptolemy's Theorem are considered.

Chapter 7 (25 pp.) includes double-ratio (cross-ratio), central projection (perspectivity between planes), a trigonometric proof of the invariance of double ratios under a central projection; harmonic range (set of 4 points), construction of fourth harmonic, pencil of lines (defined as any set of concurrent lines), harmonic pencil, complete quadrilateral (tacitly assumed to be a plane figure), complete quadrangle, principle of duality (planar), pole and polar with respect to a circle (in terms of inverse points and perpendicularity), Desargues' Theorem, and straight-edge constructions. This reviewer considers the inclusion of concepts from projective geometry very desirable, the minor deviations from standard terminology (as indicated by some of the above parenthetical remarks) unfortunate but not serious.

Chapter 8 (23 pp.) includes radical axis, power of a point with respect to a circle, coaxial circles, the nine-point circle, Euler line, inscribed and escribed circles, Feuerbach's Theorem, Pascal's Theorem, isotomic and isogonal lines, and Miquel point.

Chapter 9 (8 pp.) is based on Steiner's Theorem.

Chapter 10 (15 pp.) involves the construction of special quadrilaterals.

Chapter 11 (13 pp.) is concerned with concurrent lines; Chapter 12 (5 pp.) with impossible ruler and compasses constructions.

Very few typographical errors were noted. The emphasis upon relations among theorems and concepts is commendable. The restriction to plane geometry is conventional. The modification of the traditional content of "college geometry" seems to this reviewer to be a much-needed move in the proper direction. Whether the change is sufficiently extensive will depend upon the tastes of the instructor. This reviewer considers the text to be a good start.

BRUCE E. MESERVE
Montclair State College

Introduction to Multivariate Statistical Analysis. By T. W. Anderson. Wiley, New York, 1958. 374 pp. $12.50.

This is an excellent introductory text. To read it one needs to know the calculus, including the calculus of functions of several variables, elementary matrix theory, including characteristic vectors and roots, and the contents of a course in probability and mathematical statistics having the calculus as prerequisite and going through elementary regression theory, limiting distributions, and an introduction to statistical inference including such notions as maximum likelihood estimates, power, invariance, Bayes and minimax estimates and decision theory. It is written for statisticians.

The topics covered are well indicated by the chapter headings. After the introductory chapter come chapters on the multivariate normal distributions; estimation of mean vector and covariance matrix; sample correlation coefficients; generalized T^2; classification; the distribution of the sample covariance matrix and generalized variance; the general linear hypothesis; independence of sets of variates, hypothesis of equality of mean vectors and covariance matrices; principal components, canonical correlations; characteristic roots and special topics.

There are many worked-out examples in the text and at the ends of the chapters appear a large number of carefully chosen exercises. A full bibliography is given.

The book emphasizes the distributional aspects of multivariate analysis. This is still the most developed part of the subject, and its value in applied work has been considerably increased by the use of electronic computers.

For those with an adequate background the book is well written and provides an excellent summarization of much of the work in this area.

The reviewer wishes that the index were more complete, and that the author had written a brief overview of statistical inference in the same spirit as that in which he provided an appendix on matrix theory. But these are minor quibbles in commenting on an excellent book.

William G. Madow
Stanford University and Stanford Research Institute

Introduction to the Theory of Sets. By Joseph Breuer (translated by Howard F. Fehr). Prentice-Hall, Englewood Cliffs, N. J., 1958. viii+108 pp. $4.25.

Helped on by Sputnik, more nonmathematicians (including many secondary school teachers) are interested in taking an additional mathematics course. The real difficulty is to provide a worthwhile course, something that does not require calculus, something that is "new" and interesting. Harold Franklin Fehr's translation of Joseph Breuer's *Introduction to the Theory of Sets* provides an excellent text for the introduction to real and "new" mathematical ideas to the nonspecialist. Set theory, by adding to the concept of finite number the con-

cepts of transfinite numbers, should add to the appreciation of elementary mathematics.

A brief and accurate description of the aim of the book and of the manner in which the topics are developed is given by the translator in the Preface: "There is now a need for a treatment of set theory in English, from a less-than-abstract axiomatic approach, sufficiently elementary to serve as an introduction to the subject for college and high school instructors, college students, and interested laymen. This book meets that need. A naive approach, which depends upon observation of the concrete world for its development and meaning, is a natural way to introduce the subject, and this procedure is used in the following exposition. Little by little, certain properties and principles are developed, which in turn are used to prove further theorems concerning sets as collections of abstract entities. Thus one is led from concrete finite sets, to cardinal numbers, to infinite cardinals, and thence to ordinals via the use of ordinal-types. Abstract set theory based on an axiomatic system is not treated here. . . . The axiom of choice and its relation to the theorem of well-ordering have had tremendous effect on the whole development of set theory, but these are matters of concern to the mathematician rather than to the neophyte."

The chapters are: Finite Sets, Infinite Sets (cardinal numbers), Ordered Sets, Point Sets, Conclusion (paradoxes and a discussion of formalism and intuitionalism), and an Appendix (including a brief historical outline and a bibliography). All material is introduced first with examples, and most proofs that are given are directly based on an example or are illustrated in detail by one. In addition, at the end of each short section (*i.e.* every five or six pages) there is a short set of exercises asking about further examples or about details of previous explanations. Complete answers for all the exercises are given in the back of the book. Facts with proofs beyond the limited scope of the book (as the Well-Ordering Theorem or the nonexistence of sets M and N with M equivalent to no subset of N, and conversely) are stated, explained and used.

The inclusion of the historical material, and the discussion of different viewpoints in mathematical thought, together with the statement of several unsolved problems and the mention of areas where much is still unknown, are ways in which this book seeks to broaden the reader's outlook on mathematics.

There is some danger that the reader will feel that he is being "talked down to" in the early pages of the book and then in the closing section feel discouraged by the difficulty of understanding some comments or that he will miss the significance of them. This is particularly true in the section on Point Sets, offered as an application of set theory. This section is not as detailed in its presentation as the other material and contains many ideas that are stated and not pursued nor closely linked as applications. One hopes that the user of this book realizes the introductory and incomplete nature of much of the material and is spurred on to further study of axiomatic set theory.

LIDA K. BARRETT
University of Utah

Introductory Probability and Statistical Inference for Secondary Schools. (An Experimental Course, Preliminary Edition.) Prepared for the Commission on Mathematics, College Entrance Examination Board, 1957. x+182 pp. $1.00.

It is a pleasure to read this anonymous little book and to speculate on how a first course in statistics in college could maintain the interest which must surely be developed by such a high school course. The authors clearly expose the principal concepts in the modern approach to probability and statistical inference using a minimum of mathematical apparatus and using examples and exercises which are almost uniformly interesting in themselves and which should indicate to the student the potential practical usefulness of statistical inference.

After acclimatizing the reader with 42 pages on frequency distributions, histograms, ogives, means, standard derivations and Chebyshev's theorem, 28 pages are given over to the introduction of concepts in probability including sample spaces, events, independence, conditional probability. The next 20 pages go over these concepts again with definitions and theorems expressed more formally. For a 42 page introduction to statistical inference, the binomial distribution is introduced and used in testing hypothesis in acceptance sampling with emphasis on operating characteristic curves. The three appendices are described briefly below.

In the selection of topics to introduce the student to statistical inference, the present book is in many ways similar to S. S. Wilk's *Elementary Statistical Analysis* (Princeton University Press, 1948, 1951). All the topics in the present text are included in Wilks' and each major topic occupies about the same amount of space. Wilks' text is written for a college level course, is longer, and assumes more previous study of mathematics. The present text spends more space on fundamental concepts, using examples and exercises which develop the meaning of these concepts without getting into complex mathematical manipulations.

For the most part the authors have been careful to say what they mean. In most places where an advanced reader can detect ambiguities, the correct idea will be conveyed to the reader at whom the book is aimed. More nearly complete statements might well convey no meaning at all. The most serious error the reviewer found was in attributing the use of cream to Fisher's tea-taster. Of course, she used milk in her tea.

The mathematical prerequisites to an understanding of the text include knowledge of roots of quadratic equations, some manipulative ability with inequalities, and some understanding of the relation between single valued functions and their graphs. Two other prerequisites to the main body of the text are covered in appendices: elementary set theory and permutations and "selections." A third appendix treats mathematical induction. The appendices are more complete than necessary for the text proper and one might suspect that the authors have some convictions as to what every high school student should know about mathematics.

College students of statistical inference will be pleased to find a book in

which fundamental concepts are made clear. The present reviewer is incompetent to guess the number of high schools which will in the near future add a course in probability and statistical inferences to their offerings. Those which do, and we can hope that the number is large, will find the present text excellent.

KENNETH J. ARNOLD
Michigan State University

Initiation à la Logique. (Collection de Logique Mathématique, Série A, XIII.) By Le R. P. Dubarle. Paris, Gauthier-Villars, 1957. 90 pp. 1400 fr. (about $3.50).

The author calls this an "inventory and classification" of the principal subjects of mathematical logic for those seeking only an introduction and a minimal acquaintance with the field. The book is divided into three major sections. The first attempts to explain the spirit of symbolization that led from classical logic to contemporary symbolic logic and the relation of this change to the formalization of mathematics. The author's philosophical orientations reveal themselves in these pages. He views the whole development as due to a nominalistic interpretation of mathematical entities. The second section provides the inventory of the concepts of mathematical logic, which have become so widely used in courses in mathematical foundations and modern algebra. The outline ends with a reference to multivalued logic, intuitional logic, modal logic, natural deduction, and combinatory logic. The final section discusses the limits of the implications of logic and mathematics and refers to Gödel's and Church's theorems on incompleteness and consistency.

It is difficult to evaluate this book. Although it achieves its aim of being an inventory quite adequately, it has all the defects of an inventory.

One point of Dubarle's is worthy of being underlined. Formalism is valuable, but its function is to represent results already achieved. Its use in "discovering" proofs is quite limited.

Typographical errors: Page 43, the definition of the existential quantifier has "$E(x)A(x)$" and should read "$(Ex)A(x)$."

The axiom 2.22 has "$\sim(x)A(x)$" and should read "$\vdash(x)A(x)$."

Page 84, the reference to Tarski should read "Einführung" and not "Fin Führung."

L. O. KATTSOFF
Harpur College

Business Mathematics (4th Ed.). By C. C. Richtmeyer and J. W. Foust. McGraw-Hill, New York, 1958. xii+412 pp. $5.75.

This college textbook is designed for students who are preparing to teach commercial arithmetic in secondary schools or to enter commercial fields and for other students who wish to become acquainted with the applications of elementary mathematics in business. The first four chapters review operations with integers, common fractions, decimal fractions, approximate numbers and

percentage. Other chapters pertain to simple interest and bank discount, graphs, measurement, common logarithms, arithmetic progressions and short-term buying, geometric progressions and compound interest, annuities, equations and statistics. This is an appropriate selection of topics and the sequence is excellent from the pedagogical point of view.

The simple, concise exposition includes many illustrations. Each chapter contains numerous verbal problems, a self-test and supplementary references. The problems are varied and interesting and supply additional information about current business practices. Tables are provided in the text. The format is excellent.

On the whole this is a text which the student can read with understanding. However, the instructor should note that in some instances the terminology is not precise; for example, the word "number" should be replaced by "integer" in some of the statements of section 11.

It is the opinion of this reviewer that chapters 1–3 are too elementary and that this treatment of numbers does not provide adequate preparation for the teaching of arithmetic in the secondary school. The chapters pertaining to graphs, installment buying, compound interest, annuities and statistics are excellent and may be recommended to all teachers of arithmetic.

Those who are familiar with earlier editions will find that in this revision the authors have extended and rewritten the problem material as well as the exposition of certain topics.

EDITH R. SCHNECKENBURGER
University of Buffalo

Intermediate Algebra for Colleges. By Gordon Fuller. Van Nostrand, New York, 1958. iv+258 pp. $3.90.

The claim is made that this book is an entirely new presentation of Intermediate Algebra. Such a strong statement is bound to be questioned, considering the number of books published on this subject. There are some outstanding features in this book. The author carefully builds on the foundation of the first chapter so that all the material follows readily, except for special topics. He has broken down most of the chapters into a large number of sections so that a minimum number of new ideas are encountered before a set of problems. In sections where rules are the end result, he gives examples that clearly bring out the rules. The justification of the laws of signed integers is well done. His explanation of stated problems is excellent. The chapter on logarithms is well done. The chapter on functions brings out implicit as well as explicit functions with a lucid explanation of the concept. Finally, he has a great number of problems from which to choose in nearly all the sections.

From a look at the chapters, one can see that the author has included the standard topics of Intermediate Algebra. The one possible exception to this is the lack of a discussion of rationals in either decimal or fractional form. One might criticize the book for the lack of this and other special topics. However,

there are always some topics omitted that some think should be included. On the whole, this is an above-average text.

LEONARD E. FULLER
Kansas State University

Zahlwort und Ziffer, Bd. I. By Karl Menninger. Vandenhoeck and Ruprecht, Göttingen, 1957. 221 pp. $4.00 (DM 16.80). Paperbound.

The author has rewritten and extended his original treatise on *A Cultural History of Number* into two volumes, of which this book is the first. Herein is treated the historical development of the number sequence (Zählreihe) and the number language. The second volume will treat the origin and development of number symbols, counting boards, and computation. The first part of this book presents a very scholarly and interesting picture of the cultural-historical growth of the sequence of number words or counting words. The words are traced back to their origin and the principles underlying their construction are supported with photographs, illustrations and primary historical sources.

The second half of the book deals with the origins of the spoken form of the number words in the many languages. Again there are ample illustrations and many tables of ordered number words to aid the reader. The author has taken special efforts to bring some light into the often cloudy region of etymology of number language. That the German word *zahl*, and the English words *tale* and *teller* all have the same root *tal*, meaning *to tell*, hence, *number*, is illustrative of much that is done with the other number words. There is detailed reference to the literature and an extensive bibliography of 112 references. The German is written in a simple and clear manner and can be read by a second year student of the language without too much difficulty. There is no comparable book in the English language.

HOWARD F. FEHR
Teacher's College, Columbia University

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

Bradley University: Professor A. E. Gault has retired as Chairman of the Department of Mathematics and is succeeded by Professor M. G. Moore; Mr. Jack Barr, University of Iowa, and Captain G. C. Carlstedt, U. S. Coast Guard, Retired, have been appointed Instructors.

Vassar College: Associate Professor Winifred Asprey has been promoted to Professor;

Associate Professor Janet McDonald has been awarded a National Science Foundation Faculty Fellowship for 1959–1960 and will spend the year on leave at Indiana University; Miss Edith Moss, Wellesley College, has been appointed Instructor.

Mr. Maurice Anderson, University of Nebraska, has been appointed Instructor at the University of Wichita.

Mr. N. F. Beach, Eastman Kodak Company, Rochester, New York, has been promoted to Assistant Manager of film emulsion and plate manufacturing.

Dr. W. J. Berger, Raydist Navigation Corporation, Hampton, Virginia, has accepted a position as Research Scientist with Lockheed Missiles & Space Division, Palo Alto, California.

Assistant Professor R. C. Boles, North Carolina State College, has been appointed Professor of Mathematics and Chairman of the Mathematics Department at Tennessee Polytechnic Institute.

Mr. H. H. Brown, Lockheed Aircraft, Palo Alto, California, has accepted the position of Senior Mathematician with the Corporation for Economic and Industrial Research, Arlington, Virginia.

Assistant Professor P. L. Butzer, on leave from McGill University, is a Visiting Professor at the Technical University, Aachen, Germany.

Mr. F. P. Callahan, Jr., Philco Corporation, Philadelphia, Pennsylvania, has accepted a position as Mathematician with General Atronics, Bala-Cynwyd, Pennsylvania.

Dr. E. H. Connell, Lockheed Aircraft Corporation, Palo Alto, California, has been appointed Assistant Professor at the University of Miami.

Mr. E. L. Ellis, U. S. Army Ballistics Research Laboratory, Aberdeen, Maryland, has accepted the position of Mathematician-Programmer with Bendix Systems Division, Ann Arbor, Michigan.

Assistant Professor R. I. Fields, University of Louisville, has been promoted to Associate Professor and has been appointed Co-director of the University's Computing Laboratory.

Mr. A. L. Gilmore, Jr., Eglin Air Force Base, Florida, has accepted the position of Mathematician-Chief Programmer at the Electronic Computer Branch, Waterways Experiment Station, Vicksburg, Mississippi.

Dr. Arthur Grad, Office of Naval Research, has been appointed Program Director for Mathematical Sciences, Division of Mathematical, Physical, and Engineering Sciences of the National Science Foundation.

Assistant Professor R. T. Gregory, University of California, Goleta, has been appointed Associate Professor at the University of Texas.

Dr. J. J. Harton, Jr., Motorola, Inc., Riverside, California, has accepted the position of Senior Dynamics Engineer with Convair, San Diego, California.

Mr. D. L. Klippenstein, Korea Mennonite Central Committee, Taegu, Korea, has accepted the position of Programmer in the Computer Section of the Babcock and Wilcox Company, Barberton, Ohio.

Dr. U. R. Kodres, Iowa State College, has accepted a position as Associate Engineer for International Business Machines Corporation, Poughkeepsie, New York.

Professor Djuro Kurepa, head of the Institute of Mathematics, University of Zagreb, Jugoslavia, will lecture in the 1959 Summer Session at Teachers College, Columbia University.

Mr. S. L. Lida, formerly Applied Science Representative for the International Business Machines Corporation, New York, New York, has been appointed Manager, Communications Systems Marketing for the I.B.M. Corporation, White Plains, New York.

Visiting Professor G. G. Lorentz, University of Tubingen, Germany, has been appointed Professor at Syracuse University.

Mr. B. J. McDonald, Research Division of the U. S. Navy Mine Defense Laboratory in Panama City, Florida, is now associated with the Mathematical Analysis Section of Eglin Air Force Base, Florida.

Dr. Katsumi Okashimo, University of Toronto, is now a Defence Research Board Representative at the Computation Centre, McLennan Laboratory, University of Toronto.

Dr. Gideon Peyser, Newark College of Engineering, has been appointed Associate Professor at Pratt Institute.

Mr. E. H. Primoff, Institute of Mathematical Sciences, New York University, has accepted a position as Programmer with G. C. Dewey, New York City.

Mr. D. G. Russell, Shell Oil Company, Midland, Texas, has accepted the position of Reservoir Engineer in the Production Research Division of Shell Development Company, Bellaire, Texas.

Dr. D. R. Shreve, Carter Oil Company, Tulsa, Oklahoma, has been appointed Associate Professor at Oklahoma State University.

Mr. Irwin Stoner, Raytheon Manufacturing Company, Bedford, Massachusetts, has accepted the position of Senior Engineer with the Computation Laboratory, Arma Division, American Bosch Arma Corporation, Garden City, New York.

Dr. W. E. Wilson, Pratt Institute, has been appointed Chairman of Engineering, Harvey Mudd College of Science and Engineering.

Associate Professor O. M. Rasmussen, University of Denver, died on June 20, 1958. He was a member of the Association for eleven years.

SUMMER SESSIONS

Michigan State University, June 24–September 4: Professor Hocking, sets and abstract spaces, topology III; Professors Larcher and Doyle, theory of equations; Professor Parkus, thermal stresses; Professor Stelson, differential equations, advanced mathematics for engineers; Professor Wasserman, vector mechanics, numerical analysis; Professor Weeg, matrices and groups. June 24–July 30: Professor Hall, potential theory; Professor Norman, partial differential equations; Professor Stewart, theory of numbers, higher algebra III. July 31–September 4: Professor Doyle, complex variables II.

University of Buffalo, June 29–August 8: Professor Olson, history of mathematics; Professor Schneckenburger, infinite series, modern secondary mathematics.

University of California, Los Angeles, June 18–July 28: Visiting Professor Mackey, mathematical foundations of quantum mechanics; Professor Horn, introduction to advanced analysis; Visiting Professors Johnson and Lay, fundamental mathematical concepts (this last course is designed for secondary school teachers).

University of Illinois, June 15–August 8: Professor Suzuki, linear algebra; Professor Day, elementary geometry from a modern viewpoint; Professor Ketchum, functions of real variables. In addition, the following courses will be given: fundamental concepts, advanced algebra, introduction to higher algebra, complex variables and applications, introduction to higher analysis, advanced statistics, introduction to numerical analysis.

University of Texas, June 16–August 20: Professor Cooper, theory of functions of a complex variable; Professor Ettlinger, foundations of differential equations with applications, topics in teaching problems in mathematics; Professor Greenwood, boundary value problems; Profesor Guy, Fourier and Laplace transforms, topics in modern mathematics; Mr. Hurt, vector and tensor analysis; Professor Lane, mathematical statistics; Professor Lubben, introduction to modern projective geometry, topics in modern algebra; Professor Moore, theory of sets; Professor Wall, infinite processes, functions of a complex variable; Mrs. Barnes, topics in elementary mathematics; Staff, conference course, thesis, dissertation.

ANNUAL MEETING OF ASEE

The Mathematics Division of the American Society for Engineering Education will meet on Tuesday, June 16, 1959 at the University of Pittsburgh in conjunction with the 67th annual meeting of the ASEE. The following program will be presented.

Luncheon and Business Meeting at 12 noon: *A mathematician in an engineering school*, W. G. Warnock, Rensselaer Polytechnic Institute.

Joint Meeting with Physics Division at 2:00 P.M.: *An experimental program in teaching physics to engineers*, R. Resnick, Rensselaer Polytechnic Institute; *The teaching of modern mathematics to engineers*, J. P. Russell, Polytechnic Institute of Brooklyn; *Developments in the teaching of physics to engineers at Yale*, W. W. Watson, Yale University; *Coordinating the contents of mathematics and physics courses*, P. H. Cook, Pratt Institute.

Joint Dinner with Industrial Engineering Division at 6:30 P.M.: *Systems simulation*, Richard Bellman, Rand Corporation.

INTERNATIONAL CONGRESS OF MATHEMATICIANS 1962

The Secretary of the International Mathematical Union has announced that:

The Committee authorized by the final plenary session of the Congress in Edinburgh to determine the location of the International Congress 1962 has accepted an invitation from the Swedish National Committee for Mathematics and the Swedish Mathematical Society in the following terms:

"To mathematicians of all countries.

The Swedish National Committee for Mathematics and the Swedish Mathematical Society have the honour of inviting you to the next International Congress of Mathematicians, to be held in Stockholm during the Summer of 1962.

We will do our best to make the Congress scientifically successful and enjoyable, hoping that it will stimulate the interaction between mathematicians in different fields and countries. Ake Pleijel, Chairman of the Swedish National Committee for Mathematics, and Goran Borg, Chairman of the Swedish Mathematical Society."

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary Treasurer, announces that the following 270 persons have been elected to membership by the Board of Governors on applications duly certified.

MOSTAFA A. ABDELKADER, B.S. (Cairo) 9, Sh. Hamadan, Giza, Egypt.

ALLEN T. ADAMSON, M.A. (Whitman) Head of Dept., Cashmere High School, Washington.

ORVILLE W. ADDINGTON, M.A. (Geo. Washington) Asso. Professor, Virginia Polytechnic Institute.

CHARLES W. ALBERT, Student, University of British Columbia.

JACK ALVAREZ, B.E.E. (Georgia Tech.) Engineer, Boeing Airplane Co.

JEAN A. ANCTIL, M.A. (Laval) Head of Dept., College Militaire Royal de St. Jean.

ALLEN E. ANDERSEN, Ph.D. (Harvard) Head of Dept., University of Massachusetts.

JOHN D. ARRISON, M.A. (Michigan S.U.) Asst. Professor, Iowa Wesleyan College.

VERE AULIE, M.A. Teacher, South Tahoe High School, Altahoe, California.

GEORGE H. AUSTIN, M.S. (Wisconsin) Analyst, Investors Diversified Services.

RAYMOND G. AYOUB, Ph.D. (Illinois) Asso. Professor, Pennsylvania State University.

B. LOU ANN BARNO, Student, Oklahoma State University.

STANLEY L. BASIN, Student, San Jose State College.

BRO. CHRISTOPHER BEAULIEU, F.S.C. B.A. (Catholic) Teacher, St. Augustine High School, Brooklyn, New York.

CAMERON C. BOGUE, M.A. (Michigan) Instr., California State Polytechnic College.

JAMES C. BOLEN, M.S. (A. & M. Coll. of Texas) Instr., Agricultural and Mechanical College of Texas.

ROBERT W. BOLL, B.S. (St. Louis) Grad. Fellow, St. Louis University.

HAROLD E. BOWIE, M.A. (Maine) Professor and Head of Dept., American International College.

MARK C. BREITER, M.A. (Geo. Washington) Mathematician, Wright-Patterson Air Force Base.

KAREN D. BRENDER, Student, State University of Iowa.

HARRIET BROOKS, M.A. in Ed. (Clark) Chairman of Dept., Auburn High School, Massachusetts.

MRS. CLARA D. BROWN, M.A. (Florida) Head of Dept., Crestview High School, Florida.

JAMES P. BROWN, Ed.S. (Geo. Peabody) Chairman of Dept., Southwest High School, Atlanta, Georgia.

O. ROBERT BROWN, JR., Student, Oberlin College.

JAN R. BRUNDIN, Student, University of Alaska.

ALFRED L. BUNKE, M.A. (Columbia) Principal Statistician, New York State Dept. of Labor.

EVANGELINE A. BURMAN, A.B. (Southern California) Technical Staff, Hughes Aircraft Co.

REV. SIMEON CAPIZZI, O.F.M., B.S. (Illinois) Asst. Prof., St. Bonaventure University.

ROBERT W. CARLSON, B.S. (Roosevelt) Teacher, Martinsville High School, Missouri.

JAMES W. CASPERS, M.S. (Washington) Electronic Scientist, U.S. Navy Electronics Lab.

CARL S. CAVE, M.A. (Missouri) Asst. Professor, Missouri School of Mines.

JAMES T. H. CHAO, B.S. (Southern California) Electrical Engineer, Haughton Elev. Co.

STEVEN J. CHULAY, M.S. (Wisconsin) Staff Engineer, Massachusetts Institute of Technology Instrumentation Lab.

ADRIAN N. CLARK, M.S. (M.I.T.) Vice President, D. Van Nostrand Co.

CHARLES A. CLARK, A.B. (Boston) Program Analyst, AVCO Research & Advanced Development.

C. ROBERT CLEMENTS, A.B. (Hamilton) Teacher, Choate School.

JAMES B. COLLINS, C.E. (Cooper Union) Asso. Professor, St. Peter's College.

JOSE CONTRERAS, B.A., B.S. in Ed. (Southwest Texas S.C.) Teacher, Falfurrias High School, Texas.

PHILIP COOPERMAN, Ph.D. (New York) Asst. Professor, University of Pittsburgh.

RUSSEL B. COOVER, A.M. (Columbia) Head of Dept., Western High School, Washington, D. C.

ARLING L. CORDELL, M.A. (North Texas S.C.) Superintendent and Teacher, Gordon, Texas.

ERNEST A. COVEY, Student, Abilene Christian College.

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HENRY H. CRAPO, B.A. (Michigan) Teaching Asst., Massachusetts Institute of Technology.

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LINDOLPHO DE CARVALHO DIAS, Eng. (National School of Eng., Brazil) Asst. Professor, Centro Brasileiro de Pesquisas Fisicas.

HOWARD C. DICKEY, Analyst, Chicago Pneumatic Tool Co.

RAFAEL DOMINGUEZ, M.D., B.A. (Rosario) Head, Dept. of Research, Pathology, St. Luke's Hospital.

MRS. EDITH S. DONAHUE, A B. (Coll. for Teachers, Albany) Teacher, New Hartford Central School, New York.

JOHN W. DUNWOODIE, M.A. (Michigan) Teacher, Detroit Board of Education, Michigan.

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LESTER V. HOFFMAN, Student, Columbia University.

JESSE HOLBERT, Student, Orange County Community College.

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LT. WILLIAM M. MCKEEMAN, A.B. (California, Berkeley) USN, AEWRON Fifteen, NAS, Patuxent River, Maryland.
CATHERINE D. MEEHAN, M.A. (Wayne S.U.) Head of Dept., Port Huron High School, Michigan.
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CHARLES L. WILLIS, M.A. (N. Texas S.C.) Instr., Tarleton State College.

ERMA A. WOOD, M.Ed. (Houston) Head of Dept., Spring Branch Senior High School, Houston, Texas.

COMMANDER WALTER M. A. WYNNE, M.S. (Yale) Adjunct Professor, Polytechnic Institute of Brooklyn.

ANDREW C. YORKE, Student, Rutgers University.

ALBERT ZADRAVETZ, B.S. (Illinois Inst. of Tech.) Chief Chemist, Savoy Drug & Chem. Co.

ERVIN E. ZYZAK, B.S. (Indiana) Asst. Research Engineer, Boeing Airplane Co.

REPORT OF THE TREASURER FOR THE YEAR 1958

Following is a summary of the report of Professor H. M. Gehman as Treasurer of the Association for the year 1958. The complete report has been approved by the Finance Committee and accepted by vote of the Board of Governors. Any member of the Association who wishes the complete report of the Treasurer may obtain it by writing to the Buffalo office of the Association.

There was a surplus of $5,701 in the Current Fund of the Association for 1958. The balances in the regular funds of the Association have increased during 1958 except for the Chace Fund. The cost of printing Slaught Papers and the third edition of Professional Opportunities in Mathematics has caused a small decrease in this fund.

Assets of the Association	January 1, 1958	December 31, 1958
M & T Trust Co., Buffalo	$ 25,703.17	$ 19,687.37
Savings Accounts	75,999.51	108,363.39
Securities	100,974.12	169,620.23
	$202,676.80	$297,670.99

Funds of the Association		
Current Fund	$ 325.80	$ 1,027.14
Carus Fund	20,235.25	26,706.76
Chace Fund	8,808.28	8,760.58
Houck Fund	11,309.92	13,832.45
Chauvenet Fund	1,328.43	1,620.95
Dunkel Fund	15,889.55	19,451.65
General Fund	38,414.63	51,428.83
	$ 96,311.86	$122,918.26
Visiting Lecturers Fund	$ 48,424.53	$ 38,285.38
Fund for Committee on Undergraduate Program	55,669.95	88,691.79
Fund for Committee on Films	85.66	—
Fund for Committee on High School Contests	2,184.80	427.34
Washington Conference Fund	—	2,337.42
Survey of Non-Teaching Mathematical Employment	—	1,218.43
Secondary School Lecturers Fund	—	16,054.92
Fund for Committee on Production of Films	—	27,827.45
	$202,676.80	$297,670.99

THE NOVEMBER MEETING OF THE INDIANA SECTION

The Fall meeting of the Indiana Section of the Mathematical Association of America was held at Marian College, Indianapolis, on November 6, 1958. The program was as follows:

Professor Judah Rosenblatt of Purdue University gave a one hour lecture on *Statistics and Aircraft Warning Systems.*

Professor Charles Brumfiel of Ball State Teachers College reported on the 13th annual national T.E.P.S. conference at Bowling Green State University.

Professor J. C. Polley, Wabash College, described the activities of the Indiana School and College Committee on Mathematics.

A panel composed of Professors Charles Brumfiel, Melvin Henriksen of Purdue University, Donald Lewis of the University of Notre Dame, and George Whaples of Indiana University, and moderated by Professor Merrill Shanks of Purdue University, discussed proposals of the M.A. A. Committee on the Undergraduate Program.

CHARLES BRUMFIEL, *Secretary*

THE JANUARY MEETING OF THE NORTHERN CALIFORNIA SECTION

The twenty-first annual meeting of the Northern California Section of the Mathematical Association of America was held at Stanford University, January 17, 1959. Professor B. J. Lockhart, Chairman of the Section, presided at the morning session and Professor G. C. Preston, Vice-Chairman of the Section, presided at the afternoon session. There were 134 persons in attendance, including 96 members of the Association.

At the business meeting the following officers were elected for the coming year: Chairman, Professor G. C. Preston, San Jose State College; Vice-Chairman, Professor S. P. Hughart, Sacramento State College; Secretary-Treasurer, Professor Roy Dubisch, Fresno State College.

By invitation of the section, Professor J. L. Snell, Dartmouth College and Stanford University, delivered an address at the morning session entitled *Markov Chains and Their Applications.* An abstract of this address follows:

Recent applications of mathematics to the social sciences have given a renewed interest to the study of finite Markov chains. A procedure developed with J. G. Kemeny for systematically computing many of the basic descriptive quantities for a Markov chain was described. A discussion of some of the new applications of Markov chains was given.

At the end of the afternoon session a panel discussion was held by the newly-appointed Committee to Study the Activities of the Section consisting of Professor David Blakeslee, Chairman, Professors Henry Alder, Roy Dubisch, Harley Flanders, J. G. Herriot, Marjorie Hoffman, Brooks Lockhart, G. C. Preston, and Messrs. Kenneth Skeen and E. H. Swift. Some of the topics discussed were: The type of program most desirable for the regular meetings; what can we do to cooperate more fully with other organizations; the possibility of joint meetings with the California Mathematics Council; what would constitute the most workable executive committee; should we continue the lectureship program; should dues be collected and activities increased; should we issue a bulletin; how can we best extend our activities to include Hawaii; how can we cooperate with the Academy of Science in planning mathematics projects; and how can we work for the improvement of instruction in high schools.

Also, at this time, reports were given on the high school contest and lectureship program.

The following papers were presented:

1. *Acceptability in mathematics*, by Professor C. C. Torrance, U. S. Naval Postgraduate School, Monterey.

Is it *necessary* that a method be "logical" for it to be acceptable? Is it *sufficient* that a method produce "the right answer" for it to be acceptable? It is claimed here that the answer to both questions is *no*; consequent difficulties are discussed, and a method of resolution is indicated.

2. *Medial quasigroups*, by Professor D. A. Norton, University of California, Davis.

A short review of the concept of mediality, its significance and elementary properties.

3. *Roots and canonical forms of compound matrices*, by Dr. C.M. Ablow and Dr. J. L. Brenner, Stanford Research Institute.

Let A be a matrix in which the ith row R_i is obtained by applying the $(i-1)$th power of a permutation P to the sequence of elements $\{a_j\}_1^n$ of the first row. If P is the circular permutation of

$a_j \to a_{j-c}$, c an integer (subscripts being reduced modulo n), then A is called a c-circulant. The a_j can be numbers, or even $m \times m$ square matrices; in the latter case A is an $mn \times mn$ matrix. Properties of c-circulants are derived. In particular, some power of each latent root of any such matrix is shown to be obtainable as the root of a matrix of lower order. The results have applications in physics and chemistry.

4. *Strategies and learning models*, by Professor S. J. Bryant, Fresno State College, introduced by the Secretary.

In some learning experiments of the kind analyzed by Estes, it is shown that if the subject follows a certain type strategy his behavior will be similar to, and in some cases identical with, the behavior predicted by the Estes model.

5. *Some recent changes in the undergraduate mathematics program at the University of California, Berkeley*, by Professor Harley Flanders, University of California, Berkeley.

Several changes and experimental changes in the calculus and advanced calculus sequences were discussed, as was the new honors program in mathematics. Proposed additions to the upper division course list were mentioned.

ROY DUBISCH, *Secretary*

THE EARLE RAYMOND HEDRICK LECTURES

At the St. Louis meeting in December 1952 the Board of Governors of the Association voted to institute the custom of having a series of three expository lectures given at the summer meetings of the Association. It seemed appropriate to name these lectures after the late Professor Earle Raymond Hedrick, one of the founders and first president of the MAA. Each Hedrick lecturer is encouraged to publish his lecture as a Carus Monograph, a Slaught Paper, an article in this MONTHLY, or in other forms.

The following series of Hedrick Lectures have been delivered:

1952. Professor Tibor Rado, Ohio State University: "Derivatives and Jacobians."

1953. Professor P. R Halmos, University of Chicago: "Axiomatic Set Theory."

1954. Professor L. H. Loomis, Howard University: "Convex Sets."

1955. Professor Mark Kac, Cornell University: "Familiar Things from an Unfamiliar Point of View."

1956. Professor J. C. Oxtoby, Bryn Mawr College: "Category and Measure."

1957. Professor Leo Zippin, Queens College: "Topological Transformation Groups."

1958. Dr. A. S. Householder, Oak Ridge National Laboratory: "Some Mathematical Problems Arising in Computations with Matrices."

1959. Professor William Feller, Princeton University (Invited).

THE CHAUVENET PRIZE

The Chauvenet Prize of the Association is awarded at three-year intervals for a noteworthy expository paper published during a preceding three-year period by a member of the Association. It is expected that the next prize will be awarded at the Annual Meeting to be held in January 1960 for a paper published during the period 1956–1958. The prize of $100 is awarded for a paper which will come within the range of profitable reading for members of the Association. The purpose of the prize is to stimulate expository contributions in mathematical journals on the part of the younger American scholars.

The Chauvenet Prize Fund of the MAA was established in 1925 by a contribution of $100 by Professor J. L. Coolidge, then President of the Association. There were subsequent gifts of $500 from Professor W. B. Ford and $100 from Professor Dunham Jackson. At various times the Association has also transferred money from its general funds to the Chauvenet Prize Fund.

The Chauvenet Prize has been awarded eleven times as follows:

1925 G. A. Bliss, "Algebraic Functions and Their Divisors," Annals of Mathematics.
1929 T. H. Hildebrandt, "The Borel Theorem and Its Generalizations," Bulletin of the American Mathematical Society.
1932 G. H. Hardy, "An Introduction to the Theory of Numbers," Bulletin of the American Mathematical Society.
1935 Dunham Jackson, "Series of Orthogonal Polynomials" and "Orthogonal Trigonometric Sums," Annals of Mathematics: "The Convergence of Fourier Series," this MONTHLY.
1938 G. T. Whyburn, "On the Structure of Continua," Bulletin of the American Mathematical Society.
1941 Saunders MacLane, "Modular Fields," and "Some Recent Advances in Algebra," both in this MONTHLY.
1944 R. H. Cameron, "Some Introductory Exercises in the Manipulation of Fourier Transforms," National Mathematics Magazine.
1947 P. R. Halmos, "The Foundations of Probability," this MONTHLY.
1950 Mark Kac, "Random Walk and the Theory of Brownian Motion," this MONTHLY.
1953 E. J. McShane, "Partial Orderings and Moore-Smith Limits," this MONTHLY.
1956 R. H. Bruck, "Recent Advances in the Foundations of Euclidean Plane Geometry," this MONTHLY.

BY-LAWS OF THE MATHEMATICAL ASSOCIATION OF AMERICA (INC.)

(As amended to February 1, 1959)

ARTICLE I—NAME, PURPOSE AND CORPORATE SEAL

1. This organization shall be known as

THE MATHEMATICAL ASSOCIATION OF AMERICA (INCORPORATED)

2. Its object shall be to assist in promoting the interests of mathematics in America, especially in the collegiate field, by holding meetings in any part of the United States or Canada for the presentation and discussion of mathematical papers, by the publication of mathematical papers, journals, books, monographs, and reports, by conducting investigations for the purpose of improving the teaching of mathematics, by accumulating a mathematical library and by coöperating with other organizations whenever this may be desirable for attaining these or similar objects.

3. The Corporate Seal of the Association shall have inscribed thereon the name of the Association and the words "Corporate Seal—Illinois."

ARTICLE II—MEMBERSHIP

1. Any person who is interested in the field of collegiate mathematics shall be eligible for election to membership in the Association.

2. Election to membership shall be by vote of the Board upon written application from the individual seeking admission endorsed by two members of the Association.

3. Those who were admitted to membership in The Mathematical Association of America (unincorporated) prior to October 1, 1920, and were in good standing as such on that date, were thereby admitted to membership in this Association (Incorporated).

ARTICLE III—BOARD OF GOVERNORS AND OFFICERS

1. The Officers of the Association shall be a President, a First Vice-President, a Second Vice-President, an Editor-in-Chief of the Official Journal (hereinafter called the "Editor"), a Secretary, a Treasurer, and an Associate Secretary.

2. There shall be a Board of Governors (hereinafter called the "Board"), to consist of the Officers, the Ex-Presidents for terms of six years after the expiration of their respective presidential terms, and of additional elected members (hereinafter called "Governors"). It shall be the function

of the Board to supervise all scholarly and scientific activities of the Association, to administer and control these activities, and to authorize expenditures of funds of the Association, except that at the demand of ten or more members of the Board, or at the demand of forty or more members of the Association, any proposal to alter or initiate a matter of policy shall be referred to the general membership of the Association for its decision. All members of the Board shall hold over until their respective successors are selected or appointed and qualify.

3. There shall be an Executive Committee, advisory to the Board, and consisting of the President, the two Vice-Presidents, the Editor, the Secretary and the Treasurer. It shall be the function of this Committee to review continually the policies and activities of the Association, to plan and organize new activities, to formulate in broad outline the programs of meetings and of publications, and in general to consider all matters of importance or of interest to the Association. This committee shall prepare the agenda for meetings of the Board, and shall analyze the implications and aspects of all matters which are to come before the Board for decision. It shall present to the Board the viewpoints suggested by such analyses, as well as all such facts as may seem pertinent, or as may in any way facilitate the Board's work.

4. A statement regarding any proposed action of the Board which makes or alters a question of policy shall be published in the official journal, or notice of such proposed action shall be mailed to each member, before final action has been taken, so that members of the Association may make known to the Board their individual views.

5. The Board shall have authority to fill vacancies *ad interim* in any office, including vacancies in the Board, and to make any other appointments necessary for the transaction of the business of the Association.

6. At all meetings of the Board of Governors a quorum shall consist of not less than five (5) members and no business may be validly transacted at a meeting at which less than a quorum is present; *provided* that any meeting of the Board, whether or not a quorum be present, may be adjourned to a specified time and place by a majority of the members present without notice to the members at large other than announcement at such meeting. Informal action based on a mail ballot by the members of the Board, if ratified at a properly convened meeting of the Board, shall be as valid and effective as if originally authorized at such meeting.

7. There shall be a Finance Committee responsible to the Board; at the direction of the Board. it shall receive and administer the funds of the Association, control its properties and investments, make its contracts, and exercise such powers as may be delegated to it by the Board. This committee shall consist of four members, including the Secretary and the Treasurer.

8. (a) The Officers and Governors of the Association shall be elected in part by the Board, in part by the general membership, and in part by the membership in the Sections of the Association or by the membership in constituencies authorized by the Board for territory where Sections do not exist.

(b) The membership at large shall elect in alternate years respectively a President and a First Vice-President, each for a term of two years, and shall elect each year two Governors, for terms of three years.

(c) The membership in each Section shall elect triennially a Governor for a term of three years. For these elections, at least two nominations shall be submitted to the members by a committee appointed for that purpose by the Chairman of the Section.

(d) The Board shall elect at appropriate times by ballot and for the terms stated: a Second Vice-President for two years; an Editor, a Secretary, a Treasurer, and an Associate Secretary, each for five years; and members of the Finance Committee (other than the Secretary and the Treasurer) for four years.

(e) The President shall be ineligible for reëlection. The Vice-Presidents, the Editor, and the Governors shall be eligible for reëlection only after an interim equal to their respective terms of office.

(f) Elections by the Board shall be made from nomination by the Executive Committee. At least two nominations shall be made for each office to be filled in the case of the Second Vice-

President and the members of the Finance Committee, and the Board may in any case reject all nominations made and call for a new list.

(g) The names of members to be printed upon the ballots, together with blank spaces in the case of elections by the general membership, shall be determined by a Nominating Committee to be appointed annually for that purpose by the President with the approval of the Board. Approximately six months before the date of the annual meeting all members shall be given an opportunity to nominate by mail a candidate for each office to be filled by the members for the ensuing year. Approximately one month before the annual meeting the Nominating Committee shall select a nominee for President out of the three persons who received the most votes for this office in the nominations; the Nominating Committee shall furthermore select two candidates for each other office to be filled by the members, one being the person who received the highest vote in the nominations and the other being selected from among the several nominees next in order. The election shall be by mail or in person and shall close on the day of the annual meeting.

9. The President shall be the Executive Officer of the Association, shall preside at all meetings of the Board of Governors and at the annual meeting of the Association. He shall have the usual duties pertaining to his office and such other duties as may from time to time be assigned him by the Board of Governors.

10. In the absence of the President, the First Vice-President (or in his absence the Second Vice-President) shall have and exercise the powers of the President. The Board of Governors may assign to the Vice-Presidents such duties as may from time to time be determined.

11. The Secretary shall have the usual duties pertaining to his office, including the custody of the records of the Association and of its Corporate Seal, the keeping of minutes of the meetings of the Board of Governors and of the annual meeting and special meetings and the giving of due notice of all regular and special meetings of the Association and of the Board of Governors. The Secretary shall also have the duty of seeing that whenever Governors are elected, including the election of Governors to fill vacancies, a Certificate, under the Seal of the Association, giving the names of those elected and the term of their office, shall be recorded in the Office of the Recorder of Deeds for Cook County, Illinois. Such Certificates shall be signed by the Secretary and verified by oath of the President.

12. The Treasurer shall have the usual duties pertaining to his office including the collection of dues and the supervision and safekeeping of the funds of the Association.

ARTICLE IV—MEETINGS

1. A meeting of the Association shall be held annually, at such time and place as the Board may direct. Special meetings of the Association may be called from time to time by the Board, or while the Board is not in session by the President of the Association, to be held at such time and place as may appear from the call.

2. The Board shall hold a meeting each year immediately preceding the annual meeting of the Association. Further meetings of the Board may be held from time to time at the call of the President or of any three (3) members of the Board.

3. Notice of any meeting of members of the Association shall be given by the Secretary at least thirty (30) days prior to the date set for each meeting. Notice of all meetings of the Board other than the regular meetings provided in Section 2 shall be given to each member of the Board at least fifteen (15) days prior to the date set therefor.

4. Any member of the Association or of the Board may waive notice with the same effect as if due notice had been given him.

5. At all meetings of the Association a quorum shall consist of not less than twenty-five (25) members and no business may be validly transacted at a meeting at which less than a quorum is present; *provided* that any meeting of the Association, whether or not a quorum be present, may be adjourned to a specified time and place by a majority of the members present without notice to the members at large other than the announcement at such meeting.

6. Members may take part and vote in person or by proxy at all meetings of the Association.

Article V—Sections

1. Any group of not less than ten (10) members of this Association may petition the Board for authority to organize a Section of the Association for the purpose of holding local meetings. The Board shall have power to specify the conditions under which such authority shall be granted. The by-laws of each Section when organized and any subsequent changes in these by-laws must be approved by the Board. The Board shall maintain general supervision over the activities of all Sections.

2. The Association shall not be obligated to pay from its treasury any of the expenses of such Sections except as the Board may provide.

Article VI—Official Publications

1. The Association shall publish an official journal, which shall be sent free to all members of the Association in accordance with Article VII.

2. The Board shall have full control of the publication and sale of the official journal and of all other official publications.

3. There shall be appointed by the Board a body of Associate Editors who shall give assistance in connection with the official journal.

4. The Board shall from time to time, as the need arises, make special provision for the management of any other official publications.

5. The Board shall fix the price of the official journal and of any other official publications of the Association, but in no case shall the journal be sold to non-members for less than the annual dues of individual members.

Article VII—Dues

1. Members of the Association shall pay an initiation fee of two dollars ($2) at the time of election. The Board of Governors may authorize the admission to membership of individuals and classes of applicants without payment of the admission fee.

2. The annual dues of each member shall be five dollars ($5), including a subscription to the official journal.

3. All dues shall be payable on the first of January of each year. Should the annual dues of any member remain unpaid beyond a reasonable time, his name shall be dropped from the list after due notice.

4. New members entering the Association after April 1 of any year shall have their dues prorated for the balance of the year, except when they desire to receive the full current volume of the official journal.

5. Any member who because of age is no longer in active service, who is in good standing at the time of his retirement and who has been a member of the Association for twenty years, may, upon notifying the Secretary of said retirement, be exempt from the payment of dues, with the privilege of obtaining the official journal at an annual cost of two dollars ($2).

Article VIII—Amendments to the Articles of Association and By-Laws

1. Changes in the Articles of Association or amendments to the By-Laws may be made at any annual meeting of the Association, or at any adjourned session, thereof, or at any special meeting of the Association called for such purpose, by a two-thirds ($\frac{2}{3}$) vote of those present and entitled to vote; *provided* that due notice concerning such amendment shall have been printed in the official journal, or mailed to each member, at least one (1) month before the date of such meeting. The Secretary shall give such due notice when so instructed by a vote of the Board of Governors or when so petitioned by at least forty members of the Association.

2. No changes in the Articles of Association shall have legal effect until a certificate thereof, verified by oath of the President and under Seal of the Association, attested by the Secretary, shall be filed in the office of the Secretary of State of the State of Illinois and recorded in the office of the Recorder of Deeds for Cook County, Illinois.

PERIODS OF SERVICE OF FORMER OFFICERS OF THE ASSOCIATION AS OF FEBRUARY 1, 1959

(Except for the offices of President and Secretary-Treasurer, this list includes only the names of those who have held office since January 1, 1954. For information about preceding years, consult the American Mathematical Monthly for March 1957.)

President

E. R. Hedrick	1916	Arnold Dresden	1933–1934
Florian Cajori	1917	D. R. Curtiss	1935–1936
E. V. Huntington	1918	A. J. Kempner	1937–1938
H. E. Slaught	1919	W. B. Carver	1939–1940
D. E. Smith	1920	R. W. Brink	1941–1942
G. A. Miller	1921	W. D. Cairns	1943–1944
R. C. Archibald	1922	C. C. MacDuffee	1945–1946
R. D. Carmichael	1923	L. R. Ford	1947–1948
H. L. Rietz	1924	R. E. Langer	1949–1950
J. L. Coolidge	1925	Saunders MacLane	1951–1952
Dunham Jackson	1926	E. J. McShane	1953–1954
W. B. Ford	1927–1928	W. L. Duren, Jr.	1955–1956
J. W. Young	1929–1930	G. B. Price	1957–1958
E. T. Bell	1931–1932		

Vice-President

W. L. Duren, Jr.	1953–1954	R. V. Churchill	1956–1957
H. S. M. Coxeter	1954–1955	B. W. Jones	1957–1958
G. B. Price	1955–1956		

Secretary-Treasurer

W. D. Cairns	1916–1942	W. B. Carver	1943–1947

Associate Secretary

Edith R. Schneckenburger 1948–1957

Editor

C. B. Allendoerfer 1952–1956

Governor (arranged alphabetically)

I. A. Barnett	1952–1955	F. F. Helton	1955–1958
C. F. Barr	1951–1954	M. S. Hendrickson	1955–1958
M. A. Basoco	1954–1957	D. L. Holl	1953–1954
H. W. Brinkmann	1955–1957	A. S. Householder	1956–1958
J. C. Brixey	1951–1954	Aughtum S. Howard	1951–1954
B. H. Brown	1952–1955	R. C. Huffer	1954–1957
S. S. Cairns	1953–1955	Ralph Hull	1954–1956
T. F. Cope	1951–1954	C. A. Hutchinson	1954–1957
H. H. Downing	1954–1957	S. B. Jackson	1953–1956
J. M. Earl	1951–1954	R. D. James	1952–1955
P. D. Edwards	1954–1957	L. W. Johnson	1954–1957
R. M. Foster	1954–1957	P. S. Jones	1953–1956
Philip Franklin	1954–1956	F. W. Kokomoor	1955–1958
R. F. Graesser	1952–1955	D. H. Lehmer	1952–1954
E. R. Heineman	1953–1956	F. A. Lewis	1952–1955

L. L. LOWENSTEIN	1955–1957	G. B. PRICE	1952–1955
K. O. MAY	1953–1956	J. F. RANDOLPH	1955–1958
A. E. MEDER, JR.	1957–1958	C. B. READ	1955–1958
R. J. MICHEL	1952–1955	F. A. RICKEY	1953–1956
E. B. MILLER	1953–1956	E. B. ROESSLER	1951–1954
W. E. MILNE	1952–1954	M. F. SMILEY	1956–1958
C. W. MUNSHOWER	1952–1955	ERNST SNAPPER	1957–1958
W. H. MYERS	1954–1957	C. E. SPRINGER	1954–1956
IVAN NIVEN	1955–1958	F. H. STEEN	1951–1954
C. O. OAKLEY	1953–1956	G. B. THOMAS, JR.	1955–1958
E. N. OBERG	1954–1956	C. W. TRIGG	1953–1956
MORRIS OSTROFSKY	1954–1957	A. W. TUCKER	1953–1955
H. P. PETTIT	1951–1954	M. A. ZORN	1955–1957
J. C. POLLEY	1951–1954		

CALENDAR OF FUTURE MEETINGS

Fortieth Summer Meeting, University of Utah, Salt Lake City, Utah, August 31–September 3, 1959.

Forty-third Annual Meeting, Conrad Hilton Hotel, Chicago, Illinois, January 28–30, 1960.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, University of Pittsburgh, May 2, 1959
ILLINOIS, Millikin University, Decatur, May 8–9, 1959
INDIANA, Valparaiso University, May 2, 1959
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Goucher College, Towson, Maryland, May 2, 1959
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA
MISSOURI
NEBRASKA
NEW JERSEY, Princeton University, November 7, 1959
NORTHEASTERN
NORTHERN CALIFORNIA, University of California, Berkeley, January 16, 1960
OHIO, Miami University, Oxford, May 9, 1959
OKLAHOMA
PACIFIC NORTHWEST, University of Oregon, Eugene, June 19, 1959
PHILADELPHIA, University of Delaware, Newark, November 28, 1959
ROCKY MOUNTAIN, Utah State University, Logan, May 8–9, 1959
SOUTHEASTERN
SOUTHERN CALIFORNIA
SOUTHWESTERN
TEXAS
UPPER NEW YORK STATE, Hartwick College, Oneonta, May 9, 1959
WISCONSIN, Wisconsin State College, Platteville, May 2, 1959

GEORGE BANTA COMPANY, INC., MENASHA, WISCONSIN, U.S.A.

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RALPH D. JAMES, *Editor*

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A BUDGET OF CURIOSA METRICA

LEONARD M. BLUMENTHAL, University of Missouri

Nearly twenty years ago, this MONTHLY published an article by the writer devoted to those remarkable mathematical paradoxes due to Sierpiński, Mazurkiewicz, Hausdorff, Banach, and Tarski [4]. No such startling results are contained in the present paper, the title of which was obviously suggested by De Morgan's famous *Budget*, but perhaps the reader with sufficient good will might agree that at least some of its material explains (if it does not excuse) the use of the term *curiosa*.

1. A new inequality for tetrahedra. Let 1, 2, 3, 4, denote the vertices of a euclidean tetrahedron, perhaps degenerate, and e_i the sum of the two smallest angles of that face (triangle) of the tetrahedron which is opposite vertex i, $(i=1, 2, 3, 4)$. Pauc calls e_i the flatness (*aplatissement*) of the triangle, and his *Thesis* contains a theorem which is said to follow from a lemma that asserts (without proof) the inequality

$$e_i + e_j + e_k \geqq e_m,$$

where (i, j, k, m) is any permutation of (1, 2, 3, 4) [10].

Very recently Pauc mentioned to the writer that a proof of the inequality was still lacking. In discussions with my student, C. J. Penning, the inequality was established. The following is an adaptation of his argument.

PENNING-PAUC INEQUALITY. *If* 1, 2, 3, 4 *denote four points of a euclidean space, the sum of any three of the four numbers e_i $(i=1, 2, 3, 4)$ is greater than or equal to the fourth number, where e_i denotes the sum of the two smallest angles of the triangle whose vertices are obtained from* 1, 2, 3, 4 *by omitting point i, $(i=1, 2, 3, 4)$.*

Proof. If ϵ_i denotes a largest angle in the triangle whose flatness is e_i, then the inequality to be established is equivalent to

$$(*) \qquad \epsilon_i + \epsilon_j + \epsilon_k - \epsilon_m \leqq 2\pi,$$

for every permutation (i, j, k, m) of (1, 2, 3, 4).

Case 1. *Two largest edges of the tetrahedron are opposite.* Assume the labelling of the vertices so that those two edges are (1, 2) and (3, 4), and develop the tetrahedron on the plane of the face (1, 2, 3). The planar figure so obtained is formed by the triangle $\Delta(1, 2, 3)$ and the three triangles $\Delta(1, 2, 4_3)$, $\Delta(2, 3, 4_1)$, $\Delta(1, 3, 4_2)$ constructed on its sides by folding "outwards" the three triangular faces of the tetrahedron that meet at vertex 4.

The assumption characterizing Case 1 permits the identification of each of the angles ϵ_n $(n=1, 2, 3, 4)$, for it is clear that $\epsilon_1 = \max \angle(2, 3, 4) = \angle 2: 3, 4_1$, $\epsilon_2 = \max \angle(1, 3, 4) = \angle 1: 3, 4_2$, $\epsilon_3 = \max \angle(1, 2, 4) = \angle 4_3: 1, 2$, $\epsilon_4 = \max \angle(1, 2, 3) = \angle 3: 1, 2$, where $\angle i: j, k$ denotes the angle at vertex i of $\Delta(i, j, k)$. Two of the angles of the *convex* quadrilateral $(1, 4_3, 2, 3)$ are ϵ_3 and ϵ_4. Of the remaining two

angles, one equals or exceeds ϵ_1, and the other equals or exceeds ϵ_2 (since the three angles about a vertex of a tetrahedron satisfy the triangle inequality). It follows that $\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4 \leqq 2\pi$, and (*) holds, *a fortiori.*

Case 2. *Two largest edges of the tetrahedron are concurrent.* Assume the labelling so that these edges are (1, 4) and (3, 4), with dist (1, 4) $\geqq$ dist (3, 4). Then $\epsilon_1 = \angle 2: 3, 4$, $\epsilon_2 = \angle 3: 1, 4$, $\epsilon_3 = \angle 2: 1, 4$, while ϵ_4 may be any one of the angles of $\Delta(1, 2, 3)$, depending on which side of that triangle is largest.

The following twelve possibilities for the left-hand member of (*) must be treated:

$$(A_i) \qquad \sum_{k=1}^{4} \rho_k \epsilon_k,\ \rho_i = -1,\ \rho_k = 1,\ k \neq i,\ (i = 1, 2, 3, 4),\ \epsilon_4 = \angle 3: 1, 2,$$

and (B_i), (C_i), $(i=1, 2, 3, 4)$, obtained from (A_i) by replacing $\angle 3$: 1, 2 by $\angle 1$: 2, 3, $\angle 2$: 1, 3, respectively.

The desired inequality is quickly obtained for every left-hand member (A_i), (B_i), (C_i), *except* C_3, by (1) use of the triangle inequality satisfied by the three face angles at any vertex of a tetrahedron, (2) replacing the sum of any two angles of a face triangle by the supplement of the remaining angle, and/or (3) replacing an angle by another (and hence smaller) angle in the same face. None of these devices is useful in the exceptional case:

$$(C_3) \qquad \angle 2{:}3, 4 + \angle 3{:}1, 4 - \angle 2{:}1, 4 + \angle 2{:}1, 3.$$

Folding $\Delta(1, 3, 4)$ about side (1, 4) into the plane of $\Delta(1, 2, 4)$ it is easily shown that $\angle 3: 1, 4 \leqq \angle 2: 1, 4$. Hence

$$\angle 2{:}3, 4 + \angle 3{:}1, 4 - \angle 2{:}1, 4 + \angle 2{:}1, 3$$
$$\leqq \angle 2{:}3, 4 + \angle 2{:}1, 4 + \angle 2{:}1, 3 - \angle 3{:}1, 4 < 2\pi,$$

and the proof is complete.

Remark. The Penning-Pauc inequality is obviously valid if e_i denotes the sum of the two largest angles of the triangle opposite vertex i $(i=1, 2, 3, 4)$, or the sum of the largest and smallest angles of each face triangle. It is *not* valid, however, when the angle-sum is not chosen in the same way in each of the four face triangles.

2. Noncongruent tetrahedra with congruent edges. Two figures are called congruent provided there exists a one-to-one, distance-preserving correspondence between their points. Two triangles are congruent if the sides of one are congruent, respectively, to the sides of the other, *but this theorem is not valid for tetrahedra*; that is, two tetrahedra T_1, T_2 need not be congruent even though the six edges of T_1 are congruent or equal, respectively, to the six edges of T_2. In view of this fact it is pertinent to ask what is the maximum number of pairwise noncongruent tetrahedra that can be formed with six given segments for edges.

There are at most 6! tetrahedra constructible with six given edges, and each tetrahedron is congruent with 4! tetrahedra corresponding to the 4! ways of

labelling the four vertices. Hence there are at most $6!/4!=30$ tetrahedra that are pairwise noncongruent and have the same 6 edges.

DEFINITION. *Six positive numbers form a completely tetrahedral sextuple provided they are the lengths of thirty pairwise noncongruent tetrahedra.*

THEOREM 2.1. *The six positive numbers* $(a+nd)^{1/2}$, $n=0, 1, \cdots, 5$, $0<4d\leqq a$, *form a completely tetrahedral sextuple.*

Proof. Since $a\geqq 4d$, every triple of the numbers $a+nd$, $n=0, 1, \cdots, 5$, satisfies the triangle inequality, and consequently 30 pairwise noncongruent *metric* quadruples exist, the six mutual distances of the points of each quadruple being these six numbers. But it has been proved that if p_1, p_2, p_3, p_4 are a metric quadruple, then a euclidean tetrahedron exists whose edges have lengths $(p_ip_j)^{1/2}$, $(i, j=1, 2, 3, 4, i\neq j)$ ([3]; [5], pp. 130–132).

Remark 1. A metric quadruple p, q, r, s is *realizable* in euclidean three-space E_3 provided a tetrahedron exists whose vertices are congruent with them. A necessary and sufficient condition for this is that $D(p, q, r, s)\geqq 0$, where

$$D(p, q, r, s) = \begin{vmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & pq^2 & pr^2 & ps^2 \\ 1 & pq^2 & 0 & qr^2 & qs^2 \\ 1 & pr^2 & qr^2 & 0 & rs^2 \\ 1 & ps^2 & qs^2 & rs^2 & 0 \end{vmatrix}.$$

Similarly, six segments are realizable in E_3 provided there exists a tetrahedron with the six segments for edges. No "elegant" necessary and sufficient condition for this (in terms of the lengths of the segments) is known nor is likely to exist, since it would involve the 30 essentially different ways that the six lengths can be distributed in the above determinant.

Remark 2. Three coplanar edges of a tetrahedron satisfy the triangle inequality, but three concurrent edges need not do so. It is easy to show, however, that *in every tetrahedron there is at least one concurrent triple of edges that satisfy the triangle inequality, and there need not be more than one such concurrent triple.*

Remark 3. There are at most $[(1/2)n(n+1)]!/(n+1)!$ pairwise noncongruent simplices in E_n with the same $(1/2)n(n+1)$ edges, $n>1$. For $n=4$, this yields 30,240, a surprising increase of the 30 tetrahedra possible when $n=3$.

Remark 4. The preceding theorem shows how a class of completely tetrahedral sextuples may be obtained. What are other ways of generating such classes? There are completely tetrahedral sextuples of integers. Do such sextuples have interesting number-theoretic properties?

Remark 5. Every triple of the six numbers 97, 98, 99, 100, 193, 194 satisfies

the triangle inequality, but *there is no tetrahedron with these six numbers for the lengths of edges.*

3. Operations on simplices. Let $\Sigma(p_0, p_1, \cdots, p_n)$, $\Sigma(q_0, q_1, \cdots, q_n)$ denote simplices of euclidean n-space E_n, with vertices p_i, q_i $(i=0, 1, \cdots, n)$, respectively. A real function $f(x, y)$ is said to be *simplicial of the first kind*, with respect to the two simplices, provided a simplex $\Sigma(r_0, r_1, \cdots, r_n)$ of E_n exists such that $r_ir_j=f(p_ip_j, q_iq_j)$, $(i, j=0, 1, \cdots, n)$.

THEOREM 3.1. *If $\Sigma(p_0, p_1, \cdots, p_n)$, $\Sigma(q_0, q_1, \cdots, q_n)$ are simplices of E_n with $p_i=q_i$, $(i=0, 1, \cdots, n)$, then $f(x, y)=[(1/2)(x+y)]^\rho$, $0 \leqq \rho \leqq 1$, is simplicial of the first kind.*

A different formulation of this theorem was proved by Schoenberg [11]. It states, in effect, that if the edges of any simplex are raised to the power ρ, for ρ any number between zero and one, then another simplex exists with those numbers as lengths of edges, and such that corresponding edges of the two simplices are similarly situated.

THEOREM 3.2. *Let $\Sigma(p_0, p_1, \cdots, p_n)$ be any simplex, and let $\Sigma(q_0, q_1, \cdots, q_n)$ be an equilateral simplex with edge-length a. Then $f(x, y)=x+y$ is simplicial of the first kind with respect to those simplices.*

Proof. It suffices to prove that the quadratic form

$$Q = (1/2) \sum_{i,j=1}^{n} (r_0r_i^2 + r_0r_j^2 - r_ir_j^2)x_ix_j$$

is positive semidefinite, where $r_ir_j=p_ip_j+q_iq_j$ $(i, j=0, 1, \cdots, n)$; that is, $r_ir_j=p_ip_j+a$ $(i, j=0, 1, \cdots, n; i\neq j)$.

Substitution gives

$$\begin{aligned} Q = (1/2) \sum (p_0p_i^2 + p_0p_j^2 - p_ip_j^2)x_ix_j \\ + a\cdot \sum (p_0p_i + p_0p_j - p_ip_j)x_ix_j + a^2\cdot(1/2)\sum x_ix_j, \end{aligned}$$

where in each summation i and j are summed from 1 to n.

The first summand is positive semidefinite since $p_0, p_1, \cdots, p_n$ are vertices of a simplex of E_n, and the positive semidefinite character of the second summand follows from the preceding theorem applied for $\rho=1/2$. Since the third summand is obviously positive semidefinite, so is Q, and the theorem is proved.

Thus, a new simplex is formed by adding a given segment to each of the edges of a simplex.

Remark. The function $f(x, y)=x+y$ is not simplicial of the first kind with respect to two arbitrary simplices. For let p_0, p_1, p_2, p_3 be the vertices of a unit square of E_2 and $q_0=q_1$, $q_2=q_3$ the endpoints of a unit segment. It is clear that the E_3 does *not* contain four points r_0, r_1, r_2, r_3 such that $r_ir_j=p_ip_j+q_iq_j$, $(i, j=0, 1, 2, 3)$, for since $r_0r_1\cdot r_2r_3=1$, $r_0r_2\cdot r_1r_3=(1+\sqrt{2})^2$, $r_0r_3\cdot r_1r_2=4$, the four

points r_0, r_1, r_2, r_3 do not satisfy the ptolemaic inequality, which is valid for every quadruple of E_3 ([5], p. 80).

By bending the unit square a little along the diagonal joining p_0, p_2 a nondegenerate tetrahedron p_0, p_1', p_2, p_3' is obtained whose edges have the same lengths as before, except for the edge p_1', p_3', whose length $p_1'p_3'$ differs from p_1p_3 by arbitrarily little. Similarly, the unit interval $q_0=q_1$, $q_2=q_3$ may be replaced by a rectangle of arbitrarily small width, which in turn (by bending slightly along a diagonal) is replaced by a nondegenerate tetrahedron. In this way two *nondegenerate* tetrahedra are obtained for which $f(x, y)=x+y$ is not simplicial of the first kind.

It should be observed, however, that the six numbers r_ir_j, $(i, j=0, 1, 2, 3; i\neq j)$, $r_{ij}=r_{ji}$, in the example given above *are the lengths of the edges of a nondegenerate tetrahedron* $\Sigma(t_0, t_1, t_2, t_3)$ with $t_0t_1=t_2t_3=1$, $t_1t_2=t_1t_3=2$, $t_0t_2=t_0t_3=1+\sqrt{2}$. This suggests another kind of simplicial function that will be defined in this section.

THEOREM 3.3. *If* $\Sigma(p_0, p_1, \cdots, p_n)$, $\Sigma(q_0, q_1, \cdots, q_n)$ *are simplices of* E_n, *then* $f(x, y)=[x^{2\rho}+y^{2\rho}]^{1/2}$ *is a simplicial function of the first kind for every* ρ, $0\leqq\rho\leqq 1$.

Proof. Substituting $r_ir_j=[(p_ip_j)^{2\rho}+(q_iq_j)^{2\rho}]^{1/2}$ in the quadratic form Q of the preceding theorem gives

$$Q = (1/2)\sum [(p_0p_i)^{2\rho}+(p_0p_j)^{2\rho}-(p_ip_j)^{2\rho}]x_ix_j$$
$$+ (1/2)\sum [(q_0q_i)^{2\rho}+(q_0q_j)^{2\rho}-(q_iq_j)^{2\rho}]x_ix_j,$$

i and j being summed from 1 to n.

It follows from Theorem 3.1 that each of the two quadratic forms in the above equality is positive semidefinite. Hence $Q\geqq 0$ and the E_n contains a simplex $\Sigma(r_0, r_1, \cdots, r_n)$.

Remark. Let $\Sigma(p_0, p_1, \cdots, p_n)$, $\Sigma(q_0, q_1, \cdots, q_n)$ be simplices of E_n and let δ denote a given correlation between the edges of the two simplices. A function $g(x, y)$ is called *simplicial of the second kind* provided a simplex $\Sigma(r_0, r_1, \cdots, r_n)$ of E_n exists such that $r_ir_j=g(p_ip_j, \delta(p_ip_j))$, where $\delta(p_ip_j)$ denotes the length of that edge of $\Sigma(q_0, q_1, \cdots, q_n)$ that is correlated by δ with that edge of $\Sigma(p_0, p_1, \cdots, p_n)$ that joins p_i, p_j. Clearly simplicial functions of the first kind form a proper subclass of those defined in this remark, obtained by correlating those edges of the two simplices that join vertices with the same indices. Two simplices are called δ-additive provided for some correlation δ, $p_ip_j+\delta(p_ip_j)$, $(i, j=0, 1, \cdots, n; i=j)$ are the lengths of the edges of some simplex of E_n. What pairs of simplices of E_n are δ-additive?

4. An interesting class of arcs of Hilbert space. For each number ρ, $0<\rho<1$, the arc obtained from the unit segment $[0, 1]$ by defining the distance $\delta(x, y)$ of any two of its points x, y to be $xy^\rho=|x-y|^\rho$ is seen (with the help of Theorem 3.1) to be congruently contained in Hilbert space. Let $\{A_\rho\}$, $0<\rho<1$, denote

that class of Hilbert arcs.

(a) *Each subarc of each arc of class* $\{A_\rho\}$ *is nonrectifiable.*

If A_ρ^k denotes a subarc of A_ρ which corresponds (by congruence between A_ρ and the unit segment with distance $|x-y|^\rho$) to a subsegment of $[0, 1]$ whose euclidean length is k) the length $l(A)_\rho^k$ is given by

$$l(A_\rho^k) = \lim_{n\to\infty} n\cdot\epsilon(n),$$

where $\epsilon(n)=(k/n)^\rho$ is the distance in the new metric of two consecutive points in the subdivision of the subsegment into n equal (nonoverlapping) parts [1]. Hence $l(A_\rho^k)=k^\rho\cdot\lim_{n\to\infty} n^{1-\rho}=\infty$.

(b) Let x, y, z be three points of $[0, 1]$ with $x<y<z$, and put $xy=\lambda\cdot xz$, $0<\lambda<1$. Then $yz=(1-\lambda)xz$, and consequently

(i) $$\cos \angle x{:}y, z = [\lambda^{2\rho} - (1-\lambda)^{2\rho} + 1]/2\lambda^\rho,$$

(ii) $$\cos \angle y{:}x, z = [\lambda^{2\rho} + (1-\lambda)^{2\rho} - 1]/2\lambda^\rho(1-\lambda)^\rho,$$

(iii) $$\cos \angle z{:}x, y = [-\lambda^{2\rho} + (1-\lambda)^{2\rho} + 1]/2(1-\lambda)^\rho.$$

It is clear that none of the angles $\angle x$: y, z, $\angle y$: x, z, $\angle z$: x, y has a limit as $y, z\to x$, except that for $\rho=1/2$, $\lim \angle y{:}\, x, z=\pi/2$. It follows that *none of the arcs* A_ρ *has either a right-hand or a left-hand tangent at any point.*

(c) The arc $A_{1/2}$ has special interest. Each three of its points are the vertices of a right triangle, and though, as seen above, it does not have a unilateral tangent at any point, *it admits a parametrization in terms of analytic functions.* The functions $f_0(t)=t$, $f_{2n-1}(t)=(\cos 2\pi nt)/2^{1/2}\pi n$, $f_{2n}(t)=(\sin 2\pi nt)/2^{1/2}\pi n$, $n=1, 2, \cdots$, are easily seen to map the unit t-interval onto an arc of Hilbert space so that if $t_1, t_2\in[0, 1]$ and p_1, p_2 are their respective corresponding points, then $p_1p_2=|t_1-t_2|^{1/2}$.

It may be conjectured that *each* arc A_ρ, $0<\rho<1$, is representable by an infinite sequence of analytic (but not elementary) functions defined on $[0, 1]$.

(d) *If* $\rho<1/2$ *each of the three angles defined in* (i), (ii), (iii) *is acute.* Clearly $\angle y$: x, z is the largest angle, and putting $2\rho=1-\epsilon$, $0<\epsilon<1$, gives

$$\begin{aligned}\lambda^{2\rho} + (1-\lambda)^{2\rho} - 1 &= \lambda\cdot\lambda^{-\epsilon} + (1-\lambda)(1-\lambda)^{-\epsilon} - 1\\ &> \lambda + (1-\lambda) - 1 = 0.\end{aligned}$$

Similarly, it is seen that for $\rho>1/2$, every triangle of A_ρ is obtuse. *But these obtuse angles are not bounded away from* $\pi/2$. For

$$\lim_{\lambda\to 0} [\lambda^{2\rho} + (1-\lambda)^{2\rho} - 1]/2\lambda^\rho(1-\lambda)^\rho = 0.$$

Indeed, the above limit is zero for every ρ, $0<\rho<1$, and so each A_ρ contains a triangle whose maximum angle differs from $\pi/2$ by arbitrarily little.

An arc A has the Marchaud property provided a positive number δ_0 exists such that for each point p of A and each number δ, $0<\delta<\delta_0$, there are at most two points of A with distance δ from p. Marchaud proved that each euclidean arc with this property is rectifiable [8]. Clearly each of the nonrectifiable arcs A_ρ, $0<\rho<1$, has the Marchaud property, and so his theorem is not valid for arcs of Hilbert space.

5. Maximum angles of metric triples. If r, s, t are three pairwise distinct points of a metric space, let r', s', t' denote three points of the euclidean plane E_2 that correspond to them, respectively, by means of a congruence, and define the angles of the triple (r, s, t) to be the corresponding angles of the planar triple (r', s', t'). The conditions (a) $\max \angle(r, s, t)>\pi/3$, (b) $\max \angle(r, s, t)\geqq\pi/2$, (c) $\max \angle(r, s, t)\geqq\theta_0$, $\theta_0>\pi/2$, applied to triples of a (compact) metric continuum, have interesting consequences.

THEOREM 5.1. *If each three pairwise distinct points r, s, t, of a metric continuum M are such that* $\max \angle(r, s, t)>\pi/3$, *then M is an arc.*

This theorem was first conjectured by the writer in the case of locally connected metric continua, and it was proved under that restriction by Milgram [9]. Theorem 5.1 follows from results due to Choquet [6].

THEOREM 5.2. *Let C be a euclidean continuum, $p\in C$, and $\delta>0$ such that $r, s, t\in C\cdot U(p; \delta)$, $r\neq s\neq t\neq r$, imply* $\max \angle(r, s, t)\geqq\pi/2$, *where* $U(p; \delta) = [q\in C \mid pq<\delta]$. *Then C is a rectifiable arc in a neighborhood of p (that is, an open set $V(p)$, containing p, exists such that $C\cdot \overline{V}(p)$ is a rectifiable arc, where $\overline{V}(p)$ denotes the closure of $V(p)$).*

This theorem is proved in [2]. It follows from (a) and (d) of Section 4 that the restriction "euclidean" cannot be deleted from the hypothesis even if the equality sign in $\max \angle(r, s, t)\geqq\pi/2$ be dropped.

THEOREM 5.3. *Let p be a point of a metric continuum M, and suppose $\delta>0$ and $\theta_0>\pi/2$ exist such that $r, s, t\in M\cdot U(p; \delta)$, $r\neq s\neq t\neq r$, imply* $\max \angle(r, s, t) \geqq\theta_0$. *Then M is a rectifiable arc in a neighborhood of p.*

This theorem is proved in [2]. It is noteworthy that requiring the maximum angles of triples with sufficiently small diameters to be *bounded away from* $\pi/2$ (instead of demanding merely that such triples be obtuse) is all that is needed to extend the validity of Theorem 5.2 to *all* metric continua.

6. Ratio of chord to arc in metric space. The curves dealt with in classical differential geometry have the property that the limit, as two points of the curve approach a third point (along the curve), of the ratio of the length of the chord determined by the two points to the length of the intercepted arc of the curve is unity. Since examples of curves that do not possess this important feature are easily constructed, one is lead to inquire what curves do possess it at every point. It is clear that rectifiability of the curve in a neighborhood of the point is a

necessary condition, and hence only rectifiable curves enter into consideration.

Let A be a rectifiable metric arc, parameterized according to arc length; that is $A=f[0, s_0]$, where s_0 is the length $l(A)$ of A, and f maps the closed interval $[0, s_0]$ homeomorphically onto A. If $s\in[0, s_0]$, the limit under investigation, namely $\lim_{x,y\to s} f(x)f(y)/xy$, $x, y\in[0, s_0]$, is known as the *spread of f at s*. Denoting it by $f^*(s)$, it has been shown [7] that

$$(*) \qquad s_0 = l(A) = \int_0^{s_0} f^*(s)ds.$$

Since f^* is continuous wherever it exists, and $f^*(s)\leqq 1$, $s\in[0, 1]$, then (*) yields $f^*(s)\equiv 1$ in $[0, s_0]$, and the following theorem is obtained.

THEOREM 6.1. *Let A be a rectifiable metric arc. Then at each point of A the limit of the ratio of chord to arc is unity if and only if that limit exists at each point of A.*

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COMPLETELY TETRAHEDRAL SEXTUPLES

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1. Introduction. In the preceding paper (hereinafter referred to as [B]), L. M. Blumenthal defined a *completely tetrahedral sextuple* (CTS): a set of six positive numbers forms a CTS if they are the lengths of the edges of thirty pairwise noncongruent tetrahedra. (See [B], Sec. 2.) The object of this note

is to give two simple necessary and sufficient conditions that a given sextuple be a CTS. We shall also give several rather general classes of CTS's. (See [B], Sec. 2, Remark 4.)

2. Notations and preliminary remarks. We will use the symbol $\{x_0, x_1; y_0, y_1; z_0, z_1\}$ to denote the tetrahedron in which the pairs separated by semicolons are pairs of opposite edges and whose faces are formed by the triples (x_i, y_j, z_k) with $i+j+k=0$ or 2. The symmetric determinant $|r_{ij}|$ of order five in which $r_{ii}=0$, $r_{1j}=1$ for $j>1$, and $r_{23}=x_0^2$, $r_{24}=y_0^2$, $r_{25}=z_1^2$, $r_{34}=z_0^2$, $r_{35}=y_1^2$, $r_{45}=x_1^2$, will be denoted by

$$D(x_0, x_1; y_0, y_1; z_0, z_1). \tag{1}$$

If $x_0, x_1, y_0, y_1, z_0, z_1$ are six given positive numbers, any three of which satisfy the triangle inequality, then the tetrahedron $\{x_0, x_1; y_0, y_1; z_0, z_1\}$ is realizable in E_3 if and only if $D(x_0, x_1; y_0, y_1; z_0, z_1) \geqq 0$. (See [B], Sec. 2, Remark 1.)

Remark. The determinant (1) is invariant under 24 permutations of the six arguments: (a) the letters x, y, z may be permuted in any way, and (b) the indices 0 and 1 may be transposed for any two of the three letters.

Aside from the criterion just mentioned we shall also need the following one, which can be stated in terms of two-dimensional geometry. Let P_1 and P_2 be two points in E_2, which are to serve as centers of a system of bipolar coordinates in E_2. (For the sake of simpler description later on, we shall always think of P_1P_2 as being horizontal with P_1 to the left of P_2.) If Q is any point of E_2 whose distances from P_1 and P_2 are t_1 and t_2, respectively, we shall denote Q by $[t_1, t_2]_+$ or $[t_1, t_2]_-$, according as Q lies above or below the line through P_1 and P_2. (If Q lies on that line either sign may be used as subscript.) The notations $|[t_1, t_2]_+ - [u_1, u_2]_+|$ and $|[t_1, t_2]_+ - [u_1, u_2]_-|$ will be used to denote the distances between the points indicated. Now if $x_0, x_1, y_0, y_1, z_0, z_1$ are, as above, six positive numbers, satisfying the triangle inequality, then the tetrahedron $\{x_0, x_1; y_0, y_1; z_0, z_1\}$ is realizable in E_3 if and only if, in a system of bipolar coordinates with $P_1P_2 = x_0$, the inequalities

$$|[y_0, z_0]_+ - [z_1, y_1]_+| \leqq x_1 \leqq |[y_0, z_0]_+ - [z_1, y_1]_-|$$

hold. This follows at once from rotating the triangle of side lengths x_0, z_1, and y_1 about its base P_1P_2.

3. A simple class of CTS's. Six positive numbers, a, b, c, d, e, f, in order to form a CTS, must necessarily be distinct from each other, and any three of them must satisfy the triangle inequality. We shall accordingly assume that

$$a > b > c > d > e > f > 0, \tag{2}$$

$$e + f \geqq a. \tag{3}$$

The two criteria of Section 2, applied to the problem of CTS's, will now be stated in the form of lemmas.

LEMMA 1. *Six numbers, satisfying conditions* (2) *and* (3), *form a CTS if and only if all thirty possible determinants* (1) *formed from them are nonnegative.*

LEMMA 2. *Six numbers, satisfying conditions* (2) *and* (3), *form a CTS if and only if, in a system of bipolar coordinates with $P_1P_2=a$, the inequalities*

$$\left| [w, x]_+ - [y, z]_+ \right| \leqq v \leqq \left| [w, x]_+ - [y, z]_- \right| \tag{4}$$

are satisfied for all permutations v, w, x, y, z of the letters b, c, d, e, f.

We now state the following sufficient condition for a CTS as a consequence of Lemma 2.

THEOREM 1. *Six numbers, satisfying condition* (2) *form a CTS if $f \geqq 2^{-1/2}a$. In this statement the constant $2^{-1/2}$ is the best possible one, that is, if $f<2^{-1/2}a$ then b, c, d, and e can be so chosen* (*in accordance with* (2)) *that a, b, c, d, e, f do not form a CTS.*

Assume that $f \geqq 2^{-1/2}a$. We note first that (3) is satisfied. In the bipolar coordinate system of Lemma 2, let S_+ denote the "quadrilateral" area above P_1P_2, that is bounded by arcs of the four circles with radii a and $2^{-1/2}a$ and centers P_1 and P_2, and similarly for S_- below P_1P_2. Then all points involved in the inequalities (4) lie in S_+ or in S_-. The diameter of S_+ equals $2^{-1}a<2^{-1/2}a \leqq f$, and the minimum distance between S_+ and S_- equals a. Therefore, all inequalities (4) are satisfied, and the first part of Theorem 1 follows from Lemma 2.

On the other hand, if $2^{-1}a \leqq f<2^{-1/2}a$ then $\left| [f, f]_+ - [f, f]_- \right| <a$, and it is therefore possible to choose c, d, and e near f, and b near a (all in accordance with (2)), in such a way that $\left| [c, d]_+ - [e, f]_- \right| <b$, which violates one of the inequalities (4).

Remark. The inequality $f \geqq 2^{-1/2}a$ is not a necessary condition for a CTS. On the contrary, the ratio f/a can be arbitrarily small. (See Sec. 4, Remark 5.)

4. Necessary and sufficient conditions. We first prove the following lemma.

LEMMA 3. *The minimum of the thirty determinants* (1) *that can be formed from the six quantities of inequality* (2) *is assumed by one of the following three*:

$$D_1 = D(a, b; c, d; e, f), \qquad D_2 = D(a, b; c, e; d, f), \qquad D_3 = D(a, b; c, f; d, e).$$

We write each of the thirty determinants in question in the unique standard form

$$D(a, v; w, x; y, z), \tag{5}$$

where v, w, x, y, z is a permutation of b, c, d, e, f and where $w>x$, $w>y$ and $w>z$. (See the Remark in Sec. 2 above.)

Now if $y<z$ then

$$(6) \quad D(a, v; w, x; y, z) - D(a, v; w, x; z, y) = 2(a^2 - v^2)(w^2 - x^2)(z^2 - y^2) > 0,$$

and this eliminates from competition for the minimum the 15 determinants (5) in which $y<z$. We consider next the 12 of the remaining 15 determinants (5) in which $w=b$ and hence $v<b$. We use the relation

$$(7) \quad \begin{aligned} &D(a, v; b, x; y, z) - D(a, b; v, x; y, z) \\ &\qquad = 2(a^2 - x^2)(b^2 - v^2)(a^2 + b^2 + v^2 + x^2 - 2y^2 - z^2). \end{aligned}$$

By (2), the right side of (7) is positive, except possibly when $y=c$ and $z=d$ (note that $y>z$), and in this case we use the relation

$$(8) \quad \begin{aligned} &D(a, v; b, x; c, d) - D(a, d; b, x; c, v) \\ &\qquad = 2(a^2 - c^2)(d^2 - v^2)(a^2 - b^2 + c^2 + d^2 + v^2 - 2x^2), \end{aligned}$$

whose right-hand member is positive. Thus (7) and (8) eliminate the 12 abovementioned determinants (5) from competition for the minimum. The three remaining determinants are those in which $v=b$ (and $y>z$), and these are the three determinants D_1, D_2, and D_3.

Remark 1. Which one of the three determinants of Lemma 3 actually assumes the minimum can be seen from applying (7) and (8) to the differences D_2-D_1 and D_3-D_2, respectively: the answer is D_1, D_2, or D_3, according as $c^2+d^2+e^2+f^2$ is greater than $2a^2+b^2$, between a^2+2b^2 and $2a^2+b^2$, or less than a^2+2b^2, respectively.

THEOREM 2. *Six numbers, satisfying conditions* (2) *and* (3), *form a CTS if and only if, in a system of bipolar coordinates with* $P_1P_2=a$, *the inequality*

$$(9) \quad \left| [c, d]_+ - [e, f]_- \right| \geqq b$$

holds.

By Lemma 2, it suffices to prove the sufficiency of (9). Let T be the "triangular" area which is bounded below by part of the segment P_1P_2 and on the right and left by circular arcs of radius c with centers at P_1 and P_2, respectively. The diameter of T is less than $c<b$. Therefore, if w, x, y, z is any permutation of c, d, e, f, we have, by (2),

$$(10) \quad \left| [w, x]_+ - [y, z]_+ \right| < b,$$

since all points occurring in (10) lie in T.

Application of (10) to $[c, d]_+$ and $[e, f]_+$ together with (9) gives $D_3 \geqq 0$. For obvious geometrical reasons, (9) and (2) imply that *a fortiori* $\left| [c, d]_+ - [f, e]_- \right| > b$; and this combined with (10), applied to $[c, d]_+$ and $[f, e]_+$, gives $D_2>0$. Finally, by (6) and (2), $D(a, b; c, f; e, d) > D_3 \geqq 0$, hence $\left| [c, e]_+ - [d, f]_- \right| > b$ and *a fortiori* $\left| [c, e]_+ - [f, d]_- \right| > b$; the last inequality, together with (10), applied to $[c, e]_+$ and $[f, d]_+$, gives $D_1>0$. Theorem 2 now follows from Lemmas 3 and 1.

THEOREM 3. *Six numbers, satisfying conditions* (2) *and* (3), *form a CTS if and only if* $D_3 = D(a, b; c, f; d, e) \geqq 0$, *that is, if and only if the tetrahedron in which the faces are formed by the triples* (a, c, d), (a, e, f), (b, c, e) *and* (b, d, f) *is realizable in* E_3.

The proof is immediate, since the fact that $D_3 \geqq 0$ implies that, in a system of bipolar coordinates with $P_1P_2 = a$, the inequality $| [c, d]_+ - [e, f]_- | \geqq b$ holds and, consequently, Theorem 2 becomes applicable.

Remark 2. We note without proof that in Theorem 3 condition (3) is not needed as part of the hypothesis; in the particular case of the determinant D_3, it can be shown that (3) follows from $D_3 \geqq 0$ and (2).

Remark 3. The determinant D_3 is the only one of the thirty determinants (5) which can serve the purpose of Theorem 3. This can be seen from the example of the sextuple 120, 110, 100, 90, 80, 61, for which $D_3 < 0$, while the other 29 determinants are positive.

The following remarks are direct applications of Theorem 3.

Remark 4. Six positive numbers $x+5$, $x+4$, $x+3$, $x+2$, $x+1$, x form a CTS if and only if x is greater than or equal to the positive root of the equation $D(x+5, x+4; x+3, x; x+2, x+1) = 0$; this root lies between 6.09 and 6.10. In particular, six successive integers form a CTS if and only if the largest of them is at least 12. (Generalization to any arithmetic progression is obvious.) The CTS 12, 11, 10, 9, 8, 7 is moreover the smallest integral CTS; there is no other integral CTS whose largest member is less than or equal to 12.

Remark 5. There are CTS's in which the ratio f/a (see (2)) is arbitrarily small. This follows from the relation $D(1, 1; 1, \epsilon; 1, 1) = \epsilon^2(3 - \epsilon^2) > 0$ for any ϵ, $0 < \epsilon < 1$. Choose $\delta = \delta(\epsilon) > 0$ so that $1 - 4\delta > \epsilon$ and $D(1, 1-\delta; 1-2\delta, \epsilon; 1-3\delta, 1-4\delta) > 0$.

Remark 6. An investigation of the case discussed in [B], Theorem 2.1, leads to the result that the sextuple $(a+nd)^{1/2}$, $n = 0, 1, 2, 3, 4, 5$, with $a > 0$, $d > 0$, is a CTS if and only if a/d is greater than or equal to a cubic irrationality, which lies between 1.91 and 1.92.

DANDELIN, LOBAČEVSKIĬ, OR GRAEFFE?

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In the Russian translation of reference [5], the designation of Graeffe's method is changed to "Lobačevskiĭ's method," though not without a conscientious footnote calling attention to the fact. Since some texts refer to the "Dandelin-Graeffe method," the question who did what and when seemed of some interest.

The "when" question is readily answered. The three authors published in 1826 [2], 1834 [6], and 1837 [4], respectively. Considering this, the natural conclusion would be that Dandelin has clear priority, and that the attachment of Graeffe's name to the method must have been one of those unfortunate historical accidents that do occur at times. But if justice is to be attempted, why pass over Dandelin for Lobačevskiĭ? An interesting, and curious, paper by Rogačenko [8] discusses the question at some length, not only with analyses of the papers by the three authors, but in quoting from writers who uncritically assign the credit to Graeffe, as well as others who have other opinions.

Rogačenko freely admits that Dandelin published in 1826 a paper in which root squaring is proposed. He seems to consider it important that although Lobačevskiĭ's *Algebra* [6] bears the date 1834, it was actually in the hands of the censor in 1832. But he builds his case upon the assertion that Dandelin's paper was concerned primarily with Newton's method, and that root squaring is introduced only as a device for speeding the convergence of the Newton iteration. It is quite true that the main part of the paper does discuss Newton's method, and it is recommended that it be accompanied by *regula falsi* so that one has always an upper and a lower bound for the root. Dandelin then considers the possibility of accelerating both processes by applying them to the equation whose roots are the squares of those of the original. The method he proposes for doing this is to form the product $f(x)f(-x)$, where $f(x)=0$ is the original equation (note that this is not quite the way one does it now). He mentions also another device for accelerating convergence, by use of osculating parabolas instead of tangents and chords. Finally, in the paper proper, he comments that one could equally well take 4th powers, 8th powers, or whatever, all of which, however, do subordinate the algebraic root squaring to the geometric tangents and chords. However, Dandelin had afterthoughts, which he recorded in four appendices, and in the second of these he goes further into the matter of root squaring, making two important observations that are not always to be found in modern treatments.

The first of these observations is also reported by Rogačenko, but the second is not. The first is that if the zeros of a polynomial are widely separated into one group of very large modulus, and one of very small modulus, then the equation which remains when final terms are dropped is approximately satisfied by the large zeros. In particular, if there is only one large zero, keep the first two terms; if there are two, keep three. The second observation is that if one considers the power 2^p sufficiently high, then form the power $1+2^p$, and one gets the zeros themselves as quotients without root extraction. It is hard to reconcile the presence of these considerations with the contention that Dandelin considered root squaring an auxiliary device only, and not a possible method in itself. It is true that our critic does level another charge against Dandelin, but the relevance is obscure. It is that Dandelin did not base his conclusions upon properties of symmetric polynomials.

In the editorial introduction to Volume 4 of Lobačevskiĭ's works, Čebotarev

[1] opines that Lobačevskiĭ deserves priority over Graeffe, but fails to make an argument on Dandelin. He tells us he does not have available the 1837 monograph of Graeffe [4] but he inserts a translation of an earlier paper [3] which does deal with symmetric functions, and proves a convergence theorem, but does not describe the method. And Lobačevskiĭ does, indeed, on the final page of this edition of his *Algebra* which had just appeared in 1834, make the following suggestion: Consider equations

$$x^n - a_1x^{n-1} + \cdots = 0, \qquad y^n - A_1y^{n-1} + \cdots = 0,$$

where $A_1 = a_1^2 - 2a_2$, $A_2 = a_2^2 - 2a_1a_3 + 2a_4, \cdots$. Then the roots of the second equation are the squares of the roots of the first. In justification of the statement he refers to the identities he had previously established relating sums of powers with the coefficients. Notably, he did not refer this to the also developed properties of the product

$$f(x)f(\omega x) \cdots,$$

where ω is a root of unity. Finally, in illustration he finds the largest root of a quintic.

Why only the largest root, and what about the others? On this there is no clue, except, possibly a negative one, which is this, that in the final equation he computes only A_1, and in the intermediate equations only those coefficients that affect A_1. If he thought of this as a method for obtaining *all* roots *simultaneously*, he gives no hint of the fact. But at least he uses symmetric functions.

And now, where does Graeffe fit in, and why did anyone ever mention him? He fits in for having thought of separating the even and odd powers. A simple notion, but effective, and it is just what everyone does today. And so, Ostrowsky, in his elaborate treatment of the method [7], attaches the name of Graeffe, and begins by summarizing the contributions of these three, along with those of Encke and several others who further investigated and elaborated the technique. Whether this is justice the reader must decide for himself.

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A GENERATING FUNCTION FOR $\sigma_k(n)$

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1. Introduction. In this paper we shall be concerned with certain arithmetical properties of formal infinite series. By formal infinite series, we mean that the convergence of the infinite series under discussion is irrelevant. For a more detailed discussion of this point of view, see [2], pages 251–253 or [4], Chapter 1.

It is well known ([1], pp. 279–280) that

$$\sum_{n=1}^{\infty} \frac{nx^n}{1-x^n} = \sum_{n=1}^{\infty} \sigma(n)x^n, \qquad \sum_{n=1}^{\infty} \frac{x^n}{1-x^n} = \sum_{n=1}^{\infty} \tau(n)x^n, \tag{1.1}$$

where $\sigma(n)$ is the sum of the divisors of n and $\tau(n)$ is the number of divisors of n.

In addition, Harris has published [3] the identity:

$$\sum_{n=1}^{\infty} \frac{5-(-1)^n}{2} \frac{nx^{n-1}}{1-x^n} = \sum_{n=1}^{\infty} \sigma(2n)x^{n-1}. \tag{1.2}$$

The purpose of this paper is to generalize the identities (1.1) and (1.2). Our generalization proceeds as follows: We seek an arithmetic function, $f_k(an+b)$, with $a>0$ and $b\geqq 0$ so that

$$\sum_{n=1}^{\infty} \frac{f_k(an+b)x^{an+b}}{1-x^{an+b}} = \sum_{n=1}^{\infty} \sigma_k(an+b)x^{an+b}, \tag{1.3}$$

where $\sigma_k(n)$ is the sum of the kth powers of the divisors of n. We wish to find the conditions under which $f_k(an+b)$ exists, and when it does, to characterize it.

In Section 2 are various lemmata which help us attain our goal. A principal result is that if there is an arithmetic function $f_k(an+b)$ satisfying (1.3) with $b>0$, then $a=(a, b)$.

Section 3 leads us to a characterization of $f_k(an+b)$, when it exists, together with certain of its properties. Equating corresponding coefficients on both sides of (1.3) yields the novel identity of Theorem 4.

In Section 4 are presented several examples. In particular a practical system for computing a table of the sum of the kth powers of the divisors of n is devised. The method used is similar to the use of the sieve of Eratosthenes in computing number-divisor tables. The main feature of the system is that it carries its own check with it.

2. Some lemmata. The first lemma is a result of Cesàro ([1], p. 127).

LEMMA 1. *If $f(n)$ is an arithmetic function, then*

$$\sum_{n=1}^{\infty} \frac{f(n)x^n}{1-x^n} = \sum_{n=1}^{\infty} F(n)x^n,$$

where $F(n)=\sum_{d|n} f(d)$.

The next lemma is a slight modification of a well-known result. A proof can be found in [2], page 235.

LEMMA 2. *If a divides m and $\mu(m)$ is the Möbius function, then*

$$\sum_{d\mid(m/a)} \mu(d) = \begin{cases} 1 & \text{if } m = a, \\ 0 & \text{if } m > a. \end{cases}$$

Lemma 3 requires the introduction of the following definition: If $\mu(n)$ is the Möbius function, we define

$$\bar{\mu}(m/a) = \begin{cases} \mu(m/a) & \text{if } a \mid m, \\ 0 & \text{if } a \nmid m. \end{cases}$$

LEMMA 3. *With $\bar{\mu}(m/a)$ defined as above, we have*

$$\sum_{m=1}^{\infty} \frac{\bar{\mu}(m/a)x^m}{1 - x^m} = x^a.$$

Proof. By Lemma 1,

$$\sum_{m=1}^{\infty} \frac{\bar{\mu}(m/a)x^m}{1 - x^m} = \sum_{m=1}^{\infty}\left(\sum_{d\mid m} \bar{\mu}(d/a)\right)x^m = \sum_{\substack{m=1 \\ a\mid m}}^{\infty}\left(\sum_{d\mid(m/a)} \mu(d)\right)x^m.$$

Then by Lemma 2,

$$\sum_{\substack{m=1 \\ a\mid m}}^{\infty}\left(\sum_{d\mid(m/a)} \mu(d)\right)x^m = x^a.$$

LEMMA 4. *If there is an arithmetic function $f_k(an+b)$ with $b>0$ such that* (1.3) *is satisfied, then $a=(a, b)$.*

Proof. Set $g=(a, b)$ so that $1=(a/g, b/g)$. There is an N_0 so that $(a/g)N_0+(b/g)=p$, where p is a prime. Certainly x^{gp} occurs on the right side of (1.3) with nonzero coefficient. Expanding the left side of (1.3), we see that the coefficient of x^{aN+b} is $\sum f_k(an+b)$ summed over all n for which $(an+b)\mid(aN+b)$. Thus the coefficient of x^{gp} in the expansion of the left side of (1.3) is just $f_k(aN_0+b)$ and hence is not zero.

Now consider the coefficient of x^{2gp} in the expansion of the left side of (1.3). This coefficient is $\sum f_k(an+b)$ summed over all n for which $(an+b)\mid(2gp)$. This includes $n=N_0$ but may include other values of n. If $n\neq N_0$ and $(an+b)\mid(2gp)$, then a short and obvious argument leads us to $a=g$. On the other hand, if N_0 is the only n for which $(an+b)\mid(2gp)$, we have the coefficient of x^{2gp} as just $f_k(aN_0+b)$, which we have just shown is not zero. Hence x^{2gp} must occur on the right side. This gives us an n_1 such that $an_1+b=2gp$. Combining $an_1+b=2gp$ and $aN_0+b=gp$ yields $(a/g)(n_1-N_0)=p$. Thus a/g divides p, which implies $a=g$. In all cases, then, $a=g$.

3. Theorems. The first theorem reveals that a unique $f_k(an+b)$ satisfying (1.3) exists if $b=0$.

THEOREM 1. *If $f_k(a, n)$ is an arithmetic function for which*

$$\sum_{n=1}^{\infty} \frac{f_k(a, n)x^n}{1 - x^n} = \sum_{n=1}^{\infty} \sigma_k(an)x^n,$$

then

$$f_k(a, n) = s^k\sigma_k(r)n^k, \tag{3.1}$$

where r is the largest factor of a for which $(r, n)=1$ and $a=rs$, and conversely.

Proof. By Lemma 1,

$$\sum_{d|n} f_k(a, d) = \sigma_k(an).$$

By the Möbius inversion formula,

$$\begin{aligned} f_k(a, n) &= \sum_{d|n} \sigma_k(an/d)\mu(d) \\ &= \sum_{d|sn} \sigma_k(an/d)\mu(d) \end{aligned} \tag{3.2}$$

because $\mu(d)=0$ if d is not square free. Continuing we can write

$$f_k(a, n) = \sigma_k(r) \sum_{d|sn} \sigma_k(sn/d)\mu(d) \tag{3.3}$$

since r and sn/d are relatively prime.

Now consider

$$\sum_{d|m} d^k = \sigma_k(m).$$

From the Möbius inversion formula

$$m^k = \sum_{d|m} \sigma_k(m/d)\mu(d). \tag{3.4}$$

Substituting from (3.4) into (3.3) gives us $f_k(a, n)=\sigma_k(r)(sn)^k$ as was to be proved. Retracing steps yields the converse.

THEOREM 2. *Let us now set*

$$g_k(t; a, m) = f_k(a, m) - \sum_{\alpha=1}^{t} \sigma_k(\alpha a)\bar{\mu}(m/\alpha)$$

with the usual restriction that the sum is zero if $t=0$. Then

$$\sum_{m=1}^{\infty} \frac{g_k(t; a, m)x^m}{1 - x^m} = \sum_{m=t+1}^{\infty} \sigma_k(am)x^m.$$

Proof. By Theorem 1

$$\begin{aligned}\sum_{m=t+1}^{\infty} \sigma_k(am)x^m &= \sum_{m=1}^{\infty} \sigma_k(am)x^m - \sum_{\alpha=1}^{t} \sigma_k(a\alpha)x^\alpha \\ &= \sum_{m=1}^{\infty} \frac{f_k(a, m)x^m}{1 - x^m} - \sum_{\alpha=1}^{t} \sigma_k(a\alpha)x^\alpha.\end{aligned}$$

With the aid of Lemma 3,

$$\begin{aligned}\sum_{m=t+1}^{\infty} \sigma_k(am)x^m &= \sum_{m=1}^{\infty} \frac{f_k(a, m)x^m}{1 - x^m} - \sum_{m=1}^{\infty} \frac{\sum_{\alpha=1}^{t} \sigma_k(a\alpha)\bar{\mu}(m/\alpha)x^m}{1 - x^m} \\ &= \sum_{m=1}^{\infty} \frac{g_k(t;\, a, m)x^m}{1 - x^m}.\end{aligned}$$

COROLLARY. *If $m \leqq t$, then $g_k(t;\, a, m) = 0$. Hence*

$$\sum_{m=t+1}^{\infty} \frac{g_k(t;\, a, m)x^m}{1 - x^m} = \sum_{m=t+1}^{\infty} \sigma_k(am)x^m.$$

Proof. If $m \leqq t$, then

$$\sum_{\alpha=1}^{t} \sigma_k(\alpha a)\bar{\mu}(m/\alpha) = \sum_{d|m} \sigma_k(ad)\mu(m/d) = \sum_{d|m} \sigma_k(am/d)\mu(d) = f_k(a, m) \text{ by (3.2).}$$

THEOREM 3. *If $b = at$, set $h_k(t;\, a, n) = g_k(t;\, a, n+t)$. Then*

$$\sum_{n=1}^{\infty} \frac{h_k(t;\, a, n)x^{an+b}}{1 - x^{an+b}} = \sum_{n=1}^{\infty} \sigma_k(an + b)x^{an+b}.$$

Proof. Theorem 3 is merely Theorem 2 reworded. It is set up as a separate theorem because it exhibits an arithmetic function, $h_k(t;\, a, n) = f_k(an+b)$, which satisfies (1.3) if $a = (a, b)$. In view of Lemma 4, we now have proved the existence of $f_k(an+b)$ if and only if $a = (a, b)$.

THEOREM 4. *If $b = at$, set*

$$\delta_k(a, m) = \begin{cases} 0 & m \leqq t, \\ \sigma_k(am) & m > t. \end{cases}$$

Then

$$\sum_{d|m} g_k(t;\, a, d) = \delta_k(a, m).$$

Proof. By Theorem 2 and the definition of $\delta_k(a, m)$, we can write

$$\sum_{m=1}^{\infty} \frac{g_k(t;\, a, m)x^m}{1 - x^m} = \sum_{m=1}^{\infty} \delta_k(a, m)x^m.$$

Then by Lemma 1

$$\delta_k(a, m) = \sum_{d|m} g_k(t; a, d).$$

THEOREM 5. *Set* $\phi_k(a, n) = f_k(a, n)/n^k$, *where* $f_k(a, n)$ *is defined by* (3.1). *If* P *is the product of the distinct prime factors of* a (*including* 1), *then* $\phi_k(a, n)$ *is periodic in* n *with least period* P.

Proof. First of all it is easy to establish that if g is any factor of a such that $(g, n) = 1$, then $(g, n + P) = 1$ and conversely. Thus (3.1) shows $\phi_k(a, n) = \phi_k(a, n+P)$ for all n, and therefore $\phi_k(a, n)$ is periodic with least period $R \leqq P$.

If $\phi_k(a, n)$ has least period R, then $\phi_k(a, n) = \phi_k(a, n+R)$ for all n. In particular take $n = a$ to obtain

$$a^k\sigma_k(1) = s^k\sigma_k(r),$$

where r is the largest factor of a such that $(r, a+R) = 1$ and $a = rs$. Thus we have $\sigma_k(r) = (a/s)^k = r^k$ which implies that $r = 1$. Therefore unity is the largest factor of a which is relatively prime to $a+R$. It follows that every prime factor of a divides R. Hence $R \geqq P$ whereupon $R = P$ as asserted.

4. Examples. The first example illustrates Theorem 1. Let us take $k = 2$ and $a = 12$. By Theorem 5, $f_2(12, n)/n^2$ is periodic with least period 6. Hence if we set

$$\gamma_n = 210,\ 160,\ 189,\ 160,\ 210,\ 144$$

when

$$n \equiv 1, \quad 2, \quad 3, \quad 4, \quad 5, \quad 0, \quad (\text{mod } 6),$$

respectively, then

$$\sum_{n=1}^{\infty} \frac{\gamma_n n^2 x^n}{1 - x^n} = \sum_{n=1}^{\infty} \sigma_2(12n)x^n.$$

The second example is on Theorem 4. Take $a = 15$, $k = 1$, $t = 3$, and $m = 12$. By Theorem 4

$$\sum_{d|12} g_1(3; 15, d) = \sigma(180).$$

By direct computation

$$\sum_{d|12} g_1(3; 15, d) = \sum_{\substack{d|12 \\ d \geqq 4}} g_1(3; 15, d)$$

$$= 4\sigma(15) + \sigma(30) + 18\sigma(5) - \sigma(15) + \sigma(30) + \sigma(45) + 36\sigma(5) - \sigma(30)$$

$$= 546 = \sigma(180).$$

For the last example we shall describe how to construct a table for $\sigma_k(n)$. By Theorem 1

$$\sum_{n=1}^{\infty} \frac{f_k(a, n)x^n}{1 - x^n} = \sum_{n=1}^{\infty} \sigma_k(an)x^n.$$

Equating corresponding coefficients yields the following systematic scheme:

x	x^2	x^3	x^4	$\cdots$
$f_k(a, 1)$	$f_k(a, 1)$	$f_k(a, 1)$	$f_k(a, 1)$	
	$f_k(a, 2)$		$f_k(a, 2)$	
		$f_k(a, 3)$		
			$f_k(a, 4)$	
$\sigma_k(a)$	$\sigma_k(2a)$	$\sigma_k(3a)$	$\sigma_k(4a)$	$\cdots$

If a is the prime p, then by the definition of $f_k(p, n)$ in (3.1), and because $f_k(p, n)/n^k$ is periodic with least period p by Theorem 5, we have

$$f_k(p, n) = \begin{cases} p^k n^k \text{ if } p \text{ divides } n, \\ (p^k + 1)n^k \text{ otherwise.} \end{cases}$$

The considerations of the last two paragraphs are illustrated by computing $\sigma_2(an)$ for $a = 5$ and $n = 1, 2, 3, 4, 5$, and 6.

an	5	10	15	20	25	30
	26	26	26	26	26	26
		104		104		104
			234			234
				416		
					625	
						936
$\sigma_2(an)$	26	130	260	546	651	1300

If for each prime p we compute $\sigma_k(p)$, $\sigma_k(2p)$, $\sigma_k(3p)$, $\cdots$, then $\sigma_k(n)$ will be computed at least twice unless n is unity or a prime power, but these special cases are easily checked directly.

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ON FORMING PARTIAL DIFFERENTIAL EQUATIONS*

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1. Introduction. Although there are many texts dealing with the formation of partial differential equations (p.d.e.'s) by the elimination of arbitrary functions, there is no systematic theory on the subject for equations of order greater than 1, and no conditions by which it can be guaranteed that a given equation involving arbitrary functions will have a p.d.e. of lowest order or of a specified order, *e.g.*, the same order as the number of arbitrary functions.

Since a p.d.e. for most functions can always be found by a sufficient number of differentiations it will be the purpose of this paper to try to determine the p.d.e. of lowest order in each case.

The letters p, q, r, s, and t will have their usual connotations and all work will be done in three dimensions.

The first part of this paper deals with explicit functions and the second part with implicit functions.

2. Explicit functions. Consider

$$z = \sum_{i=1}^{n} u_i(x, y)F_i[v(x, y)], \tag{1}$$

where the $u_i(x, y)$ and $v(x, y)$ are given functions of class C^n in both x and y and the F_i are arbitrary functions of class C', for $i=1, \cdots, n$.

From (1) we obtain

$$v_y p - v_x q = \sum_{i=1}^{n} (u_{ix}v_y - u_{iy}v_x)F_i,$$

and setting $z_2 = u_1(v_y p - v_x q) - z(u_{1x}v_y - u_{1y}v_x)$, $\phi_i = u_i(u_{ix}v_y - u_{iy}v_x) - u_i(v_y p - v_x g)$, $i=2, \cdots, n$, $z_2 = \sum_{i=2}^{n} \phi_i F_i$, which has the same form as (1) but contains one less arbitrary function. Proceeding in the same manner, we can eliminate the remaining $n-1$ arbitrary functions yielding a p.d.e. for (1) of at most nth order.

Functions of the form

$$z = \prod_{i=1}^{n} \{G_i[v(x, y)]\}^{u_i(x,y)} \tag{2}$$

are special cases of (1); for (2) may be written $\log z = \sum_{i=1}^{n} u_i \log G_i$, which takes the form of (1) upon setting $z_1 = \log z$ and $F_i = \log G_i$, $i=1, \cdots, n$.

In general, we can find a p.d.e. for functions of the type

$$z = \sum_{i=1}^{n} u_i(x, y)F_i[v_i(x, y)] \tag{3}$$

* This problem was suggested to the author by Professor H. J. Zimmerberg of Rutgers University.

by taking a sufficient number of partial derivatives. The totality of equations obtained by taking partial derivatives of (3) to the jth order is $(j+1)(j+2)/2$ for $j=1, \cdots, n$. Since each order of differentiation of (3) introduces n new arbitrary functions, there are at most $n(j+1)$ arbitrary functions to be eliminated from the $(j+1)(j+2)/2$ equations. Since $(j+1)(j+2)>2n(j+1)$ for $j>2n-2$, the p.d.e. of (3) will be at most of order $2n-1$. However we shall investigate the functions of the form (3) for which it is possible to obtain a p.d.e. of at most order n.

For $n=1$ it is always possible, and the p.d.e. for (3) is $u(v_y p - v_x q) - (u_x v_y - u_y v_x)z = 0$.

Consider $n=2$. After obtaining the partials of (3) up to and including the second order we have (with equation (3)) six equations involving linearly F_i, F_i', F_i'', $i=1, 2$. A necessary and sufficient condition that these equations be functionally dependent is that the 6×6 determinant of the F_i and their derivatives be identically zero. Let D denote this determinant.

After some lengthy but straightforward calculations*,

$$\begin{aligned}\frac{D}{(u_1u_2)^3} = J\Bigg[J\Bigg\{&v_{1y}v_{2y}\left[\left(\frac{u_{2x}}{u_2}\right)^2 - \frac{u_{2xx}}{u_2} - \left(\frac{u_{1x}}{u_1}\right)^2 + \frac{u_{1xx}}{u_1}\right] \\ &+ (v_{1x}v_{2y} + v_{1y}v_{2x})\left[\frac{u_{1x}u_{1y}}{u_1^2} - \frac{u_{1xy}}{u_1} - \frac{u_{2x}u_{2y}}{u_2^2} + \frac{u_{2xy}}{u_2}\right] \\ &+ v_{1x}v_{2x}\left[\left(\frac{u_{2y}}{u_2}\right)^2 - \frac{u_{2yy}}{u_2} - \left(\frac{u_{1y}}{u_1}\right)^2 + \frac{u_{1yy}}{u_1}\right]\Bigg\} \\ &+ P(1, 2, y)\left(\frac{u_{2x}}{u_2} - \frac{u_{1x}}{u_1}\right) - P(1, 2, x)\left(\frac{u_{2y}}{u_2} - \frac{u_{1y}}{u_1}\right)\Bigg],\end{aligned}$$

where

$$\begin{aligned} J &= v_{1y}v_{2x} - v_{1x}v_{2y}, \\ P(i, j, a) &= v_{1y}v_{2y}(v_{ia}v_{jxx} - v_{ja}v_{ixx}) + v_{1x}v_{2x}(v_{ia}v_{jyy} - v_{ja}v_{iyy}) \\ &\quad + (v_{1y}v_{2x} + v_{1x}v_{2y})(v_{ja}v_{ixy} - v_{ia}v_{jxy}). \end{aligned}$$

Observe that $\partial/\partial a(u_{ia}/u_i) = (u_i u_{iaa} - u_{ia}^2)/u_i^2$, $\partial/\partial y(u_{ix}/u_i) = \partial/\partial x(u_{iy}/u_i) = (u_i u_{ixy} - u_{ix}u_{iy})/u_i^2$, $u_{ia}/u_i = \partial/\partial a(\log u_i)$, $a=x, y$; $i=1, 2$; substitution of these relations into the above equation gives, after some simplification

$$D/(u_1u_2)^3 = J\{RD_x^2 w + SD_xD_y w + TD_y^2 w + PD_x w + QD_y w\} \equiv 0, \tag{4}$$

where $w = \log(u_2/u_1)$, $R = -v_{1y}v_{2y}J$, $S = (v_{1x}v_{2y} + v_{1y}v_{2x})J$, $T = -v_{1x}v_{2x}J$, $P=P(1, 2, y)$, and $Q=-P(1, 2, x)$.

If $J\equiv 0$, then F_1 and F_2 are functionally dependent and a second-order p.d.e. exists. Assume $J\not\equiv 0$. Then J is at most conditionally zero which, from the con-

* It is not recommended that the reader attempt to perform the calculations.

tinuity of J, implies that there is a region where $J \neq 0$.

The part of (4) in braces is a second-order linear p.d.e. which can be solved by means of Laplace's transformation (*c.f.* [7], p. 182), using $x=x(u, v)$, $y=y(u, v)$. Using the notations of [7], and substituting in $R\theta^2+S\theta+T=0$ we obtain $\theta=v_{1x}/v_{1y}$, v_{2x}/v_{2y}. It is apparent that the simple integrals $u=v_1$, $v=v_2$ will satisfy equations (7) and (8) of [7].

Substituting in the system (5) of [7] we obtain $R'=T'=P'=Q'=0$ and $S'=J^3$. (4) of [7] then becomes $S's'=0$, where $s'=\partial^2 w/\partial u \partial v$.

Therefore we have as a solution,

$$u_2(x, y) = u_1(x, y) \exp \{f_1[v_1(x, y)] + f_2[v_2(x, y)]\}, \tag{5}$$

where the f_i are any functions of class C^2.

We have therefore proved the following

THEOREM 1. *A necessary and sufficient condition that a partial differential equation of at most second order exists for* (3) *with* $n=2$, *where the* $u_i \neq 0$, $v_i \neq 0$ *are given functions of class* C^2 *in* x *and* y, *and the* F_i *are arbitrary functions of class* C^2 $(i=1, 2)$ *in some region* S, *is that either* $v_{1y}v_{2x}-v_{1x}v_{2y}\equiv 0$, *in which case the* F_i *are functionally dependent, or* (5) *holds.*

The general solutions of the second order reducible homogeneous p.d.e 's $(\beta_1 D_x-\alpha_1 D_y-\gamma_1)(\beta_2 D_x-\alpha_2 D_y-\gamma_2)z=0$, for all possible values of the constants α_i, β_i, γ_i can readily be shown to form a proper subset of the general family (5) obtained by setting $v_i=a_i x+b_i y$, $i=1, 2$.

Consider (3) for $n>2$. Taking all of the partial derivatives of (3) up to and including the nth order gives $(n+1)(n+2)/2$ equations involving linearly $n(n+1)$ arbitrary functions. Consider the $[(n+1)(n+2)/2]\times n(n+1)$ matrix M of the coefficients of these $n(n+1)$ functions. Since $(n+1)(n+2)/2<n(n+1)$ for $n>2$, we have: A necessary and sufficient condition that the p.d.e. of (3) is at most of order n is that the rank of M be less than $(n+1)(n+2)$.

3. Implicit functions. For

$$F(u_1, \cdots, u_n) = 0, \tag{6}$$

$u_i=u_i(x_1, \cdots, x_n, z)$, F arbitrary, u_i given, Ince ([4], p. 9) has shown that the p.d.e. is first order and exhibits it. For $n=2$, $x_1=x$, $x_2=y$, [2] p. 227, [6] p. 212, [7] p. 64, and [11] p. 294 show that the p.d.e. is of first order and exhibit it.

Consider $\phi(x, y, z)=\sum_{i=1}^{n} u_i(x, y, z)F_i[v_i(x, y, z)]$, where ϕ, u_i, v_i are given and F_i arbitrary suitably differentiable functions. For $n=1$, a first order p.d.e. clearly exists. For $n=2$, a result analogous to Theorem 1 holds, with $u_i(x, y)$, $v_i(x, y)$ replaced by $u_i(x, y, z)$, $v_i(x, y, z)$ in (5). For $n>2$ the discussion at the end of Section 2 is true also for the implicit functions.

4. Conclusion. The partial listing of texts on p.d.e.'s in the bibliography except [7] contain only problems which are all special cases of (3) for $n=1$, Theorem 1, or (6) for $n=2$. The exceptions in [7] are problems 19 and 20 on

page 65, and it can be readily verified that they satisfy the condition stated at the end of Section 2.

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NOTE ON THE METHOD OF CONTRACTANTS

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An apparently new and interesting method for the evaluation of determinants was recently described by Macmillan in [1]. It was stated there that this method (i) is very easy to learn and use, (ii) involves as few arithmetical operations as the most rapid method of pivotal condensation, (iii) can be programmed easily for a digital computer. We have applied this method to a number of cases, and have found it in general to be as attractive as indicated above.

The technique is easily described as follows: Let

$$A = \begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} \cdots & a_{nn} \end{bmatrix} \tag{1}$$

be a matrix whose determinant $\det A$ is to be determined. This may be accomplished by the calculation of a sequence of matrices of successively lower order: First

$$A_1 = \begin{bmatrix} \overset{(1)}{a_{11}} & \overset{(1)}{a_{12}} & \cdots & \overset{(1)}{a_{1,n-1}} \\ \overset{(1)}{a_{21}} & \overset{(1)}{a_{22}} & \cdots & \overset{(1)}{a_{2,n-1}} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \overset{(1)}{a_{n-1,1}} & \overset{(1)}{a_{n-1,2}} & \cdots & \overset{(1)}{a_{n-1,n-1}} \end{bmatrix} \tag{2}$$

is obtained from A by means of all the second order minors:

$$a_{ij}^{(1)} = [a_{ij}a_{i+1,j+1} - a_{i,j+1}a_{i+1,j}] \tag{3}$$

$i, j=1, \cdots, n-1$. Next A_1 is contracted into A_2, where

$$A_2 = \begin{bmatrix} a_{11}^{(2)} & a_{12}^{(2)} & \cdots & a_{1,n-2}^{(2)} \\ a_{21}^{(2)} & a_{22}^{(2)} & \cdots & a_{2,n-2}^{(2)} \\ \vdots & \vdots & & \vdots \\ a_{n-2,1}^{(2)} & a_{n-2,2}^{(2)} & \cdots & a_{n-2,n-2}^{(2)} \end{bmatrix} \tag{4}$$

and

$$a_{ij}^{(2)} = \frac{1}{a_{i+1,j+1}} [a_{ij}^{(1)} a_{i+1,j+1}^{(1)} - a_{i,j+1}^{(1)} a_{i+1,j}^{(1)}] \tag{5}$$

$i, j=1, \cdots, n-2$.

Thus, in general, $A_{k+1} \equiv (a_{ij}^{(k+1)})$ is obtained from the $A_k \equiv (a_{ij}^{(k)})$, $i, j=1, \cdots, n-k$; $k=0, 1, \cdots, n-1$, by the simple algorithm

$$a_{ij}^{(k+1)} = \frac{1}{a_{i+1,j+1}^{(k-1)}} [a_{ij}^{(k)} a_{i+1,j+1}^{(k)} - a_{i,j+1}^{(k)} a_{i+1,j}^{(k)}] \tag{6}$$

for $i, j=1, \cdots, n-k-1$; with $a_{ij}^{(-1)}=1$, $a_{ij}^{(0)}=a_{ij}$. It is found that $A_{n-1}=a_{11}^{(n-1)} = \det A$.

As pointed out in [1], the correctness of the method may be deduced from a theorem regarding symmetrical rearrangements within determinants, as described, for example, by Aitken [2].

A count of arithmetical operations reveals that $(1/3)(n-1)n(2n-1)$ multiplications, $(1/6)(n-2)(n-1)(2n-3)$ divisions, and approximately $(1/4)(n-1)n(2n-1)$ additions are required to perform the entire contraction. The number of multiplications is thus identical with that required for the pivotal condensation method [3]. As equations (6) indicate, a symmetric matrix A is thus contracted into a sequence of symmetric matrices A_k, so that in this case the computational labor is approximately halved.

A point requiring further elaboration concerns the necessary removal of "interior" zeros, *i.e.*, zero elements $a_{ij}^{(k)}$ that may occur anywhere in A_k except in the outside rows or columns.

It is easy to devise appropriate techniques for the removal of such zeros. Thus one may proceed as follows:

I. *Removal of interior zeros from the first row.* Let us first examine the first row R_1 of A. If no zeros occur in R_1, we may proceed to step II. If, on the other hand, there are zeros in R_1, let the first zero element in R_1 be a_{ij}, $j \leqq n-1$. Then we examine the column C_j of A. Let the first nonzero element in C_j be a_{ij}, $i \leqq n-1$. Consider now the $j-2$ equations for the parameter λ_k:

$$\lambda_k a_{1k} + a_{ik} = 0, \qquad k = 2, \cdots, j-1.$$

Thus $\lambda_k = -a_{ik}/a_{1k}$, for these k. Among these λ_k let the nonvanishing ones be denoted by $\mu_1, \cdots, \mu_s$. Then we put $\Lambda_1 \neq \mu_1^{-1}, \cdots, \mu_s^{-1}$, and replace in A the row R_1 by

$$R_1^{(1)} = R_1 + \Lambda_1 R_i. \tag{7}$$

Clearly then $a_{1k}^{(1)} \neq 0$ for $k=2, \cdots, j$, so that the string of nonvanishing elements in the first row $R_1^{(1)}$ has been extended by at least one element beyond that in R_1. Certainly, $\det(R_1^{(1)}, R_2, \cdots, R_n) = \det(R_1, R_2, \cdots, R_n) = \det A$. This procedure is continued in the same manner until all the interior zero elements have been removed from the first row of the matrix; let this row be denoted by $R_1^{(s)}$, $s \geqq 1$.

II. *Removal of interior zeros from* $R_2, R_3, \cdots, R_{n-1}$. Assume, now, that in R_i, $i=2, \cdots, n-1$, the elements $a_{ik} \neq 0$, $k=2, \cdots, r-1$, but that $a_{ir}=0$, with $r \leqq n-1$. As above, consider

$$\rho_k a_{1k}^{(s)} + a_{ik} = 0, \qquad 2 \leqq k \leqq r.$$

Thus $\rho_k = -a_{ik}/a_{1k}^{(s)}$. Choosing $\rho \neq \rho_k$, $2 \leqq k \leqq r$, replace R_i by

$$R_i^{(1)} = R_i + \rho R_1^{(s)}. \tag{8}$$

Now $a_{1k}^{(1)} \neq 0$ for $2 \leqq k \leqq r$, extending the string of nonzero elements in $R_i^{(1)}$ by at least one element. Further,

$$\det(R_1^{(s)}, R_2, \cdots, R_i, \cdots, R_n) = \det(R_1^{(s)}, R_2, \cdots, R_i^{(1)}, \cdots, R_n) = \det A.$$

In this manner there will be obtained a matrix $A^{(t)}$ which contains no interior zero elements, and for which $\det A^{(t)} = \det A$.

III. *Occurrence of interior zeros in* A_k. It may happen that in A_k a particular element of R_i vanishes, $2 \leqq i, j \leqq n-k-2$, $0 \leqq k \leqq n-4$.

In such a case one may simply return to $A^{(t)}$, replace R_i by $R_i + \sigma R_1$, and start the calculation over again, choosing the parameter σ in such a manner that the corresponding element in the transformed matrix A_k becomes nonzero while the nonvanishing of the previous interior elements remains preserved. Obviously a change of R_i in $A^{(t)}$ will in general produce changes of R_{i-k}, $R_{i-k+1}, \cdots, R_i$ in A_k.

The following example illustrates the procedure. Suppose it is desired to calculate $\det A$, where

$$A = \begin{bmatrix} 1 & -1 & 2 & 0 & 0 \\ -1 & 0 & 3 & 0 & 2 \\ 0 & 3 & -2 & 1 & -1 \\ 0 & -3 & -11 & -2 & 0 \\ 2 & 0 & 1 & 0 & 2 \end{bmatrix}.$$

Putting first $R_1^{(1)}=R_1+2R_3$, then $R_2^{(1)}=R_2+R_1^{(r)}$, there results

$$A^{(1)}=\begin{bmatrix} 1 & 5 & -2 & 2 & -2 \\ 0 & 5 & 1 & 2 & 0 \\ 0 & 3 & -2 & 1 & -1 \\ 0 & -3 & -11 & -2 & 0 \\ 2 & 0 & 1 & 0 & 2 \end{bmatrix}.$$

It follows that

$$A_1=\begin{bmatrix} 5 & 15 & -6 & 4 \\ 0 & -13 & 5 & -2 \\ 0 & -39 & 15 & -2 \\ 6 & -3 & 2 & -4 \end{bmatrix},$$

and

$$A_2=\begin{bmatrix} -13 & -3 & -4 \\ 0 & 0 & -40 \\ 78 & 3 & 28 \end{bmatrix}.$$

Here the element $a_{22}^{(2)}=0$. By III we may remove this defect by starting afresh, replacing in $A^{(1)}$ the row R_2 by $R_2+\sigma R_1$. For $\sigma=1$ there results

$$A^{(2)}=\begin{bmatrix} 1 & 5 & -2 & 2 & -2 \\ 1 & 10 & -1 & 4 & -2 \\ 0 & 3 & -2 & 1 & -1 \\ 0 & -3 & -11 & -2 & 0 \\ 2 & 0 & 1 & 0 & 2 \end{bmatrix},$$

$$A_1=\begin{bmatrix} 5 & 15 & -6 & 4 \\ 3 & -17 & 7 & -2 \\ 0 & -39 & 15 & -2 \\ 6 & -3 & 2 & -4 \end{bmatrix},$$

$$A_2=\begin{bmatrix} -13 & -3 & -4 \\ -39 & -9 & 16 \\ -78 & 3 & 28 \end{bmatrix},$$

$$A_3=\begin{bmatrix} 0 & -12 \\ 21 & -20 \end{bmatrix},$$

$$A_4=\det A=28.$$

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INVERSE ELLIPTIC FUNCTIONS AND LEGENDRE POLYNOMIALS

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Series expansions for the complete elliptic integrals $K(k)$ and $E(k)$ which involve Legendre polynomials have been known for quite some time. However, most of these expansions involve the Legendre polynomial of argument $\cos\theta$ (θ real), and here we shall obtain expansions of $K(k)$ in which the argument of the Legendre polynomials which appear is greater than one. The series for $K(k)$ appear as special cases of power series expansions of three of the Jacobian inverse elliptic functions: $\mathrm{sn}^{-1}(x, k)$, $\mathrm{cn}^{-1}(x, k)$ and $\mathrm{tn}^{-1}(x, k)$.* Throughout the following $K(k)$ denotes the complete elliptic integral of the first kind with modulus k, $0<k<1$. The complementary modulus k' is, as usual, defined to be $\sqrt{(1-k^2)}$.

If n is a nonnegative integer, $0<|x|\leqq 1$, $\lambda=\frac{1}{2}(x+x^{-1})$, and $P_n(\lambda)$ denotes the nth Legendre polynomial, then we define the polynomial $R_n(x)$ to be $x^nP_n(\lambda)$. Using Laplace's first integral for $P_n(\lambda)$ [4], we obtain after some simplification

$$R_n(x) = \frac{2}{\pi}\int_0^{\pi/2}(x^2\sin^2\theta+\cos^2\theta)^n d\theta > 0. \tag{1}$$

On expanding the integrand and integrating we have

$$R_n(x) = \frac{1}{2^{2n}}\sum_{k=0}^{n}\binom{2n-2k}{n-k}\binom{2k}{k}x^{2k}, \tag{2}$$

but we shall not need this explicit expression for $R_n(x)$.

The Jacobian inverse elliptic function $\mathrm{sn}^{-1}(y, k)$ is defined by the integral

$$\mathrm{sn}^{-1}(y, k) = \int_0^y [(1-t^2)(1-k^2t^2)]^{-1/2}dt$$

if $|y|\leqq 1$ and $0<k<1$. Differentiating with respect to y and setting $ky^2=h$ and $\lambda=\frac{1}{2}(k+k^{-1})$ we have

$$\frac{d}{dy}\,\mathrm{sn}^{-1}(y, k) = (1-2\lambda h+h^2)^{-1/2} \text{ if } |y|<1.$$

From the theory of Legendre polynomials [3]: If $F(z)=z\pm\sqrt{(z^2-1)}$, then

$$(1-2zh+h^2)^{-1/2} = \begin{cases} \displaystyle\sum_{n=0}^{\infty} P_n(z)h^n, & \text{if } |h| < \min|F(z)|, \\ \displaystyle\sum_{n=0}^{\infty} P_n(z)h^{-(n+1)}, & \text{if } |h| > \max|F(z)|. \end{cases} \tag{3}$$

* Here we are following the notation used in [2]; some books write $\mathrm{sc}^{-1}(x, k)$ rather than $\mathrm{tn}^{-1}(x,k)$.

Since $\min |F(\lambda)| = k$ it follows that

$$\frac{d}{dy}\operatorname{sn}^{-1}(y, k) = \sum_{n=0}^{\infty} P_n(\lambda)k^n y^{2n} \text{ if } |y| < 1.$$

Integrating both sides of this equation, $0 \leqq y \leqq x < 1$, we obtain

$$\operatorname{sn}^{-1}(x, k) = \sum_{n=0}^{\infty} P_n(\lambda)k^n \frac{x^{2n+1}}{2n+1}, \qquad |x| < 1. \tag{4}$$

Now $k^n P_n(\lambda) = R_n(k) > 0$ and by a theorem due to Pringsheim on the converse of Abel's Theorem ([1], p. 256) it follows that (4) is valid for $x = 1$ and hence $|x| \leqq 1$.

If we take $x = 1$ in (4) we obtain the special case

$$K(k) = \sum_{n=0}^{\infty} P_n(\lambda) \frac{k^n}{2n+1} \cdot$$

If we take $x = (1+k')^{-1/2}$ and use the fact that $\operatorname{sn}^{-1}((1+k')^{-1/2}, k) = \frac{1}{2}K(k)$ ([2], p. 31), then we have

$$\sqrt{(1+k')}K(k) = 2\sum_{n=0}^{\infty} P_n(\lambda)\left(\frac{1-k'}{1+k'}\right)^{n/2} \frac{1}{2n+1} \cdot$$

Expansions for the other two inverse elliptic functions are obtained similarly but conditions for the convergence of the series are more complicated. Differentiation of the integral

$$\operatorname{cn}^{-1}(y, k) = \int_y^1 [(1-t^2)(k'^2 - k^2t^2)]^{-1/2} dt, \qquad \text{if } |y| \leqq 1,$$

with respect to y together with the substitutions $ky^2 = ik'h$ and $\mu = \frac{1}{2}(k'/k - k/k')$ yields

$$\frac{d}{dy}\operatorname{cn}^{-1}(y, k) = -\frac{1}{k'}(1 - 2i\mu h + h^2)^{-1/2}, \qquad \text{if } |y| < 1.$$

Now

$$\min |F(i\mu)| = \begin{cases} k/k' & \text{if } 0 < k \leqq 1/\sqrt{2}, \\ k'/k & \text{if } 1/\sqrt{2} < k < 1, \end{cases}$$

so from (3) we have that

$$\frac{d}{dy}\operatorname{cn}^{-1}(y, k) = -\frac{1}{k'}\sum_{n=0}^{\infty} P_n(i\mu)(-i)^n(k/k')^n y^{2n}. \tag{5}$$

The series converges and represents the function on the left if (1) $0 < k \leqq 1/\sqrt{2}$ and $|y| < 1$, or if (2) $1/\sqrt{2} < k < 1$ and $|y| < k'/k$. From Laplace's first integral

for the Legendre polynomials it is easily shown that $P_n(i\mu)=i^nS_n(\mu)$ where $S_n(\mu)$ is a polynomial of degree n in the symbol μ. These polynomials satisfy the recurrence relation

$$(n+1)S_{n+1}(\mu) = (2n+1)\mu S_n(\mu) + nS_{n-1}(\mu),$$

from which it is seen that $S_n(\mu)>0$ if $\mu>0$. Clearly, $S_n(\mu)$ is the polynomial obtained by changing all minus signs in $P_n(\mu)$ to plus signs.

If $0<k\leqq 1/\sqrt{2}$ and we integrate (5), $0\leqq y\leqq x<1$, we have

$$\text{(6)} \qquad \operatorname{cn}^{-1}(x, k) - K(k) = -\frac{1}{k'}\sum_{n=0}^{\infty} S_n(\mu)(k/k')^n \frac{x^{2n+1}}{2n+1}\cdot$$

Since $S_n(\mu)>0$, (6) is valid for $0\leqq x\leqq 1$ by the theorem of Pringsheim noted above. Also, it is clear that (6) is valid for $|x|\leqq 1$. If $1/\sqrt{2}<k<1$, then (6) is valid for $0\leqq x<k'/k$.

If $0<k\leqq 1/\sqrt{2}$ and $x=1$ we have

$$k'K(k) = \sum_{n=0}^{\infty} S_n(\mu)(k/k')^n \frac{1}{2n+1},$$

and if $x=\sqrt{\{k'/(1+k')\}}=\sqrt{(k'/k)}\sqrt{\{(1-k')/k\}}$, then from the relation $\operatorname{cn}^{-1}(\sqrt{\{k'/(1+k')\}}, k)=\frac{1}{2}K(k)$ ([2], p. 31) we obtain

$$K(k) = \frac{2}{k}\sqrt{\left(\frac{1-k'}{k'}\right)}\sum_{n=0}^{\infty} S_n(\mu)\left(\frac{1-k'}{k}\right)^n \frac{1}{2n+1}\cdot$$

In the case of $\operatorname{tn}^{-1}(y, k)$ we obtain two series, one of which is a generalization of Gregory's series for $\tan^{-1} x$. Since

$$\operatorname{tn}^{-1}(y, k) = \int_0^y [(1+t^2)(1+k'^2t^2)]^{-1/2}dt, \qquad \text{if } 0\leqq y\leqq \infty,$$

differentiation of the integral together with the substitutions $k'y^2=h$ and $\lambda'=\frac{1}{2}\{k'+(1/k')\}$ yields

$$\frac{d}{dy}\operatorname{tn}^{-1}(y, k) = [1-2(-\lambda')h+h^2]^{-1/2}.$$

Now min $|F(-\lambda')|=k'$ and max $|F(-\lambda')|=1/k'$ so from (3) we have

$$\text{(7)} \qquad \frac{d}{dy}\operatorname{tn}^{-1}(y, k) = \sum_{n=0}^{\infty} P_n(\lambda')k'^n(-1)^n y^{2n} \text{ if } |y| < 1,$$

and

$$\text{(8)} \qquad \frac{d}{dy}\operatorname{tn}^{-1}(y, k) = \sum_{n=0}^{\infty} \frac{P_n(\lambda')(-1)^n}{k'^{n+1}y^{2n+2}} \quad \text{if } \quad |y| > 1/k'.$$

Integrating (7), $0\leqq y\leqq x<1$, we have

$$\text{(9)} \qquad \operatorname{tn}^{-1}(x, k) = \sum_{n=0}^{\infty} P_n(\lambda')k'^n(-1)^n \frac{x^{2n+1}}{2n+1}\cdot$$

If $0 \leqq x \leqq 1$ the series (9) is absolutely convergent and represents $\operatorname{tn}^{-1}(x, k)$ since the series obtained for $\operatorname{sn}^{-1}(x, k')$ is convergent in the same interval. The series on the right side of (8) converges uniformly for $1/k' \leqq y \leqq x_0$; integrating (8), $1/k' \leqq x \leqq y \leqq \infty$, we find*

$$\text{(10)} \qquad \operatorname{tn}^{-1}(x, k) = K(k) - \sum_{n=0}^{\infty} \frac{P_n(\lambda')(-1)^n}{k'^{n+1}x^{2n+1}} \frac{1}{2n+1} \quad \text{if} \quad x \geqq 1/k'.$$

Formally, if $k=0$ then (10) implies the expansion of $\tan^{-1} x$ known as Gregory's series for $\tan^{-1} x$. If we take $x = 1/k'$ in (10) then we have

$$\text{(11)} \qquad \operatorname{tn}^{-1}(1, k) = \sum_{n=0}^{\infty} P_n(\lambda')k'^n(-1)^n \frac{1}{2n+1}$$

since $\operatorname{tn}^{-1}(1/k', k) + \operatorname{tn}^{-1}(1, k) = K(k)$ ([2], p. 32). Formally, if $k=0$ then $k'=1$, $P_n(\lambda')k'^n = R_n(1) = 1$, $\operatorname{tn}^{-1}(1, 0) = \pi/4$, and from (11) we obtain a well-known series for $\pi/4$.

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THE DIRAC MEASURE AS APPLIED TO THE SOLUTION OF DIFFERENCE EQUATIONS BY MEANS OF TRANSFORMS

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It is known that transforms (Laplace, Fourier, *etc.*) can be used to solve difference equations.§ In this note it is shown that by aid of the Dirac measure of L. Schwartz† (in essence the integral of Stieltjes) one can obtain the solution of linear difference equations.

Let z_k, $k=1, \cdots, n$ be a set of complex points on a contour C, and let $d\mu = a_k$ at $z = z_k$, $k=1, \cdots, n$, $d\mu = 0$ elsewhere. We define the Stieltjes' integral

* For a justification of this step see [1] p. 500.

§ H. Bremmer and B. van der Pol, Operational Calculus, Cambridge, 1950; G. Doetsch, Handbuch der Laplace-Transformation, vol. 3, Basel and Stuttgart, 1950.

† L. Schwartz, Théorie des Distributions, tome 1, Paris, 1950.

of $f(z)$ along C by

$$\int_C f(z)d\mu \equiv \sum_{k=1}^{n} a_k f(z_k). \tag{1}$$

As a special case with $d\mu=1$ at $z=0$, $d\mu=0$ otherwise (the Dirac-measure of L. Schwartz), one has $\int_{-\infty}^{\infty} f(x)d\mu=f(0)$.

One can obtain a generalization of (1) as follows: Let $y=f(x)$, $-\infty<x<\infty$, $f(x)$ single-valued, represent a curve in the xy-plane, and define the measure $d\mu(x, y)$ by

$$\begin{aligned} d\mu(x, y) &= \rho(x) \quad &&\text{for } y = f(x), \\ d\mu &= 0 &&\text{otherwise.} \end{aligned} \tag{2}$$

We define the integral of $g(x, y)$ relative to $d\mu(x, y)$ by

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x, y)d\mu \equiv \int_{-\infty}^{\infty} g(x, f(x))\rho(x)dx \tag{3}$$

provided the Riemann-integral on the right of (3) exists. In particular,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (y - f(x))F(x, y)d\mu = 0 \tag{4}$$

for $d\mu$ defined by (2), with $\rho(x)$ arbitrary.

We now make use of these definitions to find the solutions of three simple homogeneous difference equations.

Let us determine a general solution of

$$H_{\nu-1}(z) + H_{\nu+1}(z) = -2H_\nu'(z). \tag{5}$$

We define $H_\nu(z)$ by

$$H_\nu(z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\nu x}e^{-zy}d\mu \tag{6}$$

with $d\mu$ unspecified. Equation (5) yields (formally)

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\nu x}e^{-zy}(\cosh x - y)d\mu = 0. \tag{7}$$

From (4) it follows that $d\mu=\rho(x)$ for $y=\cosh x$, $\rho(x)$ arbitrary, $d\mu=0$ elsewhere, is a solution of (7). The definition (3) along with (6) yields

$$H_\nu(z) = \int_{-\infty}^{\infty} \rho(x)e^{-\nu x}e^{-z \cosh x}dx = \int_{-\infty}^{\infty} \sigma(x) \cosh \nu x \, e^{-z \cosh x}dx \tag{8}$$

with $\sigma(x)$ arbitrary, provided the integral of (8) exists. For $\sigma(x)\equiv 1$ one obtains the integral representation of the modified Hankel function,

$$K_\nu(z) = \int_{-\infty}^{\infty} \cosh \nu x \, e^{-z \cosh x} dx, \qquad \text{for } \operatorname{Re} z > 0.$$

A simple generalized random walk problem leads to the equation

$$\phi_{n+1}(z) = \int_{-\infty}^{\infty} \phi_n(z - t)\phi(t)dt \tag{9}$$

with $\int_{-\infty}^{\infty}\phi(t)dt = 1$, $\phi_1(z) = \phi(z)$. The Fourier integral of $\phi(t)$ is given by $\psi(y) = \int_{-\infty}^{\infty}\phi(t)e^{iyt}dt$.

We define $\phi_n(z)$ by

$$\phi_n(z) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-nx}e^{-iyz}d\mu \tag{10}$$

with $d\mu$ as yet undefined. Quite formally (9) and (10) yield

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-nx}e^{-iyz}[e^{-x} - \psi(y)]d\mu = 0. \tag{11}$$

A solution of (11) for $d\mu$ is given by $d\mu = \rho(y)$ for $e^{-x} = \psi(y)$, $d\mu = 0$ elsewhere. Thus (10) yields

$$\phi_n(z) = \int_{-\infty}^{\infty} \rho(y)[\psi(y)]^n e^{-iyz}dy. \tag{12}$$

Since

$$\phi(z) = \phi_1(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \psi(y)e^{-izy}dy,$$

it follows that $\rho(y) \equiv 1/2\pi$ so that

$$\phi_n(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} [\psi(y)]^n e^{-iyz}dy. \tag{13}$$

As an application of (1) we consider $f(s+1) + f(s) = 0$. Let $f(s) = \int_C e^{-sz}d\mu$ with $d\mu$ and the contour C as yet unspecified. It follows that $\int_C e^{-sz}(e^{-z}+1)d\mu = 0$, so that $d\mu = a_k$, a_k arbitrary, for z_k satisfying $e^{-z_k}+1 = 0$. Thus $z_k = -\pi i(1+2k)$, $k = 0, \pm 1, \pm 2, \cdots$, and the contour C is chosen as the imaginary axis. It follows that

$$f(s) = \sum_{k=-\infty}^{\infty} a_k e^{\pi i s(1+2k)} \tag{14}$$

provided the series converges.

Since the method outlined above requires formal manipulations, one must verify that the solution so obtained does satisfy the difference equation in question.

THEOREMS ON NONASSOCIATIVE NUMBER THEORY

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The *free logarithmetic* $\mathfrak{L}$ can be defined as the arithmetic of the indices of powers of the generator of a free cyclic groupoid.* *Indices of* $\mathfrak{L}$ are analogous to Etherington's noncommutative *shapes* [1], his *partitioned serials* [2] and his *indices* of the "most general" logarithmetic B [3], to elements of Robinson's *simple forest* [6] and to Evans' *nonassociative numbers* [4]. Systems of postulates for $\mathfrak{L}$ have been given by Etherington [2], Robinson [6] and Evans [4]. Some basic results in the nonassociative number theory, notably the unique factorization theorem, have been obtained by Etherington [2], Evans [4] and the author [5].

Addition and multiplication in $\mathfrak{L}$ are both noncommutative. Two indices commute additively if and only if they are equal. In Theorem 1 we give a necessary and sufficient condition that two indices of $\mathfrak{L}$ should commute multiplicatively. In Theorems 2 and 3 we solve Diophantine-like equations. In [4] Evans has proved "Fermat's Last Theorem" for nonassociative numbers. Theorem 3 generalizes this result which is then deduced as a corollary. We use the notation and nomenclature of [5].

It is convenient to extend the definition of exponentiation of an index (*cf.* [2], p. 449): We define $P^0=1$ for all indices P of $\mathfrak{L}$.

THEOREM 1. *P and Q, two indices of $\mathfrak{L}$, commute with respect to multiplication, i.e., $PQ=QP$, if and only if they are powers of the same index.*

Proof. The condition is obviously sufficient. To prove necessity denote the number of prime factors in an index X by $pr(X)$ and use induction on $pr(PQ)$. If $pr(PQ)=0$ or 1 then either P or Q is equal to 1 and either $P=Q^0$ or $Q=P^0$. Now let $pr(PQ)=a$ $(a>1)$ and assume that the condition is necessary for all pairs of commuting indices whose products contain less than a prime factors. Without loss of generality we can suppose that $pr(P)\leqq pr(Q)$. If $P=1$ then $P=Q^0$. Otherwise $PQ=QP$ gives $Q=PQ'$, where Q' is a proper left-divisor of Q or is equal to 1. It follows that $P^2Q'=PQ'P$ and therefore $PQ'=Q'P$. Now, $pr(PQ')<a$ and, by the induction hypothesis, $P=R^s$ and $Q'=R^t$ for some index R. Hence $P=R^s$ and $Q=R^{s+t}$.

THEOREM 2. *If $X^mP=Y^n$ or $PX^m=Y^n$, where X, Y, P are indices of $\mathfrak{L}$, P is prime and m, n are integers greater than* 1, *then X and Y are both powers of P.*

Proof. $X^mP=Y^n$ implies either (i) $X=Y^sY'$ or (ii) $Y=X^tX'$, where s and t are maximal in the sense that Y', X' are proper left-divisors of Y, X respectively or are equal to 1.

If (i) $X=Y^sY'$ then $XP=Y^sY'P$ and, since XP is a proper right-divisor of

* *Cf.* [5]. A *groupoid* is a set closed with respect to a binary product operation. It is *cyclic* if it is generated by one element.

Y^n and $pr(Y^s) < pr(XP) \leqq pr(Y^{s+1})$, $XP = Y''Y^s$ where Y'' is a proper right-divisor of Y or is equal to Y. But $pr(Y'') = pr(Y'P) = pr(XP) - pr(Y^s)$ and since Y'' is a left-divisor of XP it is a left-divisor of X and thus of Y. Hence $Y'' = Y'Q$ where Q is a prime index. Now, Y'' is also a right-divisor of Y so that Q is the prime right-divisor of Y. Therefore $Q = P$ and $Y^sY'P = Y'PY^s$, *i.e.*, Y^s and $Y'P$ commute. Hence, by Theorem 1, $Y^s = R^a$ and $Y'P = R^b$ for some index R. Note that P is a right-divisor of R and R is a left-divisor of Y and let $R = Y_1 \cdots Y_{k-1}P$ where the Y_i are prime. Then

$$XP = (Y_1 \cdots Y_{k-1}P)^{a+b}, \qquad X = (Y_1 \cdots Y_{k-1}P)^{a+b-1}Y_1 \cdots Y_{k-1}$$

and, as $X^mP = Y^n$ and $m > 1$,

$$(Y_1 \cdots Y_{k-1}P)^{a+b-1}Y_1 \cdots Y_{k-1}(Y_1 \cdots Y_{k-1}P)^{a+b-1} \cdots P = (Y_1 \cdots Y_{k-1}P)^g \cdots,$$

where $g \geqq 2a + 2b - 1$. Since factorization into primes is unique in $\mathfrak{L}$ the $(k(a+b))$th, $(k(a+b)+1)$th, $\cdots$, $(k(a+b)+k-1)$th prime factors on both sides are equal, *i.e.*, $Y_1 = P$, $Y_2 = Y_1$, $\cdots$, $P = Y_{k-1}$. Hence $Y_1 = \cdots = Y_{k-1} = P$ and X and Y are both powers of P.

If (ii) $Y = X^tX'$ consider first the case when $X' = 1$. Then $X^mP = X^{nt}$. Hence $P = X^{nt-m}$ and since P is prime $X = P$ and $Y = P^t$. If $X' \neq 1$ it is a proper left-divisor of X. Since P is a right-divisor of Y it is a right-divisor of X'. Let $X' = X''P$. Now, $pr(X^t) < pr(Y) < pr(X^{t+1})$ and Y is a right-divisor of X^mP; therefore $Y = X'''X^tP$, where X''' is a proper right-divisor of X or is equal to 1. But X''' is a left-divisor of Y and therefore of X. Also X'' is a left-divisor of X and $pr(X'') = pr(X''')$. It follows that $X'' = X'''$ and $Y = X^tX''P = X''X^tP$. Hence $X^tX'' = X''X^t$, *i.e.*, X^t and X'' commute and, by Theorem 1, $X^t = (X_1 \cdots X_h)^c$ and $X'' = (X_1 \cdots X_h)^d$ where the X_i are prime. Remembering that $n > 1$ we have

$$(X_1 \cdots X_h)^f \cdots P = (X_1 \cdots X_h)^{c+d}P(X_1 \cdots X_h)^{c+d} \cdots,$$

where $f \geqq 2c + 2d$. Comparing the $(h(c+d)+1)$th, $(h(c+d)+2)$th, $\cdots$, $(h(c+d)+h)$th prime factors on both sides we have $X_1 = P$, $X_2 = X_1$, $X_3 = X_2$, $\cdots$, $X_h = X_{h-1}$. Hence $X_1 = \cdots = X_h = P$ and both X and Y are powers of P.

If $PX^m = Y^n$ the proof is similar.

Note. In the statements and proofs of Theorems 1 and 2 no direct use is made of the operation of addition. It follows that these theorems are really about the free *multiplicative* logarithmetic $\mathfrak{L}^\times$, the semigroup formed by all indices and their multiplication (*v.* [5], p. 321).

I am indebted to a referee for drawing my attention to the fact that Theorems 1 and 2 actually apply to free semigroups. Indeed the free semigroup with ξ generators is isomorphic to any subsemigroup of $\mathfrak{L}^\times$ generated by ξ prime indices.

THEOREM 3. *If $X^p+Y^q=Z^r$ where X, Y, Z are indices of $\mathfrak{L}$ and p, q, r are integers greater than* 1 *then $X=2^k$, $Y=2^m$, $Z=2^n$ and $kp=mq=nr-1$.*

Proof. Let $Z=Z_1\cdots Z_n$, where the Z_i are prime. $X^p+Y^q=Z^r$ implies $X^p=(Z_1\cdots Z_n)^{r-1}Z_1\cdots Z_{n-1}Z_n'$ and $Y^q=(Z_1\cdots Z_n)^{r-1}Z_1\cdots Z_{n-1}Z_n''$ where $Z_n'+Z_n''=Z_n$ (*Cf.* [4], p. 302 or [5], p. 339). Since Z_n is prime, Z_n' and Z_n'' must be mutually left-prime.

Suppose $Z_n', Z_n''\neq 1$. Then $X=UX'$ and $Y=UY'$ where X', Y' are mutually left-prime and neither is 1. Therefore $U(X'X^{p-1}+Y'Y^{q-1})=Z^r$ and $X'X^{p-1}+Y'Y^{q-1}$ is prime. Since p, $q>1$ the potency* of $X'X^{p-1}+Y'Y^{q-1}$ is greater than that of U and hence $X'X^{p-1}+Y'Y^{q-1}$ is not a factor of U. Thus a prime occurs only once as a factor of X^p+Y^q. But every prime factor of Z^r occurs at least r times, *i.e.*, more than once and $X^p+Y^q=Z^r$ which is a contradiction. Therefore either Z_n' or Z_n'' is equal to 1.

If Z_n' is equal to 1, $X^pZ_n=Z^r$, where Z_n is prime and p, $r>1$. Therefore, by Theorem 2, $X=Z_n^k$, $Z=Z_n^n$. Thus $Y^q=Z_n^{nr-1}Z_n''$. Since $q>1$ the potency of Y is not greater than half of that of $Z_n^{nr-1}Z_n''$. Now, $nr-1>0$ and the potency of Z_n'' is less than that of Z_n, which is prime. Hence all prime factors of Y are equal to Z_n and $Z_n''=1$. Therefore $Z_n=Z_n'+Z_n''=2$ and $X=2^k$, $Y=2^m$, $Z=2^n$ where $kp=mq=nr-1$.

If $Z_n''=1$ the proof is similar.

COROLLARY ("Fermat's Last Theorem"). *$X^r+Y^r=Z^r$ implies $r=1$.*

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* The *potency* of an index P, δ_P, (*v.* [5], p. 324) corresponds to Etherington's *degree* of the index P, δ_P, and to Evans' *length* of the nonassociative number P, $|P|$.

MATHEMATICAL NOTES

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EXTENSIONS OF AN INEQUALITY OF H. S. SHAPIRO

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H. S. Shapiro's inequality [1] is

$$\sum \frac{x_i}{x_{i+1}+x_{i+2}} \geqq \frac{n}{2}, \tag{1}$$

where (and throughout this note) $x_{n+j}=x_j>0$ for all j, i is the summation suffix, and i runs over 1 to n unless otherwise indicated. This has been proved true for $n=3, 4, 5$ and false for $n=20$ (see [1]). It is also trivially true for $n=1, 2$. I extend it to

$$\sum \frac{x_i}{x_{i+1}+\cdots+x_{i+m}} \geqq \frac{n}{m}, \tag{2}$$

where now each denominator on the left consists of m terms, and prove the following results.

THEOREM 1. *The inequality* (2) *is true if*

$$\sin\frac{r\pi}{n} \geqq \sin\,(2m+1)\frac{r\pi}{n} \qquad \left(r=1,\cdots,\left[\frac{n}{2}\right]\right). \tag{3}$$

THEOREM 2. *The inequality* (2) *is true if*

$$n \mid m+2 \text{ or } 2m \text{ or } 2m+1 \text{ or } 2m+2. \tag{4}$$

It follows easily from Theorem 2 that Shapiro's inequality (1) is true for $n\leqq 6$.

I first prove Theorem 1. The left hand side of (2) is the weighted sum of $1/(x_{i+1}+\cdots+x_{i+m})$ $(i=1,\cdots,n)$ with weights $x_1,\cdots,x_n$ respectively. Thus since a weighted arithmetic mean is not less than the corresponding weighted harmonic mean we have

$$\sum \frac{x_i}{x_{i+1}+\cdots+x_{i+m}} \geqq \frac{(\sum x_i)^2}{\sum\left\{x_i(x_{i+1}+\cdots+x_{i+m})\right\}}. \tag{5}$$

Hence Theorem 1 follows from

LEMMA 1. *If* (3) *is true,*

$$(\sum x_i)^2 \geqq \frac{n}{m}\sum\left\{x_i(x_{i+1}+\cdots+x_{i+m})\right\}. \tag{6}$$

Proof. The inequality (6) holds if and only if the matrix of the quadratic form

$$(\sum x_i)^2 - \frac{n}{m}\sum \{x_i(x_{i+1} + \cdots + x_{i+m})\}$$

is positive definite or semidefinite. This matrix is clearly a circulant. Its first row can be obtained by writing down a row of n ones and subtracting, in succession, $n/2m$ from the elements (possibly modified by previous subtraction) in positions $2, \cdots, m+1; n-m+1, \cdots, n \pmod{n}$. Now this matrix is positive definite or semidefinite if and only if all its characteristic roots are nonnegative. Since the matrix is a circulant, it is seen that these roots are $\lambda_1, \cdots, \lambda_n$ where

$$\lambda_r = \sum_1^n \omega_r^i - \frac{n}{2m}\sum_1^m (\omega_r^i + \omega_r^{-i}) \qquad (r = 1, \cdots, n),$$

with $\omega_r = \exp(2r\pi i/n)$. Clearly $\lambda_n = 0$. For $r<n$,

$$\lambda_r = -\frac{n}{2m}\left\{\frac{\omega_r^{m+1} - \omega_r}{\omega_r - 1} + \frac{\omega_r^{-m-1} - \omega_r^{-1}}{\omega_r^{-1} - 1}\right\}$$

$$= \frac{n}{2m}\cdot\frac{\sin\dfrac{r\pi}{n} - \sin(2m+1)\dfrac{r\pi}{n}}{\sin\dfrac{r\pi}{n}},$$

after some simplification. It is seen that $\lambda_{n-r} = \lambda_r$. Hence $\lambda_r \geqq 0$, for all relevant r, if and only if (3) is true. The proof of the lemma is now complete.

Theorem 2 now follows from Theorem 1 by virtue of

LEMMA 2. *If* (4) *is true,* (3) *is true.*

Proof. It is seen that (3) is equivalent to

$$\sin\frac{r\pi}{n} \geqq \left\{\begin{array}{ll} -\sin\dfrac{3r\pi}{n} & (\text{if } n \mid m+2) \\ \pm\sin\dfrac{r\pi}{n} & (\text{if } n \mid 2m \text{ or } 2m+2) \\ 0 & (\text{if } n \mid 2m+1) \end{array}\right\} \qquad \left(r = 1, \cdots, \left[\frac{n}{2}\right]\right).$$

The lemma clearly follows if we note that the first of these inequalities is equivalent to

$$\sin\frac{2r\pi}{n}\cos\frac{r\pi}{n} \geqq 0.$$

Remark 1. It is easily shown that (3) is false if $n > 2m+2$. That (3) is not

(for all m) necessary for the truth of (2) follows since, if $m=1$, (2) is true for all n.

Remark 2. In (3) the range for r can be reduced to $1, \cdots, [n/2]-1$, the inequality corresponding to $r=[n/2]$ being always satisfied.

The proof of Theorem 1 can be modified to prove

THEOREM 3. *The inequality*

$$\sum \frac{a_1x_{i+1} + \cdots + a_nx_{i+n}}{b_1x_{i+1} + \cdots + b_nx_{i+n}} \geqq \frac{n\sum a_i}{\sum b_i}, \tag{7}$$

where the numerators and denominators are all positive, is true if

$$\left(\sum a_i\omega_r^i\right)\left(\sum b_i\omega_r^{-i}\right) + \left(\sum a_i\omega_r^{-i}\right)\left(\sum b_i\omega_r^i\right) \leqq 0 \qquad \left(r=1, \cdots, \left[\frac{n}{2}\right]\right)$$

where $\omega_r = \exp(2r\pi i/n)$.

This extends Theorem 1, (7) being an extension of (2) and of Shapiro's inequality (1).

I am indebted to Dr. U. C. Guha and to Prof. A. Oppenheim for valuable discussions on these inequalities.

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GROUPS AS UNIONS OF PROPER SUBGROUPS

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1. Introduction. It is evident that any group G which is not monogenic (generated by a single element) is expressible as a union of proper subgroups; for example, such a G is the union of its monogenic subgroups, which by hypothesis are all proper. Conversely, if a group is a union of proper subgroups of itself, it clearly cannot be monogenic.

It is not so easy to characterize groups which are *finite* unions of their proper subgroups. That there exist nonmonogenic groups which are not such finite unions is seen from the example of the additive group Q^+ of rational numbers. Indeed, suppose Q^+ were the union of its proper subgroups $H_1, \cdots, H_n$. Since there are only a finite number of these H's, it can evidently be assumed that this union is irredundant—that is, that none of the H's is contained in the union of all the others. It follows that any of the H's, for example H_1, contains a rational $r=m/n$ which is in no other H_i. This implies that for all integers h, r/h can be in no H_i but H_1, and hence must be in H_1. This is true in particular for h a multiple of m, say $h=dm$, which makes $r/h=m/ndm=1/nd$. But if this is in H_1, so is any integral multiple of it, and in particular so is any cn-tuple of it. Since $cn \cdot 1/nd=c/d$ is a completely arbitrary rational, this means that H_1 is the whole of Q^+, a contradiction. (This proof shows, incidentally, that Q^+ can

never be an irredundant union of even infinitely many of its proper subgroups.)

Groups which *are* finite unions of their proper subgroups are the chief subject of the present note. Specifically, some conditions are derived which restrict the minimum number of proper subgroups into which a group can be decomposed.

2. Unions of two and three subgroups. We first prove

THEOREM 1. *No group is the union of two of its proper subgroups.*

Proof. If G is the union of its proper subgroups A and B, the union must be irredundant; that is, neither A nor B can contain the other. This being the case, let x be an element of A not contained in B, and y an element of B not contained in A. But if xy is in A then $x^{-1}xy=y$ is in A; similarly, if xy is in B, x must be in B. Thus in any case we have a contradiction.

The argument used to prove Theorem 1 can be applied to the proof of the following more general

LEMMA. *Let G be the irredundant union of the subgroups H_i. Then for each i, H_i contains the intersection of all the remaining H's.*

Proof. Since the union is irredundant, H_i cannot be contained in the union H of the remaining H's. Let x be an element of H_i which is contained in none of the other H's, and let y be an element contained in all the other H's (and so in their intersection). If xy is in H, it must be in some H_j; but since y is in every H_j, it follows that x is also in H_j; a contradiction. On the other hand, if xy is not in H, it must be in H_i; but since x is in H_i, this implies that y is in H_i, which proves the Lemma.

That a group may be a union of three proper subgroups is shown by the example of the Klein 4-group ($\{e, a, b, c\}$ with the relations $a^2=b^2=c^2=e$, $ab=ba=c$, $bc=cb=a$, $ca=ac=b$), which is the union of the three subgroups $\{e, a\}$, $\{e, b\}$, $\{e, c\}$. In a sense, in fact, the 4-group is characteristic of all groups which are unions of three proper subgroups. Specifically, we have

THEOREM 2. *A group G is the union of three proper subgroups if and only if the Klein 4-group is a homomorphic image of G.*

Proof. Since the inverse image of a proper subgroup under a homomorphism is a proper subgroup, the "if" part is clear by the preceding paragraph. To prove the "only if," suppose that G is the union of the proper subgroups A, B and C. It follows from Theorem 1 that this union must be irredundant; hence the sets $A'=A-(B\cup C)$, $B'=B-(C\cup A)$, $C'=C-(A\cup B)$ are all nonempty. On the other hand, by the Lemma, $A\cap B\subset C$, $B\cap C\subset A$, and $C\cap A\subset B$, so that $A\cap B\cap C=A\cap B=B\cap C=C\cap A$; call it H. Thus evidently G is the disjoint union of H, A', B' and C'. By arguments similar to those used above, it can be shown that, in fact, H is normal; A', B' and C' are cosets of H; and G/H is the 4-group.

3. An indecomposability criterion. In setting bounds on the smallness of the number of proper subgroups of which a group G can be the union, a useful tool is the following

LEMMA. *Let the group G be the irredundant union of subgroups A_i $(i=1, \cdots, n>2)$ and set $M=A_2\cup A_3\cup \cdots \cup A_n$. Then if x is not in M, we have x^k in M for some $k=1, \cdots, n-1$.*

Proof. Clearly x is in A_1. Choose y in A_2 and not in A_1. Then x^jy is in M for $j=1, \cdots, n-1$ since x^jy in A_1 would yield y in A_1. If x^jy is in A_2 for some $j=1, \cdots, n-1$, then x^j is in $A_2\subset M$, as desired. If we have $x^jy=x^my$ for some $j, m=1, \cdots, n-1$ with $j>m$, then $x^{j-m}=e$ is in M, as desired. Hence we may assume that the $n-1$ elements $xy, x^2y, \cdots, x^{n-1}y$ are distinct and in $A_3\cup \cdots \cup A_n$. It follows that x^jy, x^my are in A_q for some $q=3, \cdots, n$ and some $j, m=1, \cdots, n-1$ with $j>m$. Then $x^{j-m}=(x^jy)(x^my)^{-1}$ is in $A_q\subset M$, and the lemma is proved.

On the basis of this lemma we can now prove

THEOREM 3. *Suppose that kth roots can be taken in the group G for every positive integer k less than a certain n. Then G is not the irredundant union of n (or fewer!) of its proper subgroups.*

Proof. Since the hypothesis for the given n implies the analogous hypothesis for any m smaller than n, the "or fewer" part is an obvious consequence of the rest of the Theorem. Suppose, then, that G is the irredundant union of exactly n subgroups, and adopt the notation of the Lemma. If the element x is not in M, clearly no root y of x can be in M. This is true, for example, for $y=$ the $(n-1)!$th root of x, which exists in G by hypothesis. On the other hand, by the Lemma, y^k is in M for some k less than n. Since $x=(y^k)^r$, where $r=(n-1)!/k$, x too must be in M; a contradiction.

If G is a finite group of order N, the hypothesis of Theorem 3 is equivalent to the requirement that $(n-1)!$ be prime to N. (See, for example, this MONTHLY, vol. 60, 1953, pp. 185–6.) This gives us the immediate

COROLLARY. *Let G be a finite group of order N, p the smallest prime dividing N. Then G is not the union of p or fewer of its proper subgroups.*

The criterion of Theorem 3 cannot be strengthened. Indeed, let G be the abelian group generated by two elements x, y with the relations $x^p=y^p=e$; then G is the union of the $p+1$ proper subgroups generated by the elements $x, y, xy, x^2y, x^3y, \cdots, x^{p-1}y$, respectively.

For finite groups, the case in which (as in the example just given) the minimum imposed by Theorem 3 is actually assumed may be partially characterized by

THEOREM 4. *With notation as in the Corollary to Theorem 3, suppose that G is the union of exactly $p+1$ proper subgroups S_i; then at least one of the S's, say*

S_j, has index p. If, moreover, this S_j is normal, then all the S_i have index p and p^2 divides N.

Proof. If none of the S's has index p, they must all have indexes greater than p; hence for each i, $o(S_i) \leqq N/p+1$. Then we have $N < \sum o(S_i) < (p+1)$. $N/(p+1) = N$; a contradiction. Hence some S_j must have index p; assume now that this S_j is normal. Then for $i \neq j$, S_iS_j is a subgroup of G, which cannot be proper because G is the union of S_iS_j and the $p-1$ proper subgroups S_k for $k=1, \cdots, p+1$, $k \neq i$, $k \neq j$. Thus $S_iS_j = G$ for $i \neq j$, and, since S_j is normal, we have $o(G)o(S_i \cap S_j) = o(S_i)o(S_j)$, or $p(o(S_i \cap S_j)) = o(S_i)$ for $i \neq j$. Set $o(S_i) = N/q_i$ for $i \neq j$, so that $q_i \geqq p$. Suppose that $q_i > p$ for some $i \neq j$. Then

$$
\begin{aligned}
N = o(G) &\leqq o(S_j) + \sum_{i \neq j} [o(S_i) - o(S_i \cap S_j)] \\
&\leqq (N/p) + \sum_{i \neq j} [(N/q_i) - (N/pq_i)] \\
&< (N/p) + p((N/p) - (N/p^2)) = N;
\end{aligned}
$$

a contradiction. Hence $q_i = p$ for every $i \neq j$, and $o(S_i \cap S_j) = N/p^2$ shows that p^2 divides N.

ON THE NUMBER OF DISTINCT ZEROS OF POLYNOMIALS

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Although a polynomial of degree n must have n zeros, we are well aware of the fact that some or even all of them may coincide. Consequently, nothing much can be said about the number of distinct zeros of an arbitrary polynomial. If, however, we consider related pairs of polynomials, certain sharp estimates of this number can be obtained. Furthermore, from these very simply proven estimates, corollaries on a class of polynomial-Diophantine equations can be extracted which seem quite difficult to prove otherwise.

Theorem 1. *If $P(x)$ is of degree n, $Q(x)$ is of degree m, and $n > m \geqq 0$, then there exist $n-m+1$ distinct x's for which either $P(x)=0$ or $P(x)=Q(x)$.*

Remark. That the estimate $n-m+1$ is sharp is shown by the example $P(x)=x^n$, $Q(x)=x^m$.

Proof. Let $A(x)$, of degree r, be the g.c.d. of $P(x)$ and $Q(x)$ and form

$$
p(x) = \frac{P(x)}{A(x)}, \qquad q(x) = \frac{Q(x)}{A(x)} \cdot
$$

Suppose that $p(x) = a_0(x-r_1)^{m_1} \cdots (x-r_k)^{m_k}$ and note that $p(x)-q(x)$ has no zeros in common with $p(x)$. It is a known theorem that the number of *distinct* zeros of a polynomial $B(x) = \deg B$—number of common zeros of B and B'. We apply this to $p(x)-q(x)$. We must, therefore, estimate the number of common zeros of $p(x)=q(x)$ and $p'(x)=q'(x)$. Since

$$\frac{p'(x)}{p(x)} = \frac{m_1}{x - r_1} + \cdots + \frac{m_k}{x - r_k},$$

it follows that the common roots must satisfy the equation

$$q(x)\left\{\frac{m_1}{x - r_1} + \cdots + \frac{m_k}{x - r_k}\right\} = q'(x).$$

This last equation when multiplied out has degree $\leqq \deg q' + k = m - r - 1 + k$. Hence, the number of common zeros is bounded by this quantity $m - r - 1 + k$. Consequently,

$$\text{number of distinct zeros of } p - q \geqq \deg (p - q) - m + r + 1 - k,$$
$$\text{number of distinct zeros of } p - q \geqq n - r - m + r + 1 - k,$$
$$\text{number of distinct zeros of } p - q \geqq n - m + 1 - k.$$

Since p and $p - q$ have no common zeros and p has k distinct zeros, it now follows that the number of distinct x's for which $p = 0$ or $p = q$ is $\geqq n - m + 1$. Still $\geqq$ to this quantity is the number of distinct x's for which $P = 0$ or $P = Q$ and the theorem is proven.

The main corollary of this theorem is

THEOREM 2. *If $R(x)$ is a given polynomial of degree $c \geqq 0$, then $P^a(x) - Q^b(x) = R(x)$ is impossible for polynomials (nontrivial) P and Q with complex coefficients if either a or b exceeds $2c$, $a > 1$, $b > 1$.*

Proof. Suppose that the equality holds, then

$$\text{if } \deg P = n \neq 0 \text{ and } Q = m \neq 0,\ \deg P^a = na,\ \deg Q^b = mb.$$

We may assume without loss of generality that $na = mb$.

By Theorem 1, $P^a = 0$ or $P^a = R$ for $na - c + 1$ distinct values of x or equivalently $P^a = 0$ or $Q^b = 0$ for $na - c + 1$ distinct values of x. The number of such distinct values, however, is $\leqq n + m$. Consequently,

$$n + m \geqq na - c + 1,$$
$$n(a + b) \geqq nb + mb \geqq nab - cb + b,$$
$$(c - 1)b \geqq n[(a - 1)(b - 1) - 1] \geqq (a - 1)(b - 1) - 1,$$
$$a \leqq c + \frac{c}{b - 1} \leqq 2c.$$

Similarly, $b \leqq 2c$.

Another statement of this latter result is that $\deg \{P^a - Q^b\} \geqq \frac{1}{2} \max (a, b)$ unless P or Q is a constant or $P^a = Q^b$.

A special case of Theorem 2 is that $P^a(x) - Q^b(x) = x$, $a, b > 1$, has polynomial solutions only if $a = b = 2$ and in this case they are given by

$$P = \frac{k^2x + 1}{2k}, \qquad Q = \pm \frac{k^2x - 1}{2k}.$$

In a similar way, we can estimate quantities such as

$$\deg \{xP^a - (2x^2 + 3)Q^b\}, \qquad \deg \{P^aQ^b - R^cS^d\},$$

etc., and thereby obtain other results for polynomial-Diophantine equations.

GEOMETRY OF TRIHEDRONS*

VICTOR THÉBAULT, Tennie, Sarthe, France

1. A necessary and sufficient condition for three lines $D\alpha$, $D\beta$, $D\gamma$, lying respectively in the faces DBC, DCA, DAB of a trihedron $(D) \equiv D-ABC$, to be coplanar is

$$\frac{\sin u}{\sin v} \cdot \frac{\sin u'}{\sin v'} \cdot \frac{\sin u''}{\sin v''} = +1, \tag{1}$$

where u, v, u', v', u'', v'' denote the angles $(D\alpha, DB)$, $(D\alpha, DC)$, $(D\beta, DC)$, $(D\beta, DA)$, $(D\gamma, DA)$, $(D\gamma, DB)$, respectively. This is a well-known theorem [1].

COROLLARY. *If the lines $D\alpha$, $D\beta$, $D\gamma$ are coplanar, then so are their isogonals $D\alpha'$, $D\beta'$, $D\gamma'$ with respect to the faces DBC, DCA, DAB.*

For, (1) implies

$$\frac{\sin v}{\sin u} \cdot \frac{\sin v'}{\sin u'} \cdot \frac{\sin v''}{\sin u''} = +1. \tag{2}$$

COROLLARY. *Let U and V, U' and V', U'' and V'' denote the dihedral angles formed by the planes $DA\alpha$, $DB\beta$, $DC\gamma$ in the dihedrals of (D) with the edges DA DB, DC, respectively. In order that the lines $D\alpha$, $D\beta$, $D\gamma$ be coplanar it is necessary and sufficient that*

$$\frac{\sin U}{\sin V} \cdot \frac{\sin U'}{\sin V'} \cdot \frac{\sin U''}{\sin V''} = +1. \tag{3}$$

In fact, the equivalence of (3) and (2) follows from that of the measures

$$DB \cdot D\alpha \sin u \cdot AA'/6 = 2DAC \cdot DA\alpha \sin U/3DA,$$
$$DC \cdot D\alpha \sin v \cdot AA'/6 = 2DAB \cdot DA\alpha \sin V/3DA,$$

of the volumes of the tetrahedrons $DAB\alpha$ and $DA\alpha C$, AA' being an altitude of the tetrahedron $DABC$, and so on.

COROLLARY. *If the lines $D\alpha$, $D\beta$, $D\gamma$ are coplanar, then so are the lines of intersection $D\alpha'$, $D\beta'$, $D\gamma'$ of the isogonal planes of $DA\alpha$, $DB\beta$, $DC\gamma$ with respect to*

* Translated by Bomshik Chang, University of British Columbia.

EDITED BY C. O. OAKLEY, Haverford College

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CONCERNING SOME INEQUALITIES

ALBERT A. MULLIN, University of Illinois

This note is concerned with the use of concave function theory in deriving a particular set of inequalities.

DEFINITION 1. *A set of points S is said to be convex* if for $x\in S$, $y\in S$, and $0\leqq\lambda\leqq 1$, all points of the form $(1-\lambda)x+\lambda y$ are elements of S.*

DEFINITION 2. *A function f is called concave if its domain is a convex set, its range is a subset of E^1, and for $0\leqq\lambda\leqq 1$, $f[(1-\lambda)x+\lambda y]\geqq(1-\lambda)f(x)+\lambda f(y)$.*

LEMMA 1.1. *The logarithmic function is a concave function.*

Proof. This follows by observing that the second derivative of $\ln x$, which is $-1/x^2$, is negative for all $x>0$. Hence, each and every chord joining two distinct points of the logarithmic curve does not lie above the logarithmic curve.

LEMMA 1.2. *If $a>0$, $b>0$, $\alpha\geqq 0$, $\beta\geqq 0$, and $\alpha+\beta=1$, then $\alpha a+\beta b$ is a convex set.*

Proof. This follows directly from the definition of a convex set.

THEOREM 1. *If $a>0$, $b>0$, $\alpha\geqq 0$, $\beta\geqq 0$, and $\alpha+\beta=1$, then $\alpha\ln a+\beta\ln b\leqq\ln(\alpha a+\beta b)$.*

Proof. The theorem follows from Definition 2 and Lemmas 1.1, 1.2.

COROLLARY 1.1.

$$\frac{a^{2\alpha}b^{2\beta}}{\alpha a+\beta b}\leqq a^\alpha b^\beta\leqq\alpha a+\beta b.$$

Proof. The right-hand inequality follows directly from Theorem 1. The left-hand inequality follows by noticing that

$$a^\alpha b^\beta\equiv\frac{a^{2\alpha}b^{2\beta}}{a^\alpha b^\beta}\leqq\alpha a+\beta b,$$

or, since $\alpha a+\beta b>0$, then by cross-multiplying by $a^\alpha b^\beta$ and then dividing through by $\alpha a+\beta b$, we obtain the required inequality.

* H. G. Eggleston, Convexity, Cambridge, 1958.

COROLLARY 1.2.

$$\frac{2ab}{a+b} \leqq (ab)^{1/2} \leqq \frac{a+b}{2},$$

that is, the harmonic mean of two positive numbers does not exceed their geometric mean, which in turn does not exceed their arithmetic mean.

Proof. Put $\alpha=\beta=1/2$ in Corollary 1.1.

LEMMA 2.1. *If* $x>0$ *then* $1-(1/x)\leqq \ln x \leqq x-1$.

Proof. Consider the right-hand inequality. First, let

$$\lambda(x) \equiv \ln x - x + 1, \qquad x > 0.$$

Then $\lambda'(x)=(1/x)-1$, $\lambda''(x)=-1/x^2$. From this it follows that $x=1$ yields the only zero of λ' and for $x=1$, $\lambda''=-1$. Hence $x=1$ yields a maximum (and necessarily the only extrema) for λ and this proves the right-hand inequality.

Now consider the left-hand inequality. From the right-hand inequality with x replaced by $1/x$ it follows that $\ln(1/x)\leqq(1/x)-1$ and hence that $\ln x \geqq 1-(1/x)$. This yields the left-hand inequality.

LEMMA 2.2. *If* $a>0$, $b>0$, $\alpha\geqq 0$, $\beta\geqq 0$, *and* $\alpha+\beta=1$, *then* $\alpha \ln a+\beta \ln b \leqq \ln(\alpha a+\beta b)\leqq \alpha a+\beta b-1$.

Proof. The left-hand inequality is Theorem 1. The right-hand inequality results from an application of Lemma 2.1.

THEOREM 2.* *If* $a>0$, $b>0$, $\alpha\geqq 0$, $\beta\geqq 0$, *and* $\alpha+\beta=1$, *then*

$$\frac{\alpha}{a}[a+(a-1)^2]+\frac{\beta}{b}[b+(b-1)^2] \geqq \alpha a+\beta b-1+\frac{1}{\alpha a+\beta b} \geqq 1.$$

Proof. Consider the left-hand inequality first. The left-hand set of terms of this inequality may be put in the form $\alpha f(a)+\beta f(b)$, where

$$f(x) \equiv x-1+(1/x),\ x>0; \qquad f'(x)=1-(1/x^2); \qquad f''(x)=2/x^3.$$

Since $f''(x)>0$ for all $x>0$, each and every chord joining two distinct points of the curve $y=f(x)$ does not lie below the curve. Such a function is called a convex function. Hence $\alpha f(a)+\beta f(b)\geqq f(\alpha a+\beta b)$ and this proves the left-hand inequality.

The right-hand inequality is a result of $f(x)\equiv x-1+(1/x)$ having a minimum value of 1 (at $x=1$).

COROLLARY 2.1. *Let* $A\equiv(a+b)/2$ *and* $H\equiv(2ab)/(a+b)$ *denote the arithmetic and harmonic mean, respectively, of* a *and* b. *Then* $A+(1/H)\geqq 2$.

* The middle set of terms of the two inequalities appearing in Theorem 2 have been included at the suggestion of Professor Harry Levy, University of Illinois.

Proof. From Theorem 2 with $\alpha=\beta=1/2$ we obtain

$$\frac{1}{2}\left[a-1+\frac{1}{a}\right]+\frac{1}{2}\left[b-1+\frac{1}{b}\right]\geqq 1,$$

$$\frac{1}{2}[a+b]+\frac{1}{2}\left[\frac{1}{a}+\frac{1}{b}\right]\geqq 2,$$

from which the required inequality follows.

COROLLARY 2.2. *If* $\lambda>0$ *then* $\lambda+(1/\lambda)\geqq 2$.

Proof. Let $\lambda=a=b>0$ in Corollary 2.1.

THE CATENARY AND THE TRACTRIX

ROBERT C. YATES, College of William and Mary

Although the facts herein have been in the public domain for a long time, our purpose is to re-establish them in an elementary way—then set them on parade in a relatively complete and ordered line of march. Note that we carefully avoid rectangular equations (*per se*) which, at least for the tractrix, have an awesome appearance. Moreover, once a start is made, facts seem to tumble over each other in their demand for recognition.

1. The catenary. A flexible and inextensible chain weighing m pounds per foot hangs from two supports A and B. Equilibrium throughout is established if the three forces: T, the magnitude of the variable tangential force at a repre-

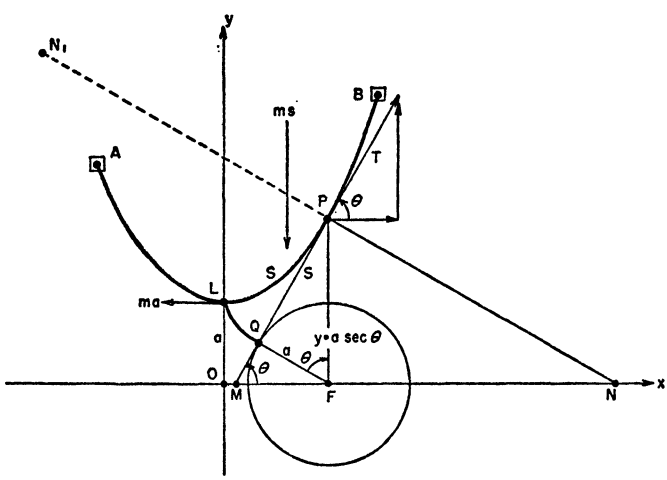

FIG. 1

sentative point P; ma, the constant* horizontal force at the lowest point L; and ms, the vertical weight of the chain length $s=\widehat{LP}$; have a resultant of zero magnitude. That is, if θ is the inclination of T:

$$T \sin \theta = ms, \qquad T \cos \theta = ma, \tag{1}$$

so that

$$s = a \tan \theta, \tag{2}$$

an *intrinsic* equation of the curve, called the catenary. Since the chain hangs steady, T is tangent to its locus at P and $\tan \theta$ is the slope of the curve.

In Figure 1, horizontal and vertical x, y axes are drawn with origin O distant a units below L. Let F be the foot of the ordinate of P and draw FQ perpendicular to the tangent line PT. Then, since $\sphericalangle QFP=\theta$,

$$PQ = ks, \qquad FQ = ka, \qquad y = ka \cdot \sec \theta.$$

But for $\theta=0$, $y=a$, and thus $k=1$. Accordingly, the tangent to the catenary at any point is also tangent to the circle with center F and radius a.

From Equation (1), the tension is

$$T = ma \cdot \sec \theta = my, \tag{3}$$

a quantity equal to the weight of a length of the chain hanging vertically from P to F.

Since $y=a \cdot \sec \theta$ and generally $dx=(\cos \theta)ds$, then $ydx=(a \sec \theta)(\cos \theta)ds$ $=a \cdot ds$ and thus

$$\text{Area } (OLPF) = \int_0^x ydx = \int_0^s ads = as.$$

That is,

$$\text{Area } (OLPF) = 2 \cdot \text{Area } (\Delta FQP). \tag{4}$$

Furthermore, since $ydx=ads$ or $\pi y^2dx=\pi ayds$, volume V_x and surface area $\sum_x$ of revolution about the x-axis have the special relation

$$2V_x = a \sum\nolimits_x. \tag{5}$$

Let PN be the normal length from P to the x-axis. Then, if R is the radius of curvature at P, we have by definition from Equation (2):

$$R = \left| \frac{ds}{d\theta} \right| = a \sec^2 \theta = \frac{y^2}{a} = PN. \tag{6}$$

The center of curvature, however, is at N_1 opposite N from P.

* This is evident if we imagine resupporting the chain with pegs at various points P. The shape of the chain does not change and thus the tension at L is constant in direction and magnitude.

2. The tractrix. In Figure 1, $PQ=s=\widehat{PL}$ and thus the locus of Q is an *involute* of the catenary, a curve called the tractrix. Its tangent length QF is con-

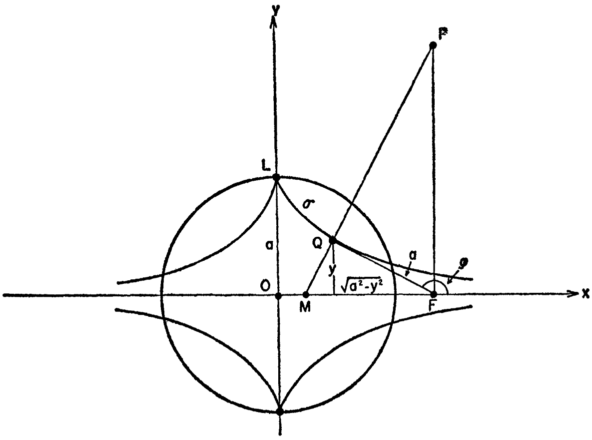

FIG. 2

stant and equal to a, and the curve can be pictured as the path of a toy wagon Q pulled along by a child F. It is quite obviously the orthogonal trajectory of circles of fixed radius a having centers on a line.

If ϕ is the inclination of the tangent, then the expression

$$\tan\phi = y' = \frac{y}{\pm\sqrt{a^2-y^2}}, \qquad y = a,\ x = 0 \tag{7}$$

defines four branches as shown in Figure 2.

Interesting and useful properties of the tractrix are now established directly from this differential equation (7) which we write as $ydx=\pm\sqrt{a^2-y^2}dy$. Note first, however, that if $\sigma=\widehat{LQ}$, $dy=(\sin\phi)d\sigma$ and, particularly here, $y=a\sin\phi$. The following measures of area A, radius of curvature ρ, volume V_x and surface area $\sum_x$ of revolution about the x-axis are immediate.

$$A = 2\int_{-\infty}^{\infty} ydx = 2\int_{-a}^{a}\sqrt{a^2-y^2}\,dy = \pi a^2;*$$

$$V_x = \pi\int_{-\infty}^{\infty} y^2dx = 2\pi\int_0^a\sqrt{a^2-y^2}\,(ydy) = \frac{2}{3}\pi a^3;$$

$$\sum\nolimits_x = 2\pi\int_{-\infty}^{\infty} yd\sigma = 2\pi\int_0^a(a\sin\phi)(\csc\phi\ dy) = 2\pi\int_0^a ady = 2\pi a^2;$$

$$\rho = \frac{d\sigma}{d\phi} = \left|\,a\cot\phi\,\right| = \left|\,-a\tan\theta\,\right| = PQ.$$

* The form $\int_{-a}^{a}\sqrt{a^2-y^2}dy$ measures the half-area of the circle $x^2+y^2=a^2$ shown in Figure 2.

The last item is recognized as the involute-evolute relation of the catenary and tractrix. From it,

$$\sigma = \left| a \ln \sin \phi \right| , \tag{8}$$

an intrinsic equation of the tractrix.

The tractrix and its surface of revolution, called the *pseudosphere*, thus have much in common with the circle and sphere. This striking analogy appears even stronger if curvatures of the two surfaces are compared.

This curvature is determined as follows. A plane containing the normal to a surface at Q intersects the surface in a curve of curvature K. As the plane turns about the fixed normal, K may attain maximum and minimum values K_1 and K_2. Their *product* K_1K_2 is defined as the curvature of the surface at Q. These values K_1, K_2 occur in sections at right angles to each other.*

For a sphere all plane sections through a normal are great circles of radius a and curvature $1/a$. A sphere then has curvature $1/a^2$.

For the pseudosphere, the section of minimum curvature at Q is made by the plane through the axis of revolution; the maximum radius of curvature is QP. The minimum radius of curvature at Q is QM, formed by the plane perpendicular to the first. Their product $(QM)(QP) = -(FQ)^2 = -a^2$ is constant and negative since the radii are oppositely directed. Thus $K_1K_2 = -1/a^2$.

The plane, the sphere, and the pseudosphere are surfaces upon which we may display the parabolic geometry of Euclid, the elliptic geometry of Riemann, and the hyperbolic geometry of Lobatschewsky and Bolyai. These are characterized by the angle-sum $A+B+C$ of triangles formed by geodesics (lines of shortest distance):

$$A + B + C \left\{ \begin{matrix} < \\ = \\ > \end{matrix} \right\} 180^\circ \left\{ \begin{matrix} \text{hyperbolic} \\ \text{parabolic} \\ \text{elliptic.} \end{matrix} \right.$$

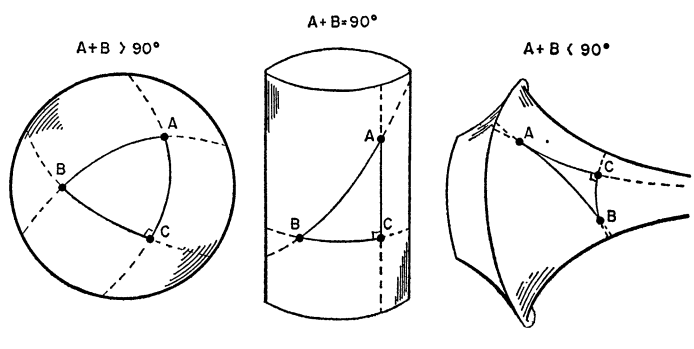

FIG. 3

* Along the *principal* directions of the surface.

A right circular cylinder may replace the plane. Figure 3 gives these surfaces showing triangles with $C=90°$. Their sides are each geodesics from point to point.

On the cylinder A and C are taken on an element, B and C on a circle. A and B lie on a *helix*.

On the sphere the sides are great circles.

On the pseudosphere A and C are on a tractrix (a meridian of the surface), B and C on a circle.

It appears that upon the sphere $A+B>90°$ and upon the pseudosphere $A+B<90°$. The fact that $A+B=90°$ on the cylinder is evident if we imagine the cylinder as a roller in a printing press. The image of ABC printed on plane paper is a triangle with straight sides. Moreover, on the cylinder

$$\widehat{AB}^2 = \widehat{BC}^2 + \overline{CA}^2.$$

As a final item, consider a pivot seated in a step. As the pivot turns, most wear occurs on the surface farthest from the axis of rotation. In time the seat

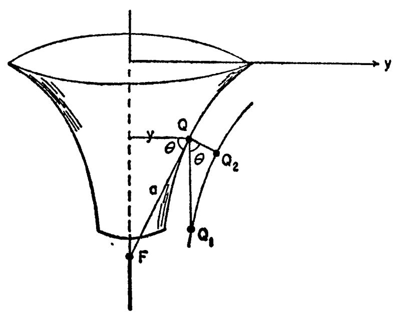

FIG. 4

and pivot become incompatible and wobble occurs. We seek the shape of a pivot such that the wear QQ_1, Figure 4, parallel to the axis of rotation is the same for all points Q of the pivot. Thus, as action and wear go on, the pivot will reseat itself.

The amount of wear QQ_2 normal to the section curve at $Q(x, y)$ is proportional to the work done by friction as the pivot turns. Let f be the coefficient of friction, p the pressure of the bearing, and n the number of revolutions per unit time. Then, if k is the factor of proportionality (a hardness constant),

$$QQ_2 = k(f)(p)(n)(2\pi y).$$

If QQ_1 is to be constant for all points Q, then

$$QQ_1 = (QQ_2) \sec \theta = (kfpn)(y \sec \theta) = \text{constant},$$

where θ is the angle between the normal and the axis. In short, $y \sec \theta = a$, a constant. Thus, if QF be drawn tangent to the curve, $QF=a$, a definitive

property of the tractrix. This is the form of the *Schiele* pivot mentioned in some books on Mechanics.

We end the account with a list of rectangular equations of the tractrix:

$$x = a \sinh^{-1} \sqrt{a^2y^{-2} - 1} - \sqrt{a^2 - y^2}$$

$$x = a \operatorname{sech}^{-1} \frac{y}{a} - \sqrt{a^2 - y^2}$$

$$x = \frac{a}{2} \ln \frac{a + \sqrt{a^2 - y^2}}{a - \sqrt{a^2 - y^2}} - \sqrt{a^2 - y^2}$$

$$x = t - a \tanh \frac{t}{a}, \qquad y = a \operatorname{sech} \frac{t}{a} \cdot$$

They seem relatively useless.

MATHEMATICAL EDUCATIONAL NOTES

EDITED BY JOHN A. BROWN, University of Delaware, AND JOHN R. MAYOR, AAAS and University of Maryland

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GEOMETRY IN THE FIRST GRADE

NEWTON S. HAWLEY AND PATRICK SUPPES, Stanford University

In the spring of 1958 we spent two and a half months in an experiment which involved teaching geometrical notions and constructions to the entire class of first grade students at Stanford Elementary School (a public school in the Palo Alto Unified School District). After a few informal talks we began a systematic development of a modified version of Book I of Euclid's *Elements*.

Our approach was to stimulate reasoning among the pupils, although no formal proofs were attempted. The propositions from Euclid which were stressed were the constructions. In each case the construction (*e.g.*, bisecting a line segment) was presented as an open problem, and the students were encouraged to attempt solutions. As much as possible we forced the children to give the reasons for rejecting an incorrect solution. Generally speaking, our pedagogical procedure closely resembled that of Socrates' interrogation of the slave in Plato's dialogue *Meno*. It is worth noting that at no point did we rely on any knowledge of arithmetic. As a consequence our program is completely independent of the standard curriculum in elementary school mathematics.

Our main modification of Euclid was to use the compasses as rigid instruments to make direct comparisons of distances. We thereby trivialized Propositions 2 and 3 of Book I.

An encouraging aspect of the experiment was the unexpected ability of the students to assimilate the technical vocabulary. Printed worksheets were distributed every other day, and the students were asked to read aloud the written formulation of the problems. The technical terms introduced were then added to the regular reading vocabulary lists ordinarily presented in the first grade. Precision of expression was improved by daily drill and continual repetition of fundamentals. Each school day one of us met with the students for a period of from fifteen to thirty minutes.

It is our conclusion that the actual physical use by each pupil of straight edge and compass played an integral part in the progress of his understanding of geometrical constructions. This physical activity, which had direct mathematical meaning, seems to us somewhat superior to the conventional physical manipulation of objects used to illustrate arithmetic.

Certain problems arose due to disparities in the abilities of the students, but these disparities were no more striking than those encountered by the regular teacher in the standard reading program.

On the basis of the encouraging results of this experiment, we are continuing work along these lines, and are also exploring possibilities in other primary grades.

THE INDIANA SCHOOL AND COLLEGE COMMITTEE ON MATHEMATICS

G. N. Wollan, Purdue University

The current wave of interest in revision of the mathematics program in our schools has led in Indiana to the organization of a joint committee of school, college and university mathematics teachers and mathematicians whose purposes, very broadly stated, are (1) to stimulate and provide opportunities on a state-wide basis for critical analysis and discussion by mathematics teachers and by other interested persons of the various proposals for changing the mathematics program and (2) to develop a program of activities of various kinds on a state-wide basis designed to improve mathematics teaching. The initial impetus for the organization of this Committee was provided by college and university mathematicians acting individually and as members of the Indiana Section of the Mathematical Association of America. In particular, Professor Carl Kossack, Head of the Department of Mathematics and Statistics at Purdue University, took the initiative by inviting representatives of the mathematics departments of other Indiana colleges and universities to a meeting at Purdue in July, 1958. This led to the formation in September of the Indiana School and College Committee on Mathematics.

Having the support from the beginning of the college and university mathematicians of the State, the Committee has sought to enlist the support and active participation of elementary and high school mathematics teachers and administrators and the State Department of Public Instruction. Currently the Committee includes about twenty representatives of the mathematics depart-

ments of fifteen Indiana colleges and universities, about the same number of junior and senior high school mathematics teachers, and officially designated representatives of the following organizations: Indiana Council of Teachers of Mathematics, Indiana Secondary School Principals Association, Indiana County Superintendents Association, and the Indiana Association of Junior and Senior High School Principals. A representative of the Indiana State Department of Public Instruction has attended some of the Committee's meetings and expressed support of the Committee and its program. The Committee has the status of a standing committee of the Indiana Academy of Science and a committee of the Indiana Section of the Mathematical Association of America. However, it has no official governmental status and no authority with respect to the school curriculum. Hopes for its being effective rest upon the assumption that its activities are effective and that if it is sufficiently broadly representative, its conclusions will get official consideration.

Subcommittees have been organized to study the mathematics programs currently being followed in Indiana. There is a subcommittee for each of the following subdivisions: (1) Grades 1–6, (2) grades 7 and 8, (3) grades 9–11, (4) grade 12, (5) grades 13 and 14 (*i.e.* freshman and sophomore college years), (6) teacher training. Some of this work is underway. It is expected that a report of this study will be made public.

The committee has undertaken to promote the organization of regional meetings throughout the State for mathematics teachers. These meetings are intended to provide an opportunity for teachers to study and discuss mathematics and problems in the teaching of mathematics. Thus far, such meetings have been organized at eight colleges. Most of the meetings have been built around a talk on some mathematical topic followed by discussion. In one case, the local school system supports the venture as an in-service training program. At some of the colleges, these meetings are being held monthly. The speakers have so far been chosen almost exclusively from the college and university group. It seems to be agreed that it would be better to try to organize the meetings along the lines of a seminar with active participation of all members.

Among the N. S. F. summer institutes to be held this summer in Indiana, two are explicitly for Indiana teachers and will attempt to develop these teachers as active participants in the program of the Committee. At least partly as a result of the work of the Committee, National Science Foundation support has been obtained for Saturday and evening graduate credit courses in mathematics to be offered next year in conveniently located centers throughout the State.

SCHOOL MATHEMATICS STUDY GROUP

In order to keep the mathematical community informed of its plans and progress, the School Mathematics Study Group will issue from time to time a newsletter. Those who wish to receive this newsletter should so indicate on a postcard addressed to: School Mathematics Study Group, Drawer 2502A, Yale Station, New Haven, Conn.

metic, the real and complex number systems.

3. *Supporting Science Courses:* Each undergraduate requirement should include a course in physics. The other hours should be chosen from the following: chemistry, biology, astronomy, geology, and meteorology.

RATINGS OF COLLEGE MATHEMATICS COURSES BY APPLIED MATHEMATICIANS

R. W. HART AND WALTER H. WOOD, Kansas State College, Pittsburg, Kansas

The present emphasis on mathematics and the demand for mathematicians by industry leads college teachers to re-evaluate their course offerings in mathematics. Most college teachers know from experience what training is needed to prepare a student to teach elementary mathematics, but only a relatively few teachers have had enough practical experience in industry to know what courses are most valuable in this field.

In an effort to determine what college mathematics courses are most desirable to a graduate going into industry, a questionnaire was sent to twenty large concerns that hire mathematicians, asking them to rate various college mathematics courses according to their desirability for prospective employees. Twenty-nine head mathematicians of the following organizations answered the questionnaire: Air Research and Development Command, USAF; Armour Research Foundation;Boeing Aircraft Corporation—Wichita Division; Continental Oil Company; Department of the Navy; Firestone Tire and Rubber Company; General Electric Company; General Motors Corporation; International Business Machines Corporation; Lockheed Aircraft Corporation; Martin Company; Midwest Research Institute; Phillips Petroleum Company; United Aircraft Corporation, Research Department; Western Electric Corporation; Westinghouse Electric Corporation.

These will be recognized as some of the largest organizations in the United States engaged in such industries as: oil, aircraft, electricity, business machines, automobiles, guided missiles and research. They are a sample of those that are now doing the research that is so vital to our national progress. The usual college courses above the sophomore level were listed, and these head mathematicians rated them as "most desirable," "desirable" or "not needed." They are here listed in the decreasing order of desirability. The numbers are percentages of the replies from the twenty-nine mathematicians. Some did not rank all of the courses, hence there is not one hundred percent of replies on all of the subjects.

	Most Desirable	Desirable	Not Needed
Differential Equations	79	21	—
Applied Mechanics	72	14	10
Advanced Calculus	69	24	7
Matrix Theory	69	28	—
Mathematical Statistics	66	28	6
Theory of Probability	62	35	3
Functions of Complex Variables	62	31	7

Numerical Analysis	62	31	7
Higher Algebra	59	38	3
Vector Analysis	55	42	3
Theoretical Physics	49	39	12
Fourier Series	46	46	8
Theory of Equations	38	48	—
Functions of Real Variables	28	55	14
Boolean Algebra	21	52	14
Advanced Analytical Geometry	17	41	38
Statistical Quality Control	14	38	45
Modern Synthetic Geometry	14	17	55
Calculus of Variations	10	76	10
Introduction to Mathematical Thought	10	38	34
Projective Geometry	10	34	45
Integral Equations	7	76	14
Number Theory	7	38	41
Differential Geometry	3	55	34
Theory of Groups	3	52	34
Non-Euclidean Geometry	3	38	48
Topology	3	31	52
Higher Plane Curves	—	41	48

The following table lists those courses that were rated as "most desirable" by more than forty per cent of the mathematicians.

	%
Differential Equations	79
Applied Mechanics	72
Advanced Calculus	69
Matrix Theory	69
Mathematical Statistics	66
Theory of Probability	62
Functions of Complex Variables	62
Numerical Analysis	62
Higher Algebra	59
Vector Analysis	55
Theoretical Physics	49
Fourier Series	46

The courses that were marked as "not needed" by more than thirty percent of the mathematicians are given in the following table:

Modern Synthetic Geometry	55
Topology	52
Non-Euclidean Geometry	48
Higher Plane Curves	48
Projective Geometry	45
Statistical Quality Control	45
Number Theory	41
Advanced Analytical Geometry	38
Introduction to Mathematical Thought	34
Differential Geometry	34
Theory of Groups	34

The choice of the term "not needed" was an unfortunate one; because, as some of those who checked the questionnaire pointed out, it is difficult to say that any course in mathematics is not needed. The expression "least desirable" probably would have been better. However, the answers do give a good indication of what subjects are used in applied work.

Some valuable suggestions were made by those who participated in this project. One of the most frequent comments was that mathematics as taught in college tends to be too departmentalized. An applied mathematician needs to correlate his work with other fields. Others stated that if an employee has good training in basic mathematics and logical thinking, their companies could give him the special work needed in their particular occupations.

These ratings should be helpful to a mathematics major who intends to go into industry. They could serve as a guide to him and his advisors in the selection of subjects for study. Further study could be made on the content of courses. It would also be worthwhile to determine what courses in other departments are most desirable for applied mathematicians.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This Department welcomes problems believed to be new and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1371. *Proposed by G. C. Thompson, Security Mutual Life Ins. Co., Binghamton, N. Y.*

On p. 457 of Vol. I of Dickson's *History of the Theory of Numbers* appears the statement: "E. Barbier asked what is the 10^{1000}th digit written if the series of natural numbers be written down." What is this digit?

E 1372. *Proposed by J. R. Munkres, Princeton University*

Let $f(x)$ be continuous on the closed interval $[0, 1]$ and have a derivative on the open interval $(0, 1)$. Suppose $|xf'(x)-f(x)+f(0)|<x^2M$ for all x on $(0, 1)$, where M is fixed. Does $f'(0)$ exist?

E 1373. *Proposed by W. A. Veech, Dartmouth College*

Find the radius of convergence and the function $f(z)$ represented by

$$f(z) = \sum_{n=0}^{\infty} s_n z^n \quad \text{if} \quad s_n = \sum_{k=0}^{n} k \binom{n}{k}.$$

E 1374. *Proposed by Yonder Page, Kitchawan, N. Y.*

What is the minimum d such that the unit square may be covered with five sets each having diameter d?

E 1375. *Proposed by L. D. Goldstone, N. Y. State Public Works Lab., Albany, N. Y.*

Construct a triangle given A, m_a, t_a, that is, given a vertex angle and the median and angle bisector issued from this vertex.

SOLUTIONS

Real and Imaginary Parts of z^z

E 1341 [1958, 774]. *Proposed by A. A. Mullin, University of Illinois*

If $w=z^z$, $z=x+iy$, express Re (w) and Im (w) in terms of x and y.

Solution by David Zeitlin, Remington Rand Univac. Assuming $z\neq 0$, let $r=(x^2+y^2)^{1/2}$ and $\theta=\text{Arctan}\,(y/x)+2m\pi$, $m=0,\ \pm 1,\ \pm 2, \cdots$. Then

$$w = z^z = e^{z \log z} = e^{(x+iy)(\log r+i\theta)} = e^{x \log r-\theta y+i(y \log r+\theta x)}.$$

Thus

$$\text{Re}\,(w) = e^{x \log r-\theta y} \cos\,(y \log r + \theta x),$$
$$\text{Im}\,(w) = e^{x \log r-\theta y} \sin\,(y \log r + \theta x).$$

Also solved by R. G. Albert, Norman Anning, M. T. Austin, Robert Bart, A. P. Boblétt, D. A. Breault, J. L. Brown, Jr., R. F. Brown and Joel Levy (jointly), P. L. Chessin, R. J. Cormier, E. S. Eby, Edwin Ellis, D. A. Freedman, Michael Goldberg, L. D. Goldstone, A. B. Harper, Jr., J. R. Holdsworth, A. M. Krall, Morris Morduchow, D. C. B. Marsh, Beckham Martin, D. L. Muench, C. S. Ogilvy, Sister Barbara Ann, W. A. Veech, and Dale Woods. Late solutions by A. E. Danese, Joseph Hammer, J. D. E. Konhauser, P. K. Maloof, J. B. Muskat, J. T. Parent, and J. W. Young.

The problem is stated in Ahlfors, ***Complex Analysis*** (1953), p. 49, and solved in Loney, ***Plane Trigonometry***, Part II (1896), p. 101.

For Positive x and y, $x^y+y^x>1$

E 1342 [1958, 774]. *Proposed by D. J. Newman, AVCO Research Division, Wilmington, Mass.*

If x and y are positive, prove that $x^y+y^x>1$.

Solution by Viktors Linis, University of Ottawa. The inequality is trivially true if either x or $y\geqq 1$. Let $0<x$, $y<1$, and put $y=kx$. Because of the symmetry we consider only $0<k\leqq 1$. Now

$$f(x) = x^{kx} + (kx)^x = (x^x)^k + k^x x^x \geqq a^k + ka,$$

where $a=\exp(-1/e)=\min(x^x)$ and $k^x \geqq k$. But $F(k)=a^k+ka$ has a unique minimum at $k_0=1-e<0$ and is increasing for $k>k_0$. Since $F(0)=1$, $F(1)=2a>1$, $f(x)>1$.

Also solved by Robert Bart, R. F. Brown and Joel Levy (jointly), E. L. Ellis, Michael Goldberg, Martin Harrow, Fritz Herzog, C. S. Ogilvy, Dale Woods, and the proposer. Late solutions by J. W. Baldwin, J. H. Bramble, D. A. Breault, G. B. Charlesworth, P. G. Hodge, Jr., L. M. Lewandowski, G. H. Meisters and J. E. Sheats (jointly), W. S. Reid, D. I. Sommerville, and an anonymous solver.

A Geometrical Probability Problem

E 1343 [1958, 774]. *Proposed by W. F. Cheney, University of Hartford, Connecticut*

Through an arbitrarily selected point within a plane triangle, a straight line is drawn which bisects the area of the triangle. What is the probability of more than one solution for a given point?

Editorial Note. This is essentially E 1228 [1957, 198]. The answer is

$$p = (3 \log 2 - 2)/4 = 0.01986 \text{ approx.}$$

Solved by Michael Goldberg, L. D. Goldstone, Viktors Linis, D. C. B. Marsh, and Helen M. Marston. Late solution by John Burr.

Convergent Subsequences of Quotients

E 1344 [1958, 774]. *Proposed by A. J. Goldman, National Bureau of Standards, Washington, D. C.*

Let $\{x_n^{(1)}\}, \{x_n^{(2)}\}, \cdots, \{x_n^{(m)}\}$ be m sequences of nonzero real numbers. Prove that there exists an integer i $(1\leqq i\leqq m)$ and an ascending sequence $\{n_r\}$ of positive integers $(r=1, 2, \cdots)$ such that each of the sequences

$$\{x_{n_r}^{(i)}/x_{n_r}^{(1)}\}, \{x_{n_r}^{(i)}/x_{n_r}^{(2)}\}, \cdots, \{x_{n_r}^{(i)}/x_{n_r}^{(m)}\}$$

is convergent.

Solution by C. H. Cunkle, Cornell Aeronautical Laboratory, Inc., Buffalo, N. Y. Take x_n such that $|x_n| = \min |x_n^{(j)}|$ for $1\leqq j\leqq m$. Then for some i $(1\leqq i\leqq m)$, the sequence $\{x_n\}$ must have a subsequence in common with $\{x_n^{(i)}\}$. Call this subsequence $\{x_{n_s}^{(i)}\}$ and let $y_{n_s}^{(j)}=x_{n_s}^{(i)}/x_{n_s}^{(j)}$. Since $|y_{n_s}^{(j)}|\leqq 1$, the sequence $\{y_{n_s}^{(j)}\}$ is bounded for each j, and every subsequence thereof contains a convergent subsequence. Let $\{y_{n_{s1}}^{(1)}\}$ be a convergent subsequence of $\{y_{n_s}^{(1)}\}$, and for $2\leqq j\leqq m$, let $\{y_{n_{sj}}^{(j)}\}$ be a convergent subsequence of $\{y_{n_{sj-1}}^{(j)}\}$. Then $\{y_{n_{sm}}^{(j)}\}$ is convergent for each j, and $\{n_{s_m}\}$ may be taken as the desired sequence $\{n_r\}$.

The property fails to be true for an infinite number of sequences $\{x_n^{(j)}\}$ even though each may be convergent itself. For example, let $x_n^{(j)}=j^{-n}$. One sees also that the nonzero hypothesis is necessary by letting $m=2$, $x_n^{(1)}=(1/n)\sin(n\pi/2)$, $x_n^{(2)}=(1/n)\cos(n\pi/2)$.

Also solved by R. F. Brown and Joel Levy (jointly), D. C. B. Marsh, W. A. Veech, and the proposer. Late solution by John Burr.

4856. *Proposed by L. A. Rubel, Institute for Advanced Study*

The growth function $h(\theta)$ of an entire function $f(z)$ of exponential type is defined by $h(\theta)=\lim\sup_{r\to\infty} r^{-1}\log|f(re^{i\theta})|$. It is well known that $f(z)\equiv f_1(z)f_2(z)$ need not imply that $h(\theta)\equiv h_1(\theta)+h_2(\theta)$. Suppose, however, that $h_1(\theta)\equiv h_2(\theta)$. Must $h(\theta)\equiv h_1(\theta)+h_2(\theta)$ in this case?

SOLUTIONS

A Theorem of Baire

4804 [1958, 633]. *Proposed by I. S. Gál, Cornell University.*

A theorem of Baire states that every lower semi-continuous function on a metric space is the pointwise limit of an increasing sequence of continuous functions. Show by an example that the theorem cannot be extended to arbitrary uniform spaces.

Solution by the proposer. Let S be a noncountable set and let $X=\{\infty\}\cup S$. Define a topology on X by specifying its closed sets: $C\subseteq X$ is closed if $\infty\in C$ or if it is countable. Under this topology X becomes a Hausdorff space. We show that X is completely normal: Let A and B be separated sets in X. If $\infty\notin A\cup B$ then A and B are open and so $O_A=A$ and $O_B=B$ are disjoint open neighborhoods of A and of B, respectively. If $\infty\in A$ and B is separated from A, then B must be closed. Indeed $\overline{B}=\cap[C: B\subseteq C]\subseteq\{\infty\}\cup B$ and so by $\overline{B}\subseteq cA$ we have $\overline{B}=B$. Hence A and B can be separated by the disjoint open neighborhoods $O_A=cB$ and $O_B=B$. Since the space is completely normal and separated, it is uniformizable.

Every real-valued function f on X is continuous at each $x\in S$. We prove that a function f is lower semi-continuous at ∞ if and only if there is an open set O_∞ containing ∞ such that $f(x)\geqq f(\infty)$ for every $x\in O_\infty$. In fact, if f is lower semi-continuous at ∞ then for every $\epsilon=1/n$, $(n=1, 2, \cdots)$, there is an open neighborhood $O_\infty^{(n)}$ of ∞ such that $f(x)>f(\infty)-1/n$ for every $x\in O_\infty^{(n)}$. Hence $f(x)\geqq f(\infty)$ for every $x\in O_\infty=\bigcap O_\infty^{(n)}$ and O_∞ is open because its complement is countable.

It follows that noncontinuous, lower semi-continuous functions exist. Also we see that f is continuous if and only if $f(x)=f(\infty)$ for every point x of an open neighborhood O_∞ of ∞. Hence if $f_1, f_2, \cdots$ is a sequence of continuous functions on X, then there exist open sets $O_\infty^{(n)}$ such that $f_n(x)=f_n(\infty)$ for every $x\in O_\infty^{(n)}$. Therefore $f_n(x)=f_n(\infty)$ for every $n=1, 2, \cdots$ and for every $x\in O_\infty=\bigcap O_\infty^{(n)}$. Now let the sequence be pointwise convergent to a limit function f. Then $f_n(\infty)\to f(\infty)$ and so $f_n(x)=f_n(\infty)\to f(\infty)$ for every $x\in O_\infty$. Hence $f(x)=f(\infty)$ for every $x\in O_\infty$ and so the pointwise limit of continuous functions is continuous.

A Double Summation

4805 [1958, 633]. *Proposed by R. R. Goldberg, Pittsburgh, Pa.*

The following relation is found to be true for $n=1, \cdots, 6$:

$$n^2 \sum_{k=0}^{n} \frac{(-1)^{n+k}}{(n-k)!} \frac{(n+k-1)!}{k!k!} \sum_{j=1}^{n+k-1} \frac{1}{j} = 1.$$

Is it true for all positive integers n?

Solution by Chih-yi Wang, University of Minnesota. The answer is in the affirmative. By writing $n^2=(n+k)(n-k)+k^2$, the left member may be transformed as follows:

$$n^2 \sum_{k=0}^{n} \frac{(-1)^{n+k}}{(n-k)!} \frac{(n+k-1)!}{k!k!} \sum_{j=1}^{n+k-1} \frac{1}{j}$$

$$= \sum_{k=0}^{n-1} \frac{(-1)^{n+k}}{(n-k-1)!} \frac{(n+k)!}{k!k!} \sum_{j=1}^{n+k-1} \frac{1}{j} + \sum_{k=1}^{n} \frac{(-1)^{n+k}}{(n-k)!} \frac{(n+k-1)!}{(k-1)!(k-1)!} \sum_{j=1}^{n+k-1} \frac{1}{j}$$

$$= \sum_{k=0}^{n-1} \frac{(-1)^{n+k+1}}{(n-k-1)!} \frac{(n+k-1)!}{k!k!} = (-1)^{n+1} \sum_{k=0}^{n-1} (-1)^k \binom{n-1+k}{n-1}\binom{n-1}{k}$$

$$= (-1)^{n+1}(-1)^{n-1}(1)^{n-1} = 1,$$

where we have used the known formula

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k}\binom{\alpha+\beta k}{n} = (-1)^n\beta^n, \qquad n \geqq 0.$$

See, for example, Gould, *Some Generalizations of Vandermonde's Convolution*, this MONTHLY, vol. 63, 1956, p. 85.

Also solved by Leonard Carlitz, Karl Goldberg, Y. L. Luke, E. Makai, Immanuel Marx D. J. Newman, F. D. Parker, R. C. Read, James Singer, and Franklin C. Smith.

Location of Zeros

4806 [1958, 633]. *Proposed by D. J. Newman, AVCO Research and Development, Wilmington, Mass.*

Let $0=a_0<a_1<a_2<\cdots$ be integers. Prove that $\sum_{n=0}^{\infty} z^{a_n}$ has no zeros in $|z|<(\sqrt{5}-1)/2$. Also, this is the best possible constant.

Solution by Leonard Carlitz, Duke University. Put $\alpha=(\sqrt{5}-1)/2$. So that $\alpha^2+\alpha-1=0$.

1. If $a_1>1$, then for $|z|<\alpha$, we have

$$\left|\sum_{n=1}^{\infty} z^{a_n}\right| < \left|\sum_{n=2}^{\infty} \alpha^n\right| = \frac{\alpha^2}{1-\alpha} = 1.$$

Hence $\sum_{n=0}^{\infty} z^{a_n}\neq 0$.

2. If $a_1=1$, consider

$$(1-z)\sum_{n=0}^{\infty} z^{a_n} = 1 - z^{b_1} + z^{b_2} - z^{b_3} + \cdots,$$

where $2 \leqq b_1 < b_2 < \cdots$. The proof is now completed exactly as above.

3. For the function $f(z) = 1 + z + z^3 + z^5 + \cdots$, $(|z| < 1)$, we have

$$f(z) = 1 + \frac{z}{1 - z^2} = \frac{1 + z - z^2}{1 - z^2}.$$

Consequently $f(-\alpha) = 0$.

Also solved by J. B. Kelly, J. C. C. Nitsche, and the proposer.

A Diophantine Equation

4807 [1958, 633]. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Determine the integral solutions of the equation

$$(m - 1)x^2 + my^2 = (m + 1)z^2. \tag{1}$$

Solution by R. Venkatachalam Iyer, Trivandrum, India. For certain values of m, but not for all, equation (1) has solutions. When, however, there exists one set of integers, α, β, γ, such that

$$(m - 1)\alpha^2 + m\beta^2 = (m + 1)\gamma^2, \tag{2}$$

then the general solution, for that value of m, can be obtained as follows. Put $x = \alpha$, $y = \beta + p\xi$, $z = \gamma + q\xi$ in (1) and solve for ξ. The result is

$$\xi = \frac{2m\beta p - 2(m + 1)\gamma q}{(m + 1)q^2 - mp^2}.$$

Hence, x, y, z may be chosen as

$$\begin{aligned} \pm x &= \alpha\{-mp^2 + (m + 1)q^2\} \\ \pm y &= m\beta p^2 - 2(m + 1)\gamma pq + (m + 1)\beta q^2 \\ \pm z &= m\gamma p^2 - 2m\beta pq + (m + 1)\gamma q^2, \end{aligned} \tag{3}$$

with p, q arbitrary, or any integers proportional to (3).

For small values of m, for which solutions exist, it is possible to determine α, β, γ by trial. In particular, if k is any integer we may take $m = k^2 + 1$, $\alpha = 1$, $\beta = k$, $\gamma = k$; or $m = 2k^2$, $\alpha = k$, $\beta = 1$, $\gamma = k$.

Also solved by W. J. Blundon and by the proposer.

Editorial Note. The proposer puts (1) in the form $m = (x^2 + z^2)/(x^2 + y^2 - z^2)$ from which solutions can be obtained with $m = x^2 + z^2$ provided $x^2 + y^2 - z^2 = 1$. Solutions of the latter equation are known: Let a, b, c, d be integers such that $ac - bd = \pm 1$ and take

$$x = ac + bd, \qquad y = ad - bc, \qquad z = ad + bc. \tag{4}$$

With these values, $m = (ac + bd)^2 + (ad + bc)^2$. Then for each m, the corresponding x, y, z give rise to an infinite set of solutions as indicated in (3) above.

To determine whether solutions exist for a proposed value of m we may employ a theorem of Legendre (*Théorie des nombres*, 1830, p. 47). *If each of the coefficients a, b, c is different from zero and has no square factor, if they are not all of the same sign and no two have a common factor, then the necessary and sufficient condition that solutions of $ax^2 + by^2 + cz^2 = 0$ exist is that the numbers $-bc$,*

$-ca$, $-ab$ *shall be quadratic residues respectively of* a, b, c. For $m>1$ the equation (1) can always easily be reduced to Legendre's form and the determination made.

To show that, for a given m, the general solution of (1) is given by (3), multiply (1) through by α^2 and replace $(m-1)\alpha^2$ by its value from (2). The result is

$$(m+1)(\gamma^2x^2-\alpha^2z^2)=m(\beta^2x^2-\alpha^2y^2).$$

Let x, y, z be any integers satisfying (1) and let p/q, $(p, q)=1$, be

$$(\beta x+\alpha y)/(\gamma x+\alpha z)$$

reduced to lowest terms. We now have

$$(m+1)q(\gamma x-\alpha z)=mp(\beta x-\alpha y), \qquad p(\gamma x+\alpha z)=q(\beta x+\alpha y),$$

from which the ratios of x, y, z are found to be as given in (3). Conversely, the values given in (3) satisfy (1) by virtue of (2).

By Legendre's theorem it is easily shown that m must be a sum of two squares. One important question remains unanswered: Is the proposer's solution (4) complete in the sense that for every m for which (1) has solutions, one of these solutions satisfies $x^2+z^2=m$ and $x^2+y^2-z^2=1$? No counterexample has been brought to light.

Bernoulli Numbers—A Finite Summation

4808 [1958, 633]. *Proposed by J. M. Gandhi, Jain Engineering College, Panchkoola, India*

Consider the expression $S_2(n, i)=\sum a_1^2\cdots a_i^2$, where the sum is taken over all possible integral choices of the a's such that $1\leqq a_1\leqq\cdots\leqq a_i\leqq n$. Prove that

$$2(2^{2n}-1)B_{2n}=(-1)^{n-1}\{(n-1)!^2-(n-2)!^2S_2(n-2, 1) \\ +(n-3)!^2S_2(n-3, 2)-\cdots(-1)^n1!^2S_2(1, n)\},$$

where the B_{2n} are the well-known Bernoulli numbers.

Solution by R. C. Read, University College of the West Indies, Jamaica. Denote the right-hand side of the equation by X_{2n}, so that

$$X_{2n}=\sum_{r=1}^{n-1}(-1)^r r!^2S_2(r, n-r-1).$$

$S_2(r, i)$ is the sum of the products i at a time, with repetitions allowed, of the squares of the numbers from 1 to r. Hence it is the coefficient of x^i in the expansion of

$$1/(1-x)(1-2^2x)(1-3^2x)\cdots(1-r^2x).$$

Consequently $S_2(r, n-r-1)$ is the coefficient of x^{n-r-1} in

$$\prod_{s=1}^{r}(1-s^2x)^{-1},$$

and $r!^2S_2(r, n-r-1)$ is thus the coefficient of x^{n-1} in

$$\prod_{s=1}^{r}\frac{s^2x}{1-s^2x}\,.$$

Hence X_{2n} equals the coefficient of x^{2n-2} in

$$\sum_{r=1}^{n-1} (-1)^r \prod_{s=1}^{r} \frac{s^2x^2}{1-s^2x^2}\cdot$$

Resolving into partial fractions in the usual way we find that

$$\prod_{s=1}^{r} \frac{s^2x^2}{1-s^2x^2} = (-1)^r + \sum_{\alpha=1}^{r} \frac{A_\alpha{}^{(r)}}{1-\alpha^2x^2},$$

where $A_\alpha^{(r)} = 2(-1)^{r-\alpha}r!^2/(r-\alpha)!(r+\alpha)!$. Hence X_{2n} equals the coefficient of x^{2n-2} in

$$(1) \qquad \sum_{r=1}^{n-1}(-1)^r + \sum_{r=1}^{n-1}\sum_{\alpha=1}^{r} \frac{2(-1)^\alpha r!^2}{(r-\alpha)!(r+\alpha)!}\,\frac{1}{1-\alpha^2x^2}.$$

Consider the expression obtained from (1) by replacing x^m by $x^m/m!$. It is

$$(2) \qquad \sum_{r=1}^{n}(-1)^r + \sum_{r=1}^{n-1}\sum_{\alpha=1}^{r} \frac{2(-1)^\alpha r!^2}{(r-\alpha)!(r+\alpha)!}\cosh \alpha x,$$

since $(1-\alpha^2x^2)^{-1}$ is replaced by $\cosh \alpha x$. (2) may now be written

$$\sum_{r=1}^{n-1} \frac{r!^2}{(2r)!}\left\{\sum_{\alpha=1}^{r}(-1)^\alpha\binom{2r}{r-\alpha}(e^{\alpha x}+e^{-\alpha x}) + (-1)^r\binom{2r}{r}\right\}$$

$$(3) \qquad = \sum_{r=1}^{n-1}\frac{r!^2}{(2r)!}(-1)^r(e^{x/2}-e^{-x/2})^{2r}$$

$$= \sum_{r=1}^{n-1}\frac{r!^2}{(2r)!}(-1)^r(2\sinh \tfrac{1}{2}x)^{2r}.$$

Hence, for $n>1$, X_{2n} is $(2n-2)!$ times the coefficient of x^{2n-2} in

$$\sum_{r=0}^{\infty}\frac{r!^2}{(2r)!}(-1)^r(2\sinh\tfrac{1}{2}x)^{2r},$$

since the expression differs from (3) only in the constant term and in terms of higher degree than $2n-2$.

Now it can easily be shown that

$$\sum_{r=0}^{\infty}(-1)^r\frac{r!^2}{(2r!)}(2z)^{2r} = \frac{1}{1+z^2} - \frac{z\sinh^{-1}z}{(1+z^2)^{3/2}},$$

and hence X_{2n} is $(2n-2)!$ times the coefficient of x^{2n-2} in $\operatorname{sech}^2\tfrac{1}{2}x - \tfrac{1}{2}x \operatorname{sech} \tfrac{1}{2}x \tanh\tfrac{1}{2}x = (d^2/dx^2)\{x\tanh\tfrac{1}{2}x\}$. But it is well known that

$$x\tanh\tfrac{1}{2}x = \sum_{n=0}^{\infty}\frac{2(2^{2n}-1)}{(2n)!}B_{2n}x^{2n},$$

so that

$$\frac{d^2}{dx^2}\{x \tanh \tfrac{1}{2}x\} = \sum_{n=1}^{\infty} \frac{2(2^{2n}-1)}{(2n-2)!} B_{2n}x^{2n-2}$$

whence $X_{2n}=2(2^{2n}-1)B_{2n}$.

Also solved by Leonard Carlitz.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

The Gyroscope; Theory and Applications. By J. B. Scarborough. Interscience, New York, 1958. 257 pp. $6.50.

This book gives a practical, unsophisticated and careful mathematical treatment of the theory and applications of the gyroscope. Two thirds of the book is devoted to a detailed engineering analysis of devices in which the gyroscopic principle is used and of situations in which gyroscopic action has a nonnegligible role. The topics covered include gyroscopic effects in car wheels, hoops, grinding mills, projectiles, propeller-type airplanes, the gyroscopic compass, the Draper hermetic integrating gyro, the Sperry gyrotron vibratory gyroscope, gyroscopic stabilizers, and astronomy. An appendix treats Schuler tuning, a technique devised to mitigate the effect of vehicle motion on a gyroscopic instrument.

The book should be usable from cover to cover by anyone who at some time has had a year of calculus and a course in engineering mechanics. Any deficiencies in his early training are supplied in the first third of the book by an eighteen-page exposition of vector algebra, a sixteen-page summary of the elements of mechanics of rigid bodies, and forty-six pages on gyroscopics in general. The reader with more background would do well to read chapters four and five of Goldstein's *Classical Mechanics* and chapter four of Sommerfeld's *Mechanics* before becoming enmeshed with the manipulative details of Scarborough's exposition.

This book has no exercises for students, but going through his many worked-out examples will serve the same purpose. I feel that the intended readers are engineers and not mathematicians or physicists. The book shows Scarborough's care and attention to detail, well known to us from his now justly famous *Numerical Mathematical Analysis.*

NATHAN GRIER PARKE, III
Parke Mathematical Labs., Inc.

Elements of Plane Trigonometry. By H. Sharp, Jr. Prentice-Hall, Englewood Cliffs, N. J., 1958. ix+274 pp. $4.95.

The first paragraph of the preface states: "The purpose of this textbook is to present trigonometry in the language and spirit of modern mathematics. The vocabulary of elementary mathematical analysis is used exclusively throughout."

The book begins with a brief introductory chapter in which the student is told that trigonometry is concerned with periodic phenomena as well as the mensuration of triangles. Next, careful definitions of *relation* and *function* are given together with enough examples to cement understanding. (Indeed, these definitions are stressed throughout the text.) Then the trigonometric functions are defined, and their elementary properties are thoroughly investigated. The main topics considered at this point are: Values at special angles, periodicity and graphs, trigonometric identities and equations, multiple angle formulas, and the inverses of trigonometric functions. (Before the latter is introduced, a careful discussion of inverse relation and inverse functions is given.) The main body of the text closes with applications of the above to solving triangles, periodic phenomena, and complex numbers and De Moivre's theorem. All this is accomplished in 168 pages. The remainder of the text consists of three appendices (scientific notation, numerical computation, and logarithms), sets of tables, an index, and answers to exercises.

The author manages to carry out his program without neglecting the standard topics, without being too brief for students at this level, and without turning a text into a treatise. (Evidently the author feels that a text should supplement rather than supplant competent instruction.) The author has done an exceedingly good job of placing virtually all of the topics normally covered in a standard trigonometry course in proper mathematical perspective.

This excellent book would be even better if it did not contain a number of minor annoyances. In some places the author uses semisymbolic jargon instead of a clear, understandable English sentence (*cf.*, *e.g.*, Examples 2 and 3, pp. 80–81).

MELVIN HENRIKSEN
Purdue University

The Tree of Mathematics. Managing Editor, Glenn James. The Digest Press, Pacoima, California, 1957. 403+xvii pages. $6.00.

This book is the joint work of some twenty contributors who have written brief expository chapters on the main branches of mathematics, beginning with high school algebra and extending into the currently active fields of the present day. These chapters appeared previously as a series of articles in the Mathematics Magazine, of which Dr. James is the managing editor.

The first seven chapters comprise a rapid account of elementary mathematics through the fundamental ideas of the calculus, and each of these chapters

concludes with a short list of exercises to test the reader's understanding. The material in these introductory chapters forms a minimum prerequisite for the comprehension of the succeeding expositions of advanced topics.

The topics covered include Differential Equations; Series; Theory of Numbers; Metric Differential, Non-Euclidean and Projective Geometry; Topology; Real and Complex Variable; Abstract Algebra; General Analysis; Integral Equations; Calculus of Variations; Matrices, Determinants, and Systems of Equations; Probability; Dynamic Programming; and the Theory of Games.

The authors have, in the limited space at their disposal, made their contributions meaningful and understandable. A reader with the prerequisite background and some natural mathematical ability should be able to grasp the central ideas of the topics presented.

The book contains an unfortunate number of misprints, some of which will confuse the reader who is seeing the material for the first time. Although the editor suggests that the book can be used for a survey course in high school or college, the reviewer believes that it is better suited for individual study by teachers and students of mathematics and related subjects who wish to extend their mathematical horizons. *The Tree of Mathematics* is a valuable contribution to mathematical expository literature.

R. A. BEAUMONT
University of Washington

Introductory Formal Logic of Mathematics. By P. H. Nidditch. University Tutorial Press, London, 1957. vii+188 pp. 12s 6d.

In a spirit of tolerance, we shall overlook the author's italicized comment "In the whole structure of mathematics there is not a single valid proof in the logical sense," as well as his generous spirit which he evidences in the remark, " . . . a part of the business of every mathematician is to give proofs; we show him how he can and ought to conduct his business." (p. v.) In a sense what Nidditch says is obvious. He considers a proof "logically valid" if *every* step taken is written down and justified by reference to logical rules, definitions, or preassigned descriptions. But this demand is also in many cases quite trivial, for there is hardly time to do this for every theorem. It becomes important when there is doubt that this construction can actually be effectuated either because of the mathematician's ability or the absence of some link in the chain. Quite obviously, however, Nidditch is not himself completely rigorous for he at times does skip steps and even assumes the consistency both of mathematics and of the rules of inference.

Be that as it may, the more direct aim of the book is to provide an introduction to formal logic using the methods and notation of the Polish logicians which are not common in the United States. This is done in a series of chapters presenting the "languages" of the propositional calculus, the quantificational calculus, the formula calculus, the calculus of set theory, elementary and complex. Each "language" is presented by listing its "alphabet" (symbols for terms, oper-

ations, and relations), the definitions (usually recursive) of its expressions and terms, the definitions of its formula, and its rules of inference. Careful explanations are given in very succinct form. In sum, for the more sophisticated formalist Nidditch accomplished his avowed aim quite respectably.

The mathematician interested in Boolean algebras will find on page 156 "five formulae" which "serve jointly to define any Boolean algebra of sets." Unfortunately he will not recognize them. This comment serves to point up a major criticism of the book so far as American logicians (for the most part) and mathematicians are concerned. It is not clear to the reviewer why, for example, Jb_1b_2 is better symbolically than $(b_1 \, v \, b_2) \cdot \sim(b_1b_2)$, or why $Eb_1b_2 = KCb_1b_2Cb_2b_1$ is to be preferred to $(b_1 \equiv b_2) \equiv (b_1 \subset b_2) \cdot (b_2 \subset b_1)$.

Indeed, the use of the operators E, K, C, *etc.* is blurred when symbols are used which too closely identify them with the term variables. In this connection it is interesting to note that the author apparently felt "English translations" of many of the formulae would help the student. These are provided in an appendix.

On page 63 Nidditch says that the requirement that all definitions and reasonings in mathematics be constructive leads the Intuitionists to eliminate "trex" (law of excluded middle). It is perhaps better to say that the Intuitionists *restrict* the use of this principle.

On page 65, Nidditch refers to the propositional, formula, and quantificational calculi as "pure logic" and the subsequent "languages" as mathematics. This is interesting, but points up what I believe is a confusion in Nidditch's point of view. As "languages" or "calculi" or better "algebras," all the formal systems are on a par. The only basis for calling L_{pc}, L_{fc} and L_{qc}, "pure logic" is in the interpretation one gives them. As abstract algebras (and I use this deliberately), they differ only in their definitions, formula, or alphabet.

Finally, it would help to know why, except for the fact that it is in fashion, these are called "languages" and not algebras.

Those interested in a more general introduction to formal logic using the Polish symbolism will find A. N. Prior's book useful. Those desiring a more extended analysis of the operators A, K, C, N, *etc.* will find the work of H. B. Curry on Schoenfinkel's Calculus worth studying.

LOUIS O. KATTSOFF
Harpur College

Problèmes Stochastiques Posês par le Phenomène de Formation d'une Queue d'Attente à un Guichet et par des Phenomènes Apparentes. By F. Pollaczek. Gauthier-Villars, Paris, 1957. 123 pp. 2,500 fr.

This little book on queueing theory with a single server, written by one of the leading workers in this field, will provide a good introduction to the subject as approached by the author's complex variables method. This approach, while not always as elegant or revealing of probabilistic intuition as those of Kendall,

Lindley, *etc.*, yields explicit results concerning many of the interesting quantities considered in queueing theory, in many cases. The student who wants a well-written example of the use of Laplace transforms, contour integration, *etc.*, in simple probabilistic applications, can find it here.

The author's approach is usually that of solving an integral equation for the Laplace transform of a limiting distribution function, or of solving a recursion relation for the transforms of a sequence of distribution functions, and then of inverting the result. He considers such things as the waiting time of the nth arrival and also the conditional distribution of waiting time, given that there is an arrival at a specified time T. In some cases of the former, an asymptotic (with n) formula and error term are given. He considers also such modifications of the usual setup as that wherein anyone delayed a positive amount by the usual queueing process also suffers an additional delay, and that wherein anyone who encounters a busy queue leaves without service. Joint distributions of several waiting times are considered, as are busy periods. The examples are often in the case of exponential service times and inter-arrival times, but a few other interesting cases are also treated in detail.

J. KIEFER
Cornell University

Elementary Statistical Methods (Rev. Ed.). By Helen M. Walker and Joseph Lev. Holt, New York, 1958. xvi+302 pp. $4.75.

The authors state that this is "a revision of the book under the same name published by the senior author in 1943," but that "it is actually a completely new piece of writing." "The authors have attempted to provide an introduction to statistical inference which is modern, intuitive, and nonmathematical." The book "is written primarily for the person whose study of mathematics ended when he left high school." The general quality of exposition in the text is good and, in general, is at the level intended. Chapter 13, "Introduction to Statistical Inference," impressed the reviewer as particularly clear and precise. The material covered is that widely agreed upon as basic in a first, nonmathematical course, and includes also such topics as Computing the Power of a Test, The Sign Test, Yates' Correction, and The z_r Transformation (of Fisher). The subject of inference is essentially postponed until the descriptive phase of statistics has included regression and correlation. Since the authors strongly advise (viii) that their sequence of topics be followed in "*this* book," those who wish to take up inference in relation to means of normal populations and then proceed to regression and correlation may find difficulty in using this text. The problems appear to be good and in sufficient supply, and answers are given. There is no numbering of sections in chapters. Overall, this should be a satisfactory text for use in a nonmathematical course, following the authors' sequence. There are a few items on the other side. The statement on page 62 concerning the ordinate

of a frequency polygon seems to be contradicted by the statement on page 63. The fourth paragraph on page 168 is completely misleading and irrelevant. On page 197 it is implied that $Z=50+10(X-\overline{X})/s$ is normally distributed if X is so distributed. There are fairly obvious misprints in the text on pages 52, 104, 172, 221; in leftmost column head for X's, page 147; and in (7.1) and (16.7).

FRANKLIN S. MCFEELY
Montana State College

Understanding and Teaching Arithmetic in the Elementary School. By E. T. McSwain and Ralph J. Cooke. Holt, New York, 1958. xi+420 pp. $5.50.

This text deals with instructional methods and, according to the authors (p. v), with the improvement of the teacher's "understanding of the meanings, vocabulary, and mathematical operations that constitute the language and science of arithmetic." The authors' way of pursuing this aim is to present a kind of cookbook of procedural rules supported by numerical examples and by explanations which are consistently unsound or inadequate. It is a case of the blind leading the blind, the result being a confusion too extensive to document here.

There is no indication that the authors understand the distinction between symbols and the referents which they denote. It is hard to tell under just what circumstances the authors regard a fraction as a number, but they are quite definite about zero: Zero is always only a placeholder symbol, never a number, not even a digit, an expression like $0\div 5$ posing an impossible problem. A *number*, we are told in formal definition, is "A mathematical symbol or symbols to show the idea of total amount of a quantity or total units in a group. It possesses place value and face value." There are 121 such definitions listed under the heading "Basic Vocabulary in Arithmetic" on pp. 367–377, of which the following are typical:

Division: "A mathematical and mental process used to change the dividend into numbers like the divisor. . . . " *Multiplicand:* "One of a given number of like numbers to be grouped and the product notated by one number." *Perimeter:* "The sum of the length of the sides of a surface plane. The outer boundary of a plane." *Subtrahend:* "A number in subtraction that identifies one of the given component numbers of a given number." *Time:* "Continuous movement. . . . "

That a text like this one can find a market; that our elementary school texts are often equally mathematically illiterate; for this a large portion of blame must be shouldered by college mathematicians, who by their neglect have left the control of lower school education to the untrained.

ROBERT L. SWAIN
State University Teachers College
New Paltz, New York

Differential Equations Applied in Science and Engineering. By Harold Wayland. Van Nostrand, Princeton, N. J., 1957. xiii+353 pp. $7.50.

This text is written for advanced undergraduates and beginning graduate students in applied science and engineering. The author intends it "to follow a two-year course in calculus and analytic geometry and a basic course in physics . . . ." His primary aim is to provide a rapid introduction to classical tools for solving the partial differential equations of theoretical physics.

The pattern followed is roughly that established some twenty years ago in mathematical books for engineers. Following a brief beginning chapter on problem formulation the author devotes forty-seven pages to a treatment of vector analysis with emphasis upon orthogonal curvilinear coordinate systems and upon coordinate-free vector properties. Chapters III and IV offer some ninety pages on methods for solving linear ordinary differential equations, including solutions by means of power series. Chapter V, on functions defined by differential equations, outlines the properties and applications of Bessel functions, Gamma functions and Legendre functions. A fifty-eight page discussion of Fourier series and orthogonal functions in Chapter VI is followed by forty pages devoted to boundary value problems. The method of separation of variables is applied to standard problems in heat flow, neutron diffusion, wave motion and the like. The final short chapter on integral transforms yields a description of finite Fourier and Hankel transforms, the Fourier transform and Laplace transform.

The author has managed to compress a surprisingly large number of mathematical facts and formulas in his book, including a selection of standard topics from advanced calculus. The informal, conversational style of the author nevertheless makes for pleasant reading without sacrificing pedagogical effectiveness. The reviewer notes with approval the inclusion of a list of some forty-seven selected references to problems in the recent technical literature, illustrating a variety of applications.

From the mathematician's point of view the results are overly informal in statement, proofs are noticeably absent, derivations are brief and heuristic and mathematical niceties and generalities are, in the main, neglected. Indeed, the headlong pace of the book allows slight opportunity to develop a proper appreciation of the power of rigorous mathematical analysis. For example, the discussion of Sturm-Liouville systems together with associated eigenvalue and expansion properties occupies but four pages in the single section on orthogonal functions. For this reason it is all the more regrettable that the chapter bibliographies do not contain many well-known works which offer excellent supplemental treatments.

The applied mathematician, in particular, will be disappointed by the lack of emphasis on initial problem formulation and by the limited attention allotted to interpretation and justification of final results. Also, he will seek in vain for a chapter on methods of numerical and approximate solution which are indis-

pensable for today's applied scientist.

For students in technical courses, preparing to cope with tomorrow's world of science, a knowledge of classical differential techniques is essential. The text under review provides a notably rapid and cursory survey of classical techniques and applications of differential equations. Because of the elementary and introductory character of the text the reviewer heartily endorses the author's recommendation that the student seek supplementary detail and news of modern mathematical developments in the established literature on applied mathematics.

B. H. Colvin
Boeing Scientific Research Laboratories
Seattle, Washington

Finite-Dimensional Vector Spaces (2nd ed.) By P. R. Halmos. Van Nostrand, Princeton, N. J., 1958. 199 pp. $5.00.

The 1943 reviews by E. R. Lorch and M. Kac of the first edition (denoted by FE) are made part of this review, for most of FE has been copied directly (occasionally with original misprint: "Negative" p. 120). Added are: (1) Fields, (2) Determinants, (3) Problems, (4) Other.

Fields: Usually not of characteristic 2, occasionally algebraically closed; these blanket assumptions make many theorems not self-contained in their statements; later in the book the field has become real or complex. As usual, one with no knowledge of fields may use real or complex scalars throughout, without losing any of the spirit of the book. (Paradoxically, finite dimensionality is never a blanket assumption.)

Determinants: These are scalars associated with operations (necessarily multiplication by a scalar) on the space (necessarily 1 dimensional) of alternating multilinear forms of highest degree. Clearly these coordinate-free methods (already the theme of FE) make this a mathematician's book as opposed to the recent plethora of "educational" texts on matrices.

Problems: Excellent! Ranging from elementary checks to a casual request for proof of the von Neumann-Jordan characterization of inner-product space. However the author announces a dangerous policy: declarative problems may be false! Fortunately, he seems to have reconsidered. Almost without exception the problems mean what they say, and in one case the phrase "true or false" is used.

Other changes: The elegant proof of Schwarz's inequality via Bessel has been borrowed from the author's book (denoted by HS) on Hilbert space; fine if one can postpone the triangular inequality so long. Tensor product is dual of space of bilinear forms on direct product; complexification is done this way and, later, directly.

Paradoxically, the proof of invertibility (if $|1-A|<1$) is more general (after omission of a redundant step, copied from FE!) than the one in HS.

With only the generalization to Hilbert space in mind, the book uses at

several points far more than the minimum needed. To give two examples: In the proof that a linear transformation must be bounded; in the Riesz representation theorem, much of which can be phrased so as to hold in a uniformly convex normed space.

Of 9 misprints noted, the most annoying is $\overline{BA}$ for $\bar{B}\bar{A}$, page 99. (Three times!) Foreign mathematicians are warned not to search in dictionaries for zeroish, zeroness, askable, ... In style, Professor Halmos follows G. H. Hardy in the role of a "missionary preaching to the cannibals."

ALBERT WILANSKY
Lehigh University

Engineering Mathematics. By Robert E. Gaskell. Holt, New York, 1958. xvi+462 pp. $7.25.

Here is another text on the ever-popular subject of advanced mathematics for engineers, this one by the head of a mathematical research laboratory of Boeing Aircraft. The traditional material is covered: differential equations, matrices, complex variables, Fourier series, numerical methods, special functions, vector analysis, Laplace transforms, boundary value problems. The methods used and physical applications made are, for the most part, the thrice familiar ones but there are praiseworthy efforts at timeliness. Rigor is average for its category and, when lacking, there is no pretence. A first-rate reference is usually furnished for omitted proofs or developments. It goes without saying that this manner of text must perforce emphasize time-tested ingredients. The individual teacher can venture farther at his own risk.

There are brief sections which are mildly and estimably novel: Dimensional analysis, analog computation, complex integration, least squares, automatic control. Uncluttered typography, simple figures, and a distaste for nonessential complications makes this book unformidable in appearance and relatively easy to absorb. Clarity is a dominant characteristic. There are plenty of exercises but the fact that the answers are given may not appeal to all teachers. Most unusual is the intelligence of the author's comments in introducing each new subject. Even the preface is a pleasure to read.

One must always have reservations. It can be argued that too much effort is devoted to the D-symbolic method for linear differential equations. More appropriate might have been the succinct all-purpose weighting function method so popular with engineers. The unit impulse, now a universal tool, is conspicuously absent, as are Fourier transforms. Graeffe's method, now that digital computers are rampant, should yield to Lin.

Complete lucidity and the accepted norm of comprehensiveness seem to be the goals set by the author and they are achieved. Just about all the standard basic material is here and in highly digestible form. A knowledgeable instructor will find it flexible enough to permit exercise of his own personality as well.

JOHN L. VANDERSLICE
Applied Physics Laboratory, The Johns Hopkins University

Essential Business Mathematics. Llewellyn R. Snyder. McGraw-Hill, New York, 1958. 470 pp. $5.50.

This book, which has been designed as a textbook, is divided into two parts. The first 105 pages are devoted to grade school arithmetic for the reader who is deficient in the fundamental skills of computation and problem solving. The second part of the book is devoted to such topics as simple and compound interest, periodic payment plans, insurance, taxes, corporate securities, profits, discount and markup, problems in retailing, stocks and bonds, and annuities. The author sticks religiously to his original intent to present the topics of his book with no mathematics beyond elementary arithmetic. In the development of annuity formulae he avoids geometric progressions by supplying the end result of progressions as applied to the compound interest problem and by accentuating the use of tables. The author presents much interesting and helpful information on effective interest rates, incentive wage formulae, social security, corporate securities, and stocks and bonds.

It is the reviewer's opinion that this book would more appropriately lend itself to review in a business periodical. It is more *business* than it is *mathematics*. There is a preponderance of devotion to "rules of thumb" rather than to the critical thinking which makes the student somewhat independent in the solution of financial problems. For a text claiming to be very practical there appears to be too much emphasis on short cuts in computation, "casting out nines," and aliquot parts tables. The text does *not* contain enough mathematics to form a sound basis for accounting, business finance, and statistics as claimed by the author. It might make a text for a general business course or a reference source for a small businessman.

TRUMAN WESTER
Central State College

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

MIDWESTERN CONFERENCE ON MECHANICS

The Sixth Conference on Fluid Mechanics and the Fourth Conference on Solid Mechanics will be held at the University of Texas on September 9–11, 1959. A series of invited lectures related to the program is being sponsored by the University of Texas. These will include the following: "Recent Progress in Rarefied Gas Dynamics Research" by Professor S. A. Schaaf, University of California, Berkeley; "New Developments in Shell Theory" by Professor L. H. Donnell, Illinois Institute of Technology; "Recent Progress in Applied Mechanics" by Professor Sydney Goldstein, Harvard University.

Correspondence regarding details of the Conference and technical papers should be sent to: Dr. M. J. Thompson, Chairman, Midwestern Conference on Fluid and Solid Mechanics, Department of Aeronautical Engineering, The University of Texas, Austin 12, Texas.

PERSONAL ITEMS

Mr. L. A. Shepp, Brooklyn College, has won the William Lowell Putnam Prize Scholarship for the eighteenth competition.

National Bureau of Standards: The following professors of mathematics have been awarded fellowships in Numerical Analysis for four-month periods; R. V. Andree, University of Oklahoma; T. A. Botts, University of Virginia,; G. C. Byers, Michigan College of Mining and Technology; R. T. Gregory, University of Texas; R. E. Lee, University of Missouri; L. F. Meyers, Ohio State University; E. P. Miles, Jr., Florida State University; B. C. Moore, Agricultural and Mechanical College of Texas; J. D. Munn, Mississippi Southern College.

Mr. R. S. Barton, Shell Development Company, Houston, Texas, has been appointed Manager of Applied Programming, The Electrodata Division of Burroughs Corporation, Pasadena, California.

Associate Professor J. W. Carr, III, University of Michigan, has been appointed Associate Professor and Director of the Research Computation Center, University of North Carolina.

Dr. C. E. Clark, Booz, Allen and Hamilton, Chicago, Illinois, has accepted the position of Mathematician with the Systems Development Corporation, Santa Monica, California.

Assistant Professor R. L. Davis, University of Virginia, has been awarded a Science Faculty Fellowship by the National Science Foundation for the academic year 1959–60 and will study at Stanford University.

Dr. Alice B. Dickinson has been appointed a Lecturer at Smith College.

Professor J. A. Dieudonne, Northwestern University, has accepted appointment as a permanent member of the staff of a new French research institute in Paris.

Assistant Professor H. A. Heckart, Simpson College, has been promoted to Associate Professor.

Professor F. F. Helton, Central College, Fayette, Missouri, has been appointed Professor at the College of the Pacific.

Dr. T. R. Horton, International Business Machines Corporation, has been appointed Manager of 700–7000 Systems Marketing, Data Processing Division, White Plains, New York.

Mr. H. H. Hunt, Oklahoma State University, has been appointed Instructor at Central State College, Edmond, Oklahoma.

Professor J. L. Katz, Rensselaer Polytechnic Institute, has been awarded a Science Faculty fellowship for a year's study at University College, London, England.

Professor D. M. Krabill, Bowling Green State University, has been awarded a National Science Foundation Science Faculty Fellowship for study in the Operations Research Program at the University of Michigan during the summers of 1959–61.

Mrs. Marilyn J. Lockhart, Programmer at International Business Machines Corporation, Poughkeepsie Research Laboratory, has been appointed Associate Mathematician in the Computing Department.

Dr. M. M. Lotkin has accepted the position of Principal Staff Scientist with Avco Research Division, Wilmington, Massachusetts.

Mrs. Ann S. Miller, Roanoke Catholic High School, Virginia, has been appointed Junior Engineer at the Lycoming Division, Avco Manufacturing Corporation, Stratford, Connecticut.

Mr. Otto Mond, Army Signal Engineering Laboratory, has accepted the position of Technical Engineer with the Aircraft Nuclear Propulsion Department, General Electric Company, Cincinnati, Ohio.

Mr. Joel Niedelman, Burroughs Corporation, Paoli, Pennsylvania, has accepted a position as Mathematician with the Bendix Aviation Corporation, North Hollywood, California.

Mr. C. E. Prince, Jr., Taejon Presbyterian College, Taejon, Korea, has been promoted to Assistant Professor.

Mr. M. B. Richins, Arizona State College, has accepted the position of Engineer with the Bell Telephone Laboratory, Murray Hill, New Jersey.

Mr. I. S. Rubin, Alwac Corporation, New York, has accepted the position of Sales Engineer with the Thomson-Ramo-Wooldridge Products Corporation, Los Angeles, California.

Mr. F. C. Sherburne, Jr., Michigan State University, has been appointed Instructor at Hope College.

Mr. W. W. Shirley, Richfield Oil Corporation, Los Angeles, California, has been appointed Mathematician-I.B.M. Programmer.

Captain M. J. Stager, Agricultural and Mechanical College of Texas, has been appointed Instructor at the United States Air Force Academy, Colorado.

Mr. Melvin Tainiter, Fairchild Astrionics Division, Wyandauch, Long Island, has accepted the position of Associate Engineer with the Arma Division, American Bosch Arma Corporation, Garden City, New York.

Professor Emeritus W. E. Cleland, Geneva College, died on February 26, 1959. He was a member of the Association for 39 years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

Professor H. M. Gehman, Secretary Treasurer, announces that the following 159 persons have been elected to membership by the Board of Governors on applications duly certified.

IRVING ADLER, M.A. (Columbia) Instr., Columbia University.
CARL M. ALBERT, A.B. (Brown) Grad. Student, Quinnipiac College.
ROGER W. ALDRICH, JR., Student, Drew University.
DONALD W. ANDERSON, Student, California Institute of Technology.
GLORIA D. ANDREWS, B.S. (Roosevelt) Teacher, Carver High School, Chicago, Illinois.
WILLIAM P. BAIR, M.A. in Ed. (Occidental) Teacher, Pasadena City Schools, California.
EUGENE H. BALSTER, B.S. (S. Dakota S.C.) Research Engineer, North American Aviation.
ERWIN J. BASINSKI, B.A. (St. Thomas) Grad. Student, University of Houston; IBM Programmer, Shell Chemical Corp.
DAVID S. BATES, Student, Butler University.
DONALD BATMAN, Student, San Francisco State College.
H. LYNNE BERNSTEIN, B.S. (Illinois Inst. of Tech.) IBM Operator, Federal Bank of Chicago.
JOHN A. BERTON, M.A. (Illinois) Asst., University of Illinois.
NORMAN BLEISTEIN, Student, Brooklyn College.
MRS. PATRICIA L. BRANDT, B.S. (Oklahoma) Head of Dept., U. S. Grant High School, Oklahoma City, Oklahoma.
HAZEL H. BREWER, M.S. (Southern California) Teacher, Pasadena Board of Education, California.
CHARLES K. BROWN, III, M.S. in Ed. (Cornell) Head of Dept., Westtown School, Pennsylvania.
WALTER D. BURGESS, Student, University of British Columbia.
CLARENCE E. BUTLER, M.Ed. (Maine) Chairman of Dept., Wilton High School, Connecticut.
WILLIAM G. CALDWELL, B.S. (U.S. Military Acad.) Teacher, San Francisco City College.
JOHN W. CALVERT, Student, University of Kentucky.
CONRAD E. CAMPBELL, M.A. (Syracuse) Grad. Asst., Syracuse University.
MICHAEL E. CHAMBREAU, Student, Stanford University.
GEORGE F. CHAPLINE, JR., Student, University of California.
DOROTHY J. CHRISTENSEN, Ph.D. (Washington) Instr., Wellesley College.
RALFE J. CLENCH, JR., B.A. (Queen's, Canada) Part time Lecturer and Grad. Student, Queen's University, Canada.
LEON COHEN, Student, City College of New York.
CARLTON J. COOK, B.S. (M.I.T.) Grad. Student, Harpur College.
WILLIAM F. COULSON, B.S. (Iowa S.C.) Teacher, Bellflower High School, California.

CAROL A. CUNNINGHAM, Student, Butler University.
MARY M. DAMARODIS, Student, Seton Hall University.
PHILIP J. DAVIS, Ph.D. (Harvard) Chief, Numerical Analysis, National Bureau of Standards.
JOHN G. DEMAS, Midshipman, United States Naval Academy.
GEORGE G. DEN BROEDER, JR., M.A. (Wayne S.U.) Instr., University of California; Research Specialist, Lockheed Missiles & Space Division.
PETER A. DIAMOND, Student, Yale University.
MRS. ROSIE L. EDWARDS, B.S. (Wiley) Teacher, Bruce High School, Gilmer, Texas.
ROBERT J. ELLIOTT, Student, New College, Oxford, England.
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ARTHUR T. WHITE, II, Student, Oberlin College.
REV. ANDREW P. WHITMAN, S.J., M.S. (Catholic) Grad. Student, Catholic University.
HOWARD L. WIENER, Student, University of Oregon.
JANICE E. WILSHIRE, Student, Baylor University.
LOWELL S. WINTON, Ph.D. (Duke) Professor, North Carolina State College.
MOON WON, B.A. (Dangook) Head of Dept., Korean Military Academy, Seoul.

REQUESTS FOR PUBLICATIONS OF CUP

The Association's Committee on the Undergraduate Program in Mathematics has arranged for the printing and distribution of the following five books:

UNIVERSAL MATHEMATICS
ELEMENTARY MATHEMATICS OF SETS—WITH APPLICATIONS
MODERN MATHEMATICAL METHODS AND MODELS, VOLUME I: MULTICOMPONENT METHODS
MODERN MATHEMATICAL METHODS AND MODELS, VOLUME II: MATHEMATICAL MODELS
CALCULUS AND ANALYTIC GEOMETRY, BY EMIL ARTIN

The contents of these books are described in advertisements printed in this MONTHLY, August–September 1958 and October 1958.

Since the demand for these books has exceeded expectations, it has been necessary to have them reprinted, and accordingly their distribution has been greatly delayed. It is expected, however, that by the time this notice appears, current orders will be filled promptly. If, therefore, your request for any of the above books was mailed before May 1 and has not yet been filled, please send a second request to the Mathematical Association of America, University of Buffalo, Buffalo 14, New York. A nominal charge will be made for future requests.

HARRY M. GEHMAN, *Secretary Treasurer*

PROGRAM OF VISITING LECTURERS, 1959–60

With the financial support of the National Science Foundation, the Visiting Lectureship Program administered by the MAA since 1954 will be continued during the academic year 1959–60. Lecturers and the regions covered are as follows:

Professors J. S. Frame of Michigan State University and S. A. Jennings of the University of British Columbia will lecture in the Western Mountain and Pacific region.

Professors David Blackwell of the University of California, Berkeley, M. R. Hestenes of the University of California, Los Angeles, and Ernst Snapper of Indiana University will lecture in the Central and Gulf region.

Dr. Philip J. Davis of the National Bureau of Standards and Professor Tibor Rado of Ohio State University will lecture in the Eastern region.

A booklet describing the Program of Visiting Lecturers may be obtained by writing to the office of the Association at the University of Buffalo.

THE MARCH MEETING OF THE SOUTHEASTERN SECTION

The thirty-eighth annual meeting of the Southeastern Section of the Mathematical Association of America was held March 20–21, 1959, at East Tennessee State College, Johnson City, Tennessee. Professors D. E. South, Chairman of the Section; T. C. Carson, Vice-Chairman; M. C. Boyce; B. G. Clark; F. A. Ficken; F. A. Lewis; and C. L. Seebeck, Jr., presided over the general and divisional sessions. There were 210 in attendance, including 140 members of the Association..

The following officers were elected for the coming year: Chairman, Professor T. C. Carson, East Tennessee State College; Vice-Chairman, Professor T. H. Lee, University of South Carolina; Secretary-Treasurer, Professor C. L. Seebeck, Jr., University of Alabama. At the business meeting Professor E. D. Nichols, Florida State University, gave a report of the High School Mathematics Contest as held in Florida. Mostly negative reports were received from the other states of the section, since many conduct their own contests.

After the Friday evening banquet, a silver tray was presented to the retiring secretary in appreciation of his service as Secretary-Treasurer of the Section since 1933.

The following program was presented:

1. *A note on spherical graphs*, by Professor W. G. Miller, Clemson College.

The paper was presented essentially as an instructional device and noted one aspect of an investigation relating to curves of infinite extent. Using a hybrid system of rectangular and polar coordinates, the infinite Euclidean plane was mapped on a sphere. Cartesian abscissas and ordinates were taken analogous to geographical longitudes and latitudes. The transformation of units was attained with successive partial sums of a geometric series, converging from zero at one pole to infinity at the other. Results were demonstrated with several graphs on a sphere of circumference 1000 mm., taking $S=500$, $a=25$, and $r=0.95$.

2. *Deciphering infinite messages*, by Dr. Basil Gordon, Army Ordnance Missile Command, Redstone Arsenal, Alabama.

Let $A=\{1, \cdots, n\}$, and let M be an infinite sequence $\cdots a_{-2}a_{-1}a_0a_1a_2 \cdots$, with a_i in A. Finally, let D be a fixed collection of "words" $(c_1, \cdots, c_k)$, c_i in A. The message M is said to be decodable if it can be divided into words of D by insertion of parentheses. If all decodable messages are uniquely decodable, the size of D is severely restricted. Inequalities on the size of D are established. The relation of these inequalities to the results of McMillan and Shannon on finite messages is discussed.

3. *Rocket trajectories—series solutions*, by Professor R. C. Meacham, University of Florida.

To investigate the degree of polynomial needed for real-time smoothing procedures during rocket flight, power series solutions of the equations of motion may be used. Such solutions are presented under various conditions of rocket flight, including spherical earth gravity field, constant altitude thrust force, "gravity turn" thrust force, and drag force.

4. *Method of orbit determination for an artificial earth satellite*, by Dr. J. R. Garrett, Rich Electronic Computer Center, Georgia Institute of Technology.

The motion of an artificial earth satellite is given by a system of three nonlinear differential equations. The equations are formulated to account for the oblateness of the earth and the atmospheric resistence. Methods of numerical integration for the system of equations are discussed. Some of the difficulties involved in precision orbit work is mentioned. The research effort currently in progress at the Rich Electronic Computer Center is the basis for comment.

5. *Retired officer program*, by Professor J. J. Gergen, Duke University.

This program is supported at Duke University by the National Science Foundation and is designed to retrain and prepare participating retired armed services officers to teach high school mathematics or college mathematics through calculus. The program leads in one year to the degree of Master of Arts in Teaching.

6. *On the number of trials necessary to produce a success*, by Professor D. E. South, University of Florida.

This paper contains a short résumé of the history of the Theory of Probability, particularly with reference to problems with varying trial probabilities. Two cases for the probability that exactly $x+r$ trials will be required to produce r successes are discussed. In the first case, the trial probability depends upon the number of previous successes, and in the second, the trial probability is a function of the number of the trial.

7. *NSF summer mathematics camp for talented high school students*, by the Director, Professor E. D. Nichols, Florida State University.

Forty mathematically-talented high school students from ten states were selected from 1100 applicants to attend The Florida State University 1958 six weeks Summer Mathematics Camp sponsored by the National Science Foundation. For about four hours daily, the students studied mathematics not usually found in the high school program. In addition they had a Russian language course, and listened to a number of lectures given by scientists from various fields. A similar camp will be held June 15 through July 24, 1959.

8. *Teaching of trigonometry on closed circuit TV*, by Professor Ayrlene M. Jones, University of Alabama.

The experiment of the University of Alabama in teaching trigonometry by closed-circuit television is described. It was found that no appreciable difference in grades resulted. Students preferred this type of instruction with an experienced teacher to small classes with an inexperienced teacher. Among other benefits noted, graduate assistants who conducted discussion TV classes learned much about teaching by observing and associating with a superior teacher.

9. *Teaching of mathematics of finance on closed circuit TV*, by Professor J. O. Reynolds, East Carolina College.

At East Carolina College this is a service course for the Business Education Department. An elementary algebra test was used to select about 40% of the better students. Daily lectures of a half period were given on TV; then a work session under a supervising instructor completed the period. Overall results were encouraging. The better students did quite well, and results were about the same as obtained in a regular classroom situation for all students.

10. *Help for the neglected student in mathematics*, by Professor W. L. Williams, University of South Carolina.

In this paper the author defined the neglected student in college mathematics today as the gifted student. He pointed out what is being done for the poorly prepared student, such as providing remedial classes for him, but emphasized that little is being done for the student of superior

ability. He suggested among other things that high schools should be encouraged to make available so-called college algebra and trigonometry to their better students so that when they enter college they can begin immediately the study of analytic geometry and calculus.

11. *The superior student program in mathematics and physics and the honors program in engineering,* by Dr. L. S. Winton, North Carolina State College.

This paper is concerned with the relation of the freshman-sophomore superior student program to the junior-senior honors program in engineering at North Carolina State College and with the content of course material. This four-semester program for the competitively-selected personnel of this group consists of a brief review of algebra and trigonometry followed by 42 weeks of integrated analytic geometry and vector calculus and ten weeks in the elementary theory of ordinary differential equations.

12. *The probability distribution of the product of n random variables,* by Professors Richard Schulz-Arenstorff and J. C. Morelock, King College, read by Professor Morelock.

This paper presents the probability distribution of the product of n random independent variables distributed with uniform density over a unit interval. As expected, the distribution becomes more platykurtic with increasing n. It is of particular interest to observe that standard methods of complex integration using characteristic functions centers all attention to one singular point at the origin where the residue is easily summed to obtain the solution.

13. *The probability that* $(n, f(n))$ *is r-free,* by Professor P. J. McCarthy, Florida State University.

An integer is said to be r-free if it is not divisible by the rth power of any prime. Let f be a function from the positive integers into the nonnegative integers. Let $f^*(x)$ be the number of y such that $f(y)=x$. Assume that f^* is finite and that f and f^* are nondecreasing, $f(x)=O(x)$ and $f^*(f(x))=O(x)$. It is then shown that if n is chosen at random the probability that $(n, f(n))$ is r-free is $1/\zeta(2r)$.

14. *On error bounds for Simpson's rule and the trapezoidal numerical integration formula,* by Professor C. L. Seebeck, Jr., University of Alabama.

To offset the deficiency in current calculus texts, several elementary derivations of error bounds for the trapezoidal law and Simpson's rule are given.

15. *The numerical solution of certain parabolic equations using finite differences,* by Dr. F. J. Witt, Oak Ridge National Laboratory.

The parabolic equation $\eta_t(x, t)=f(x, t)$ where $f(x, t)=A(x, t)\eta_{xx}(x, t)+B(x, t)\eta_x(x, t)+C(x, t)\eta(x, t)+D(x, t)$ with initial conditions and two-point boundary conditions may be solved by setting $(\eta(x, t+\Delta t)-\eta(x, t))/\Delta t=Pf(x, t+\Delta t)+Qf(x, t)+Rf(x, t-\Delta t)$, ($P, Q, R$ constants) and solving the resulting differential equations by a central difference method. The solution has been accomplished with fourth order accuracy in Δx on an electronic digital computer and a study made of the choice of P, Q, and R which produces the more accurate and converging solutions. A discussion of truncation errors, validity conditions and mesh spacing is also given.

16. *On mathematical models,* by Dr. O. E. Taulbee, Lockheed Aircraft Corporation, Marietta, Georgia.

In operations research studies the mathematical model is an attempt to simulate the process under consideration, whether existing or proposed. The pertinent features as well as the advantages and disadvantages of mathematical models are presented. Their relationship to a mathematical system is also indicated.

17. *Polynomial solutions of a class of Riccati equations*, by Professor R. W. Cowan, University of Florida.

The Riccati equation is taken in a form that has polynomial coefficients. If the equation is to have one or more polynomial solutions, at least two of the highest degree terms must be equal after the polynomial solution has been substituted in the equations. On this basis it is possible to develop criteria which limit the possible degrees of polynomial solutions. Several examples are given for the various cases, one containing polynomial solutions of three different degrees.

18. *The condition of matrices*, by Dr. C. T. Fike, Oak Ridge National Laboratory, read by title.

Turing's "condition number" $\|A\| \cdot \|A^{-1}\|$ measures the effect on A^{-1} of changes in A. A similar index of stability sometimes can be defined for characteristic roots of matrices. A relation between these two problems can be exhibited.

19. *Effect of a crack on a stress concentration in an orthotropic plate*, by Professor C. B. Smith, University of Florida.

A plate having two perpendicular axes of elastic symmetry in its plane is said to be orthotropic. A plate of this type with a small crack in its interior is considered. The plate is assumed to be in a state of plane stress due to stresses produced by a point load acting near the crack. The solution is obtained by the methods of complex variables, and is quite different from the solution for the corresponding problem for the isotropic plate.

20. *Remarks on the simplex method of linear programming*, by Professor F. A. Ficken, University of Tennessee.

This paper presents a theoretical discussion of the simplex method that is believed to have certain advantages. The problem is prepared beforehand in such a way that the first tableau has certain specific characteristics. It is then proved inductively, even in the degenerate case, that succeeding tableaux have these same characteristics, with strictly increasing values of the functional being maximized; the effectiveness of the method is then clear. The solution of the dual problem, finally, may be read off at once from the marginal entries at the bottom of the leftmost columns of the final tableau, without any renumbering of indices.

21. *High precision calculation of Arcsin x, Arccos x and Arctan x*, by Dr. I. E. Perlin, Rich Electronic Computer Center, Georgia Institute of Technology.

A polynomial approximation for Arctan x in the interval $|x| < \tan(\pi/24)$ is developed. The approximating polynomial consists of nine terms and yields an approximate value accurate to twenty decimal places. The domain $\tan(\pi/24) \leqq |x| < \infty$ is subdivided into six intervals. For each of the first five intervals, the addition formula reduces the problem of calculating Arctan x to that of computing Arctan t_i by means of the polynomial approximation. For the final interval, the calculation of Arctan x is made by Arctan $|x| = \pi/2 - \text{Arctan}(1/|x|)$. The Arcsin x and Arccos x are computed by means of relationship to Arctan x.

22. *The effects of singular collineations*, by Professor B. G. Clark, Vanderbilt University.

It was observed that in elementary studies of collineations the singular ones are usually dismissed with the statement that there is a certain singular subspace of points which have no correspondent, while the remaining points are carried into another fixed subspace. This note pointed out that there is induced in the fixed subspace a collineation which may be of any type, and proved the general theorem: *If in a space S_n the matrix of a collineation has rank $n-k$, there is a singular space S_k and all other points of S_n are carried into a fixed S_{n-k-1}; all points of the S_{k+1} determined by the singular S_k and a point B, not in S_k, are carried into one and the same point of the fixed S_{n-k-1}.*

23. *The units of the quadratic Euclidean domains*, by Professor E. H. Hadlock, University of Florida.

The units of the quadratic Euclidean domains $R(\sqrt{D})$ for $D>0$ are obtained and expressed as integral powers of the fundamental units, where the fundamental unit for a fixed value of D is either the fundamental solution or associated with that solution of one and only one of the equations $x^2-Dy^2=\pm 1, \pm 4$.

24. *Plane arcs which are the union of two similar subarcs*, by Professor M. K. Fort, Jr., University of Georgia.

This paper is concerned with plane arcs A which may be expressed as the union of two subarcs A_1 and A_2 such that: $A_1 \cap A_2$ consists of a single point, A_1 is congruent to A_2, and A is similar to A_1 and A_2. The existence of several different types, and the classification of such arcs is discussed.

25. *A class of analytic finite projective planes*, by Professor J. R. Wesson, Vanderbilt University.

A finite projective plane with $n+1$ points on each line determines a ternary system T with n elements $0, 1, a, b, \cdots$. The system T may be used to construct a projective plane with points $(1, 0, 0)$, $(x, 1, 0)$, and $(x, y, 1)$, where x, y are any elements in T. The lines of the plane have equations $z=0$, $uzy=0$, and $uz(vyx)=0$, where u, v are any elements in T. Any finite projective plane will lead to an analytic projective plane of the same order with points and lines as given above. The analytic plane so determined is analogous to that of Veblen and Wedderburn.

26. *Simplexes on the twisted cubic*, by Mr. Robert Everett, University of Georgia.

In E^3 if a 1-simplex and a 2-simplex have their vertices on the twisted cubic, a curve on which all points are in general position, under what conditions will these simplexes intersect? It is shown by the use of determinants that they intersect only if their vertices alternate along the curve.

27. *Hiroshima and Nagasaki and some of the mathematical problems they raise*, by Professor W. S. Snyder, University of Tennessee. (By invitation.)

To realize the maximum benefits atomic energy has to offer we need to know much more about the long term effects of radiation on man. The populations of the above cities may provide some information and the Atomic Bomb Casualty Commission was organized by the National Research Council to study the long term effects of radiation on the populations of these cities. The interpretation of the data obtained requires a rather accurate estimate of the radiation dose and also poses difficult statistical problems. Some of these are discussed.

28. *A universal metric for the experimental sciences*, by Professor Jose Gallego-Diaz, University of Puerto Rico.

The author asserts that a better description of certain physical phenomena would be obtained by using a metric invariant under a change of scale rather than the Euclidean metric whose characteristic property of invariance under rotation of axes is usually without significance in the representation of such phenomena. With this new Riemannian metric it is found that the curves $y=cx^m$ are geodesics. Extremals for certain integral analogues of Hamilton's principle and the Brachistochrone integral are found. Applications to biology and economics were made.

29. *A discriminant approach to the identification of conic sections*, by Professor F. S. Harper, Georgia State College of Business Administration.

By applying elementary discriminant methods to the general quadratic in x and y, criteria are established for identifying the conic sections, including all of the degenerate cases.

30. *Some new rational distance sets*, by Professor G. B. Huff, University of Georgia.

There is no nonzero rational number x such that $(x, 0)$ is at rational distance from $(0, a)$ and

(a, b) if $a, b = 1, 2$ or $4, 5$. One might conjecture that this statement holds whenever a, b are consecutive integers. On the contrary, it may be shown by a simple elementary argument that if k is an integer such that $k+1$ is a square, there is an infinite number of rational points on the x-axis, each at a rational distance from $(0, 2k)$ and $(0, 2k+1)$. This statement and similar results may be obtained from the properties of a certain elliptic cubic.

31. *The theory of relations*, by Professor Andrew Sobczyk, University of Florida.

A *structure* (in this paper) is any set B with a reflexive and symmetric relation δ. Theorems: *For any structure, there exists a partition* $B = \bigcup B_i$ *of* B *into maximal subsets* B_i *such that* $x\delta y$ *iff* x *and* y *belong to the same* B_i. *Any structure is a union of metric spaces, for which the distance* d *assumes only nonnegative integer values, and such that* $x\delta y$ *iff* $d(x, y) = 0$ *or* 1. *For any structure* B *with a finite number of elements, the partition is unique; in case of an uncountable number of elements, a counterexample shows that the partition is not always unique.* There are many more theorems, examples, and applications.

32. *Sylvester's theorem and the Vandermonde determinant*, by Cadet W. R. Alford, The Citadel.

If one borders the Vandermonde determinant by the column of elements $1, A, A^2, \cdots, A^{n-1}, P(A)$ and the row of elements $P(\lambda_L), \cdots, P(\lambda_n), P(A)$, then the determinant set equal to zero is a form of Sylvester's theorem. The λ_i's are the eigenvalues of the matrix A.

33. *Elliptic conic sections*, by Professor D. B. Goodner, The Florida State University.

Elliptic conic sections were studied by analytic methods. Equations for rigid motions of the elliptic plane were developed and used to reduce various second degree equations to standard forms. Systematic study of the standard forms led to the discovery of several basic properties of elliptic conic sections.

34. *Simple proofs of some classical determinantal relations*, by Dr. A. S. Householder, Director, Mathematics Panel, Oak Ridge National Laboratory.

Standard proofs of a number of the classical determinantal relations, both identities and inequalities, generally make use of a cumbersome index notation that is troublesome to write and more troublesome to follow. It is the purpose of this paper to point out that most of this can be dispensed with by a judicious use of partitioned matrices and of permutation matrices. Illustrations will include certain determinantal expansions, the theorem of corresponding minors, theorems on compounds, and a generalized Hadamard inequality.

35. *Present trends in undergraduate college mathematics*—a panel discussion, lead by Professor D. R. Davis, East Carolina College.

The topic was treated as follows: (1) Nature and content of lower division service courses as indicated by recent textbooks and current committees, by Professor Trevor Evans, Emory University. (2) Nature and content of lower division courses for mathematics and science majors as advocated by the Dartmouth and Kansas Summer Writing Groups, the Committee on Undergraduate Program, and current textbooks, by Professor E. A. Cameron, University of North Carolina. (3) Trends in changing content and in new courses in the upper division college curriculum as indicated by recent textbooks and by present trends in the development of both pure and applied mathematics, by Professor F. W. Kokomoor, University of Florida. A summary was given by the panel leader.

36. *Some mathematical aspects of radiation dose calculations*, by Professor W. S. Snyder, University of Tennessee. (By invitation)

If R is a convex phantom irradiated by an external source and if S is any phantom which includes R but does not shield R from the incident radiation then the maximum dose in S is not less

than the maximum dose in R. By similar considerations bounds may often be obtained for maximum dose or for integrated dose in an irradiated phantom as well as for the variation of dose within the phantom. Some isoperimetric problems are posed and discussed briefly.

37. *Some convergence properties of infinite exponentials*, by Professor W. B. Evans, Georgia Institute of Technology.

Analogous to the infinite exponential $E_1^\infty a_i$, the infinite exponential $E_{-\infty}^0 a_i$ and the double infinite exponential $E_{-\infty}^\infty a_i$ are defined. Convergence is defined for these exponentials and conditions are specified for the sequence $\{a_i\}$ which will insure convergence.

38. *A generalization of the binary system*, by Dr. Basil Gordon, Army Ordnance Missile Command, Redstone Arsenal, Alabama.

Let $K=\{k_n\}$ be a sequence of positive integers with $k_n<k_{n+1}$. Let S be the set of sums of distinct members of K. If each s in S has a unique representation in such a form, S is called a π-set with base K. Thus the set of all integers is a π-set with base $K=\{2^n\}$. It is shown that if S has measure $m>0$, then $\lim_{n\to\infty} 2^n/k_n=m$ and conversely. The problem of determining $c_n=\min_k k_n$ is also discussed.

39. *A method for the evaluation of certain integrals containing higher transcendental functions*, by Professor M. O. Gonzáles, University of Alabama.

If $L\{F(t)\}=f(s)$ and $(s+\beta)^{\alpha+1}f(s)=\sum_{n=0}^\infty c_n u^n$, where $u=(s-\beta)/(s+\beta)$, then it follows $\int_0 e^{-\beta t}F(t)L^{(\alpha)}(2\beta t)dt=c_n/(2\beta)^{\alpha+1}$ where $L_n^{(\alpha)}$ denotes the generalized Laguerre polynomial. Applications of this result to cases in which $F(t)$ is a higher transcendental function are given.

40. *Some properties of the sub-series of certain series*, by Professor M. G. Boyce, Vanderbilt University.

In a given series if some terms are deleted a sub-series is obtained. Some series have sub-series converging to every number in some interval. Examples and a necessary and sufficient condition are given. Another property possessed by some series is that of representing every rational number of an interval by a sub-series having a finite number of terms. Several general theorems are given, and special series, including $\sum 1/n$ and $\sum 1/n(n+1)$, are discussed.

41. *Combining concepts of Euclid and Dedekind to define the real number system*, by Professor E. B. Shanks, Vanderbilt University.

A geometric description of the analytic definitions employed in the paper follows. A real number is defined as a set of rationals on a finite directed line segment with zero as its only endpoint. Equality is defined as identity. Multiplication is defined by taking as the product of two real numbers the set of products whose two factors come one from each segment. Similar definitions are given for addition, subtraction, and order involving appropriate pairs of real numbers, each containing a positive rational. The definition of addition is then extended by analogy to the rational number system.

42. *On the group of a circulant of order n*, by Professor F. A. Lewis, University of Alabama.

This paper proves that the group of a circulant of order n is $n\phi(n)$ when n is odd and $(n/2)\phi(n)$ when n is even. As a corollary it follows that a circulant of order n is a symmetric function if and only if $n=3$.

43. *A special Pellian equation*, by Professor J. W. Grenier, University of Florida.

Theorems and corollaries are given describing all integral solutions of the equation $\delta x^2-dy^2=2$, with $\delta\cdot d=D$, in terms of $t_n+u_n\sqrt{D}$, solutions of Pell's equation $t^2-Du^2=1$, when t_1 is even and D is a product of distinct odd primes.

44. *Distributive lattices induced by finite groups*, by Dr. H. P. Carter, Oak Ridge National Laboratory.

Dr. Carter defines a maximal join-irreducible element of a lattice L to be a join-irreducible element A such that $A \cap B \neq A$ for all join-irreducible elements B which are distinct from A. Then he proves the theorem: If L is a finite distributive lattice, then a necessary and sufficient condition that L be induced by a finite group is that L contain n distinct maximal join-irreducible elements each connected to 0 by a single chain, and that L contain $(a_1+1) \cdots (a_n+1)$ elements, where $a_1, \cdots, a_n$ are the respective heights of these chains.

45. *A probability associated with the Euclidean algorithm*, by Professor H. S. Thurston, University of Alabama.

Given two integers α and $\beta \neq 0$ of the quadratic field $Ra(\sqrt{m})$ where $m<0$ is square-free. What is the probability that integers μ and ν can be found such that $\alpha=\beta\mu+\nu$ where $0 \leq N(\nu) < N(\beta)$? The paper solves this problem, expressing the probability as a function of m.

H. A. ROBINSON, *Secretary*

CALENDAR OF FUTURE MEETINGS

Fortieth Summer Meeting, University of Utah, Salt Lake City, Utah, August 31–September 3, 1959.

Forty-third Annual Meeting, Conrad Hilton Hotel, Chicago, Illinois, January 28–30, 1960.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA
IOWA
KANSAS, Kansas State College of Pittsburg, Spring, 1960.
KENTUCKY, University of Kentucky, Lexington, April, 1960.
LOUISIANA-MISSISSIPPI
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, American University, Washington, D. C., December 5, 1959.
METROPOLITAN NEW YORK
MICHIGAN, University of Michigan, Ann Arbor, March 26, 1960.
MINNESOTA
MISSOURI, Central Missouri State College, Warrensburg, Spring, 1960.
NEBRASKA, University of Nebraska, Lincoln, April 23, 1960.
NEW JERSEY, Princeton University, November 7, 1959.
NORTHEASTERN, Boston College, Chestnut Hill, Massachusetts, November 28, 1959.
NORTHERN CALIFORNIA, University of California, Berkeley, January 16, 1960.
OHIO
OKLAHOMA, Oklahoma City University, October 23, 1959.
PACIFIC NORTHWEST, University of Oregon, Eugene, June 19, 1959.
PHILADELPHIA, University of Delaware, Newark, November 28, 1959.
ROCKY MOUNTAIN
SOUTHEASTERN, University of South Carolina, Columbia, April 1–2, 1960.
SOUTHERN CALIFORNIA, Los Angeles State College, March 12, 1960.
SOUTHWESTERN, Air Force Missile Development Center, Holloman Air Force Base, New Mexico, April, 1960.
TEXAS, San Antonio College, April, 1960.
UPPER NEW YORK STATE, University of Rochester, Spring, 1960.
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METRIC POSTULATES FOR PLANE GEOMETRY

SAUNDERS MACLANE, University of Chicago

1. Introduction. The current interest in better high school mathematics directs attention to Euclidean geometry, as that part of the high school curriculum with the most subtle logical structure. For this very reason, there is little agreement on the arrangement of geometry for the schools. Even in its best form, the Euclidean arrangement suffered from various imperfections centering in the neglect of betweenness and separation properties; in recent years this arrangement has undergone steady attrition from all hands—those who would transfer training and those who wouldn't; those who would use geometry as a vehicle for reasoning in everyday life and those who prepare lists of "essential theorems." It is small wonder that the idea of a logical structure for elementary geometry is sometimes lost from sight.

A break with these difficulties is provided by the *Basic Geometry* of Birkhoff and Beatley [**2**]. These authors observe that the real number system is at hand in the schools; assuming it, they take the measures of distance and of angle as primitives for geometry; the result is that they build up the traditional system of geometry much more quickly than did Euclid [**7**], Hilbert [**8**], or Veblen [**12**]. The basic reason for the speed is simple: the subtleties of the betweenness relation are mastered by the use of the order relation for numbers. The particular axioms to be used for the measures of angles and distance can be varied and perhaps improved from Birkhoff's original form [**1**]. The problem is easily stated: to find a simple and intuitive set of facts on distance and angle which suffice to characterize plane geometry. An efficient metric foundation of geometry in this manner would be of such manifest utility that I present herewith a statement of one such variant, in the hope that others may be encouraged to try their hands at improving this type of axiomatics.

That axioms on distance and angle measure will suffice is clear, for distance can be used to define betweenness and this latter concept alone is a sufficient primitive for geometry in Veblen's system [**11**]. Blumenthal [**4**] and Gillam [**6**] have also considered axiom systems based just on distance. Another such system for plane geometry appears as support for a philosophical thesis in Reidemeister [**10**]; however, he takes the Pythagorean theorem as an axiom, while we would rather deduce this from more evident geometric facts.

2. Primitive notions. This axiomatics applies to the plane. Primitive terms are point, distance, line, and angle measure. The distance $d(AB)$ is intended to be the usual (nonnegative) distance from the point A to the point B. Practically, it is the quantity measured by a ruler. One sees at once that the points P on the segment from A to B are exactly those points for which $d(AB)$ is the sum $d(AP)+d(PB)$. Furthermore, the points on the ray AB consist of these points P on the segment, plus the points Q for which B is on the segment AQ. In other words, segments, rays, and, for the matter of that, lines may be defined in terms of distance.

Finally, if r and s are any two rays from the same point, the angle $\angle rs$ is to be the angle measured in degrees counterclockwise from r to s. Practically, it is the quantity measured (in degrees) by a protractor. Note especially that, with Birkhoff [1], we take angles to be directed. This fact, which is not emphasized in the book [2], seems useful for this type of axiomatics, since it gives immediate access to separation properties. For example, to show that a line l separates the plane, we take any ray r on l; the side to the left of r is then the union of all these rays s such that the $\angle rs$ is less than 180°. More detail is given below, but the basic idea is clear. Directed angles are fundamental to geometry and to elementary trigonometry, which we regard as part of geometry; they are the basis of the important facts about orientation; they should appear in beginning geometry. Even the most fuzzy-minded teenager can understand that a left is a left, not a right, glove, while those to whom this example may not appeal can surely distinguish a left turn from a right one.

3. The axioms on distance. There are four axioms on points and distances, as follows.

D1. *There are at least two points.*

D2. *If A and B are points, $d(AB)$ is a nonnegative number.*

D3. *For points A and B, $d(AB)=0$ if and only if $A=B$.*

D4. *If A and B are points, $d(AB)=d(BA)$.*

Axioms D2, D3, and D4 are those for a semimetric space. If the usual triangle axiom is added, we would have the axioms for a metric space; however we do not need the triangle axiom.

A point B is said to lie *between* the points A and C if all three points are different and $d(AC)=d(AB)+d(BC)$. If $A\neq C$, the interval, Int AC, is defined to be the set of all points B which lie between A and C. Thus Int AC is intended to be the usual open interval from A to C. If $A\neq O$ are points we now define

$$\text{Ray } OA = \{P \mid P\neq O \text{ and } d(AP) = \mid d(OA) - d(OP)\mid \}.$$

Any such set $r=$ Ray OA is called a *ray* from O; thus each ray from O is a set of points not containing O. Ray OA could also be described as the union of the interval OP with the set of all points P such that either $P=A$ or A lies between O and P. For that matter, Int OA can now be described as the set of all points P on Ray OA for which $d(OP)<d(OA)$. Also, Int OA is the intersection of Ray OA with Ray AO.

4. The axioms on lines. There are four axioms on lines, as follows:

L1. *A line is a set of points containing more than one point.*

L2. *Through two distinct points there is one and only one line.*

L3. *Three distinct points lie on a line if and only if one of them is between the other two.*

L4. *On each ray from a point O and to each positive real number b there is a point B with $d(OB)=b$.*

For distinct points $A \neq B$, we shall denote by Line (AB) the unique line l containing A and B. In view of Axiom L3 this line is the union of the following four disjoint sets: Int (AB); the set of all P with A between P and B; the set of all Q with B between A and Q, and the set consisting of A and B. In particular, Ray $AB \subset$ Line AB.

LEMMA 1. *If A, B, and C are collinear,* $d(AC) \leqq d(AB)+d(BC)$.

This asserts that the triangle axiom holds at least for the points of any one line. The proof is direct from Axiom L3. Either B is between A and C, in which case the equality holds, or one of A or C lies between, in which case the inequality applies.

LEMMA 2. *If B is between O and A, then* Int $OB \subset$ Int OA.

Proof. Take any $P \in$ Int OB; then $d(OB)=d(OP)+d(PB)$. By hypothesis, B is between O and A, so $O \neq A$ and $d(OA)=d(OB)+d(BA)$. Combining these equations and applying the triangle axiom (which holds because all points concerned lie on Line (OA)) we have

$$d(OA) = d(OP) + d(PB) + d(BA) \geqq d(OP) + d(PA).$$

But the triangle axiom also gives the reverse inequality $d(OA) \leqq d(OP)+d(PA)$. Hence equality holds, and P is between O and A.

LEMMA 3. *On each ray from a point O and to each positive real number b there is exactly one point B with $d(OB)=b$.*

Proof. Suppose, instead, that there were two such points B and B' on some Ray OA. Then O, B, B', and A all lie on a line, to wit, the line OA. For the purposes of this proof let us write $[x, y, z]$ when the three real numbers x, y, and z are such that one of them is the sum of the other two. Then O, B, B' collinear implies by Axiom L3 that $[b, b, d(BB')]$; since $B \neq B'$ and $d(BB') \neq 0$, this gives $d(BB')=2b$. Set $a=d(OA)$.

Case 1: $b>a$. Then A is between O and B and also between O and B'; hence $d(AB)=b-a=d(AB')$. Then A, B, B' collinear gives $[b-a, b-a, d(BB')]$; hence $d(BB')=2b-2a$. Since $a \neq 0$, this contradicts the previous value $d(BB') =2b$.

Case 2: $b=a$. Then $B \in$ Ray OA gives either $B \in$ Int OA or $A \in$ Int OB. If the former, then $a=d(OA)=d(OB)+d(BA)$, whence $d(BA)=0$ and $B=A$ by Axiom D3. The same conclusion holds in the latter case. All told, $B=A=B'$, in contradiction to the assumption $B \neq B'$.

Case 3: $b<a$. Here B and B' are both between O and A, so that $d(BA) = a-b = d(B'A)$. Then A, B, B' collinear gives $[a-b,\ a-b,\ d(BB')]$; hence $d(BB') = 2a-2b$. The previous value was $d(BB') = 2b$. Hence $a=2b$ and $d(OB) = d(BA) = d(AB') = d(B'O) = b$, $d(OA) = d(BB') = 2b$. In other words, O, B, A, B' are, in order, the vertices of a "square" with sides b with both "diagonals" of length $2b$.

This "square" has the property that any *three* of its points can be embedded in the ordinary real line, preserving distance, while all four cannot be so embedded. Such a configuration is known in distance geometry (*cf.* [**4**]) as a pseudo-Euclidean quadruple.

To show this impossible, we use Axiom L4 to choose a point C on Ray OA with $d(OC) = 3b$. Then A is between O and C, and B, $B' \in \text{Int } OA \subset \text{Int } OC$, by Lemma 2. Hence $d(BC) = 2b = d(B'C)$. Then B, B' and C collinear yields $[2b,\ 2b,\ 2b]$, a contradiction.

This completes the proof in all cases.

THEOREM 1. *If* $O \neq A$ *and* $B \in \text{Ray } OA$, *then* $\text{Ray } OB = \text{Ray } OA$.

In proving this we may assume $B \neq A$. Then $B \in \text{Ray } OA$ asserts that either $B \in \text{Int } OA$ or $A \in \text{Int } OB$. Since our conclusion is symmetric in A and B, it will suffice to prove, say, that $B \in \text{Int } OA$ implies $\text{Ray } OB = \text{Ray } OA$. Also, Ray OB contains a point at any given distance from O, while by Lemma 3, Ray OA contains only one such point. Hence it will suffice to prove that

$$B \in \text{Int } OA \quad \text{implies} \quad \text{Ray } OB \subset \text{Ray } OA.$$

To show this, take $P \in \text{Ray } OB$ and consider the three possible cases for P. If $P \in \text{Int } OB$, then by Lemma 2, $\text{Int } OB \subset \text{Int } OA \subset \text{Ray } OA$, and so $P \in \text{Ray } OA$. If $P=B$, then $P \in \text{Ray } OA$. There remains the case when $B \in \text{Int } OP$, so that $d(OP) = d(OB) + d(BP)$. Since $B \in \text{Int } OA$, we also have $d(OA) = d(OB) + d(BA)$. Suppose now that $P \in \text{Ray } OA$ is false; since P is on the line OA we then have O between P and A and hence

$$\begin{aligned} d(PA) &= d(PO) + d(OA) \\ &= d(OB) + d(BP) + d(OB) + d(BA) \\ &\geqq 2d(OB) + d(PA), \end{aligned} \qquad \text{(Lemma 1)},$$

a manifest contradiction to $d(OB) \neq 0$.

COROLLARY. *Any two rays from a fixed point O are equal or disjoint.*

This is immediate.

For elementary instruction some of these arguments, notably that for Lemma 3, will seem too artificial. This can be handled by making Lemma 3 a part of Axiom L4. Going further, we can add Theorem 1 as an axiom. From this Lemmas 2 and 3 can be proved easily. For example, to prove Lemma 3 observe

that Theorem 1 gives Ray $OA=$ Ray OB, so that the proof of Lemma 3 need only be given in Case 2.

5. Coordinates on a line.

THEOREM 2. *If O is any point of the line l, then l contains exactly two disjoint rays from O, and every point $B\neq O$ on l is on one of these rays.*

Proof. By Axiom L1 there is on l a point $A\neq O$. By Axiom L4, there is on Ray AO a point A' with $d(AA')=2d(OA)$. Since A' cannot be between O and A, we must then have O between A and A'. If $a=d(OA)$, then $d(AA')=2a$. Let $r=$ Ray OA, $r'=$ Ray OA'. Take $B\neq O$ on l. If B is not in r, then by Axiom L3 and the definition of a ray, O is between A and B, and $d(AB)=a+d(OB)$. If also B is not in r', then O is between A' and B, and $d(A'B)=a+d(OB)$. But A, A', and B are collinear and distinct, which implies that one of the three numbers $a+d(OB)$, $a+d(OB)$, and $2a$ is the sum of the other two, which is impossible. Therefore B is either on r or on r'. Since $A'\in r'$ and $A'\notin r$, Theorem 1 shows that r and r' can have no points in common. Any ray from O contained in l must have a point on one of r or r'; hence by Theorem 1 it must be either r or r'. We may call r' the ray *opposite* r.

We can now define coordinates on a line l. If $O\in l$, choose one of the two rays from O in l, and call it the "positive" ray r; call the other ray from O in l the "negative" ray r'. Define a function x on the set l to the real numbers by setting $x(O)=0$, $x(P)=d(OP)$, if $P\in r$, and $x(P)=-d(OP)$ if $P\in r'$. We call x the *coordinate function* on l with origin O and positive direction r. One readily proves

THEOREM 3. *Each coordinate function x on the line l is a one-one mapping of l onto the set of real numbers such that $d(PQ)=|x(P)-x(Q)|$ for all P, Q on l.*

Birkhoff assumed as an axiom the existence of a coordinate function with the property stated in Theorem 3. Since such a coordinate function is by no means unique, we have regarded it as appropriate to analyze his axiom into more primitive assertions about distance as formulated in our axioms above.

From this theorem it follows that Q lies between P and R if and only if either $x(P)<x(Q)<x(R)$, or $x(P)>x(Q)>x(R)$. Hence, also, the two rays from any point A on l are the sets of all P with $x(P)>x(A)$ and $x(P)<x(A)$, respectively. It follows that if x and y are any two coordinate functions on l, there is a real number a and a number $\epsilon=\pm 1$ such that $x(P)=\epsilon y(P)+a$ for all $P\in l$. The choice $\epsilon=-1$ amounts to a change in the "direction" of the line.

The notion of directions can be treated formally if one wishes. A *direction* D on a line l can be described as a function which assigns to each point $P\in l$ a ray $D(P)$ from P and contained in l, in such fashion that for any two points P and Q in l, the intersection $D(P)\cap D(Q)$ is one of $D(P)$ or $D(Q)$. We may then say that P comes *before* Q in the direction D if $D(P)\cap D(Q)=D(Q)$. This relation "before" yields a transitive order of the points on the line. Using the results above, one may easily prove

THEOREM 4. *A line has exactly two directions D and D'.*

Given one of these directions D, the second is described by letting $D'(P)$ be the ray opposite $D(P)$ at every point P. A direction is determined by any one of its positive rays; that is, given any ray r from a point O contained in the line l, there is exactly one direction D of l with $D(O)=r$.

6. One-dimensional geometry. The discussion hitherto has been essentially one-dimensional. It yields a set of metric axioms sufficient to characterize the one-dimensional Euclidean space: the Axioms D1, D2, D3, D4, the definition of "betweenness" and of "ray," the Axiom L4, and a rephrased Axiom L3:

L3′. *Given three distinct points, one of them lies between the other two.*

There are many other ways of setting up such axiom systems for a line; for example, one is given by Reidemeister in [10].

7. The axioms on angles. We regard an angle as the figure (or ordered triple) consisting of a point O and two rays r and s from O. Since we measure angles in degrees, it is convenient for each real number c to write $c°$ for the number c taken modulo 360.

There are four axioms on angle measure.

A1. *If r and s are rays from the same point, $\angle rs$ is a real number modulo* 360.

A2. *If r, s, and t are three rays from the same point, then $\angle rs+\angle st=\angle rt$.*

A3. *If r is a ray from O and c is a real number, then there is a ray s from O such that $\angle rs=c°$.*

If $A\neq O\neq B$ are points, we write $\angle AOB$ as usual for $\angle$(Ray OA)(Ray OB). Then

A4. *If $A\neq O\neq B$, then $\angle AOB=\angle BOA\neq 0°$ if and only if $d(AB)=d(AO)+d(OB)$.*

There are only two more axioms to come later, a *similarity* axiom and a *continuity* axiom. For the moment we observe that there are three existence axioms: the existence of points (D1); of points on a ray (L4); and of rays on a point (A3).

Axiom A2 giving the additivity of angles is possible only because we deal with directed angles. It is a most useful axiom; for example, it implies at once that $\angle rr=0$ and that $\angle rs=-\angle sr$. In Axiom A4, observe that always $\angle AOB+\angle BOA=360°=0°$; hence $\angle AOB=\angle BOA\neq 0°$ implies that $\angle AOB=180°$, so the axiom is equivalent to the following requirement;

A4′. *If $A\neq O\neq B$, then $\angle AOB=180°$ if and only if O is between A and B.*

THEOREM 5. *Let r and s be two rays from the same point O. If $\angle rs=180°$, then r and s, together with O, form a straight line. If $\angle rs=0°$, then $r=s$.*

Proof. Suppose first that $\angle rs=180°$; then $r\neq s$ and hence $r\cap s=0$. Pick points A on r and B on s; then $A\neq O\neq B$, and $\angle AOB=\angle rs=180°$; hence, by Axiom A4, O lies between A and B and therefore on the line AB. This line contains r and s; by Theorem 2 it must consist of r, s, and O, as asserted.

Suppose next that $\angle rs=0°$. By Axiom A3 there is a ray t from O with $\angle st=180°$; hence s, t, and O form a line. But then $\angle rt=\angle rs+\angle st=180°$, so that again r, t, and O form a line. It is the same line, hence r is one of s and t, by Theorem 2. Since $r\neq t$, we get $r=s$, as asserted.

COROLLARY. *If $A\neq O$, Ray OA consists of all points $B\neq O$ with $\angle AOB=0°$.*

We can now introduce polar coordinates with O as origin and prove the usual facts about the one-one correspondence between points and their polar coordinates.

8. Similar triangles. With Birkhoff we deduce the theory of parallel lines from that of similar triangles. One reason for this order of events is that the classical order, with parallels first, involves a theorem that an exterior angle of a triangle exceeds either remote interior angle; the usual proof makes tacit assumptions about sides of a line.

By an *ordered triangle* we mean the figure $\triangle ABC$ formed by three distinct points A, B, and C; if you wish, the three intervals which they determine can be regarded as part of the figure. But notice that we name the vertices in order. This saves trouble, for the same reason that the use of ordered simplices in the modern version of singular homology theory saves trouble (see, *e.g.*, S. Eilenberg, *Singular Homology Theory*, Ann. of Math. vol. 45, 1944, pp. 407–447). A less sophisticated reason is better: ordered triangles are what actually occur in the theorems of elementary geometry.

Two ordered triangles $\triangle ABC$ and $\triangle A'B'C'$ are similar if there is a positive real number k and number $\epsilon=\pm 1$ such that $\angle ABC=\epsilon\angle A'B'C'$, $\angle BCA =\epsilon\angle B'C'A'$, $\angle CAB=\epsilon\angle C'A'B'$, $kd(AB)=d(A'B')$, $kd(BC)=d(B'C')$, $kd(CA) =d(C'A')$. The triangles are congruent if this holds with $k=1$. We assume the side-angle-side (SAS) case of similarity as an axiom, in the following form.

SIMILARITY AXIOM. *If two ordered triangles $\triangle ABC$ and $\triangle A'B'C'$ have $\angle ABC=\epsilon\angle A'B'C'$, $d(AB)=kd(A'B')$, and $d(BC)=kd(B'C')$ (for $\epsilon=\pm 1$, k positive) they are similar.*

Recall that Hilbert [8] assumed a weaker version of the congruence case ($k=1$) of this axiom. We assume the case of similarity as well, so as to avoid the sophistication of an "incommensurable case" argument (*i.e.*, a continuity argument). This stronger assumption is in keeping with standard practice; for example, in a real vector space the distributive law $k(V+W)=kV+kW$ for multiples by a real scalar could be proved by continuity from the laws of addition, but we usually simply assume the general law.

Reinhold Baer has pointed out to me that the similarity axiom may be

interpreted in terms of "motions." Take $\epsilon = \pm 1$, $k > 0$, and let two rays $r =$ Ray BC, $r' =$ Ray $B'C'$ be given. There is then a "motion" M which takes r into r' with scale factor k; explicitly, M is defined as that transformation of the plane which takes each point P into the point $P' = M(P)$ with $\angle PBC = \epsilon \angle P'B'C'$ and $d(P'B') = kd(PB)$; the Axioms A3 and L3 and Theorems 1 and 5 insure that there is a unique such point P'. The similarity axiom is then the assertion that M multiplies distances by k and angles by ϵ: $d(M(P)M(Q)) = kd(PQ)$, $\angle M(P)M(O)M(Q) = \epsilon \angle POQ$. By representing this motion M as the composite of several simpler motions, one can deduce the general similarity axiom from the following three special cases: *congruence* ($k=1$, $\epsilon = \pm 1$, $r \neq r'$), *reflection* ($k=1$, $\epsilon = -1$, $r = r'$), and *stretch* ($k \neq 1$, $\epsilon = 1$, $r = r'$).

A triangle $\triangle ABC$ is said to be *degenerate* if its vertices A, B, and C are collinear. It suffices to postulate the similarity axioms for a nondegenerate triangle, since in the case of a degenerate triangle the axioms can be proved from the previous axioms.

From the similarity axiom one now derives the other congruence and similarity theorems in the following order. First ASA (angle-side-angle) for a nondegenerate triangle, then base angles in an isosceles triangle are equal, then SSS (three sides given), and finally the theorem that the sum of the angles in a triangle is 180°. Note that ASA may be false for a degenerate triangle. The details are as in Birkhoff [1]; note especially the elegant proof of the angle-sum proposition made by bisecting the sides of the given triangle. Each similarity theorem (SAS, ASA, *etc.*) includes the corresponding congruence theorems, though for elementary purposes it may be better to first prove the congruence theorems and then the corresponding ones on similarity.

9. The continuity axiom. Let X move along the line AB, and let O be on one side of AB. Our continuity axiom is to assert that $\angle AOX$ is a monotonic and continuous function of $x = d(AX)$. This can be put in more elementary form by speaking of "proper" angles—an angle $c°$ is *proper* if $0 < c < 180°$, and *improper* if $180° < c < 360°$. Clearly the supplement of a proper angle is proper, and the (additive) inverse of a proper angle is improper.

CONTINUITY AXIOM. *Let $\angle AOB$ be proper* (Fig. 1). *If D is between A and B, then $0 < \angle AOD < \angle AOB$. Conversely, if $0 < \angle AOC < \angle AOB$, then the ray OC meets the interval AB.*

The statement that $\angle AOX$ is continuous (as above) follows easily; it is indeed a very simple and illuminating exercise in elementary epsilontics. Birkhoff takes this continuity as an axiom (part of his Postulate III), though it is not quite clear whether he means to assume just continuity or continuity and monotonicity. In the book [2], "continuous" is defined to mean continuous and monotonic, and it is then assumed that $\angle AXB$ varies "continuously" when X varies along a line *or* a curve. This is troublesome, for if A and B are the endpoints of the diameter of a circle and X moves along a curve crossing the cir-

cumference of that circle several times, $\angle AXB$ does not vary monotonically.

We now sketch briefly how the continuity axiom gives the separation of the plane by lines. First observe the cyclic order of angles in a triangle, as in:

THEOREM 6. *If an ordered triangle $\triangle ABC$ has one angle $\angle ABC$ proper, then the remaining angles $\angle BCA$ and $\angle CAB$ are proper.*

For school use this might well be an axiom; the following proof, though simpler than that presented by Birkhoff, is still somewhat subtle. Take X on $\{B\} \cup$ Ray BC (Fig. 2) and set $x = d(BX)$. Now set $f(x) = 180° - \angle ABC - \angle XAB$. For all $x \geqq 0$, $f(x)$ is a continuous function. For $x > 0$, the angle-sum theorem gives $f(x) = \angle BXA$, which is neither $0°$ nor $180°$. For $x = 0$, $f(0) = 180° - \angle ABC$; hence is proper. Thus $f(x)$ is a continuous function on the segment $0 \leqq x \leqq d(BC)$; it is never $0°$ nor $180°$; therefore $f(x)$ is always proper; therefore $f(d(BC)) = \angle BCA$ is proper.

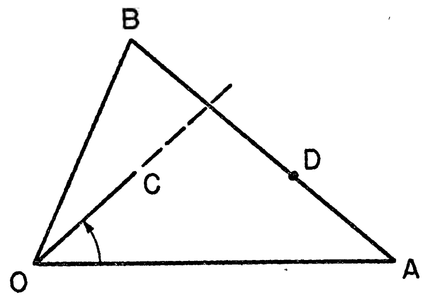

FIG. 1

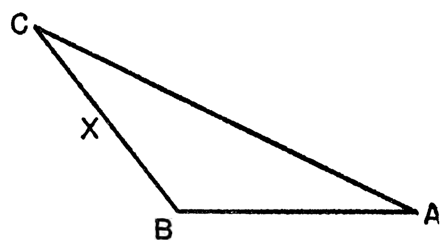

FIG. 2

We can now define the "sides" of a line l. Take a direction D on the line l, as described in Theorem 4. If A is before B in the direction D, we say that P is to the *left* of l, as directed by D, if $\angle BAP$ is proper, and to the *right* of l if $\angle BAP$ is improper. A simple application of Theorem 6 shows that this definition is independent of the choice of the points A and B; that is, if also A' is before B' in the direction D, then $\angle B'A'P$ is proper precisely when $\angle BAP$ is proper. We now can show that each line separates the plane into its left and right side.

THEOREM 7. *If l is a directed line, then points P and Q not on l lie on the same side of l if and only if l does not meet the interval PQ.*

Proof. Suppose first that $l \cap \text{Int } PQ \neq 0$, and that P is, say, to the left of l in the direction D. We must then show Q to the right of l. Let A be the point in which l meets Int PQ, and take some $B \in l$ with A before B. Then $\angle BAP$ is proper. But by Axiom A4, $\angle QAB + \angle BAP = 180°$; hence $\angle QAB$ is proper and therefore $\angle BAQ = -\angle QAB$ is improper. Hence Q is to the right of l in the direction D.

Suppose next that $l \cap \text{Int } PQ = 0$, and that P is again to the left of l. We must show that also Q is to the left of l. Again, take A before B on l in the direction D; then $\angle BAP$ is proper. For $X \in \text{Int } PQ$ the angle $\angle BAX$ is a continuous function of $d(PX)$; since $l \cap \text{Int } PQ = 0$, this function is never 0° nor 180°. Hence $\angle BAX$ is always proper; in particular, $\angle BAQ$ is proper and Q is also to the left of l.

COROLLARY (Axiom of Pasch). *Given $\triangle ABC$ and a line l with $A \notin l$, if l meets* Int BC, *then l also meets one of* Int AB *or* Int AC.

Proof. Since l meets Int BC, B and C lie on opposite sides of l. Since $A \notin l$, A lies on one of these sides, say with C. Thus A and B are on opposite sides, so that by the theorem, l meets Int AB.

10. Parallels and perpendiculars. It is convenient to start with the existence of a triangle for given ASA.

THEOREM 8. *Given an interval AB and two numbers c and d between* 0 *and* 180 *such that $c+d<180$, there exists a point C such that the ordered triangle ABC has $\angle CAB = c°$, $\angle ABC = d°$.*

Theorem 5 shows that the ray making a given angle with a given ray is unique; hence the conclusion of this theorem could be rephrased as follows.

On the interval AB (Fig. 3), *erect at B a ray BE' with $\angle ABE' = d°$ and at A, a ray AE with $\angle EAB = c°$. The rays BE' and AE must then meet.*

Proof. We first show that there exists some triangle with angles c and d. Set $e = 180 - c - d$ and construct any triangle $\triangle LMN$ with $\angle LMN = c°$. If $\angle MNL = d°$, we are done. If $\angle MNL > d°$, the continuity axiom provides a point $X \in \text{Int } LM$ with $\angle MNX = d°$, and $\triangle MNX$ is the desired triangle with angles c and d. Otherwise $\angle MNL < d°$; the angle-sum proposition then shows that $\angle NLM > e°$. In this case the continuity axiom provides a point Y on Int MN with $\angle YLM = e°$, hence with $\angle MYL = 180° - e° - c° = d°$. In any case, we then have a triangle RST with $\angle TRS = c°$, $\angle RST = d°$.

Now return to the given interval AB. By polar coordinates, there is a point C with $\angle ABC = d°$ and $d(BC)/d(BA) = d(ST)/d(SR)$. Then $\triangle ABC$ is similar to $\triangle RST$ by the axiom of similarity; hence $\triangle ABC$ has the desired angles.

Perpendicular lines are defined as usual. Axiom A3 includes the fact that one

can erect a perpendicular to a given line at any point of that line. We also prove

THEOREM 9. *If $C \notin l$, there is through C a unique line $\perp l$.*

Proof. That there is at most one such perpendicular follows from the angle-sum proposition for a triangle. To construct one, direct l from A to B so that C lies to the left of l. Then $\angle BAC$ is proper. If $\angle BAC = 90°$, we are done. Otherwise take $B' \in l$ so that ray AB is opposite ray AB'; then one of the angles $\angle BAC$, $\angle CAB'$ is acute. If $\angle BAC$ is acute, Theorem 10 gives a triangle EAC with $\angle EAC = \angle BAC$, $\angle ACE = 90° - \angle BAC$. Then $\angle CEA = 90°$, and CE is the desired perpendicular. If $\angle CAB'$ is acute, there is a similar construction.

Two lines are said to be *parallel* if they do not meet.

THEOREM 10. *If $C \notin l$, there is through C a unique line parallel to l.*

Proof. By Theorem 9, take m through C with $m \perp l$; then take a line n through C with $n \perp m$. By the angle-sum proposition n and l cannot meet; by Theorem 8 any other line through C will meet l.

The Theorem of Pythagoras now follows, say, by dropping a perpendicular from the vertex of the right angle to the hypotenuse, and using similar triangles. With this, we can introduce rectangular coordinates, prove the distance formula, and the usual description of straight lines by linear equations. This shows that the plane described by our axioms is the classical one. The trigonometric functions may also be defined. The fact that our angle measure agrees with the classical one rests on the usual proportionality between inscribed angles and arc; the proof, as in Birkhoff [1], employs the continuity axiom.

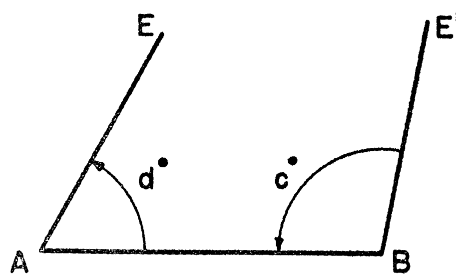

FIG. 3

11. Problems and alternatives. Our axioms suffer from at least one blemish; in pure Euclidean geometry, distance should be a magnitude with no fixed unit of measure. Such a notion of magnitude can be easily described axiomatically, but is probably not suitable for high schools; at any rate, we can easily observe that all our constructions and theorems are unaltered under change of scale (*i.e.*, when all distances involved are multiplied by some fixed positive real number). A similar remark applies to a change in the unit of angle measure.

More important is the observation that our axioms apply to an *oriented*

plane, hence cannot at once be extended to three-space. This is because there is no continuous way of choosing directions for all angles $\angle rs$ in three-space so that the addition axiom A2 will hold. It would be interesting to have an axiom system of this general type for the unoriented plane and the extension of this axiom system to space. This would allow easy comparison with the ordinary axiom systems for geometry, which are normally formulated for three dimensions.

We have not studied the independence of all of our axioms, but we can present examples to show the independence of the similarity axioms and of the continuity axiom.

To show the independence of the similarity axiom use the example of [8], page 47. Take the points, lines, and angle measures to be the usual points, lines, and angle measures of a plane Π, situated in three-space, but take $d(AB)$ to be the Euclidean distance $d(A'B')$, where A' and B' are the respective projections of A and B on some other plane Π' cutting Π at an acute angle. It is then easily seen that all the axioms except the similarity axiom hold. In view of this example it is convenient to speak of a system of points, lines, distances, and angle measures which satisfy the axioms D1—D4, L1—L4, and A1—A4 as a *pre-Euclidean plane.*

To show the independence of the continuity axiom use a Hamel basis of the real numbers to construct an automorphism ϕ of the additive group of real numbers such that $\phi(a)=a$ for every rational number but $\phi(b)=3b$ for some irrational number b. Now take the points, lines, and distances to be the points, lines, and distances of an ordinary plane, but define a new angle measure by $\alpha(rs)=\phi(\angle(rs))$ for any two rays r and s from the same point. Since ϕ is the identity on rational numbers, the new angle measure α has $\alpha(rs)=180 \pmod{360}$ if and only if $\angle rs=180°$. Hence the elementary axioms, and especially Axioms A1–A4 on angle measure, are still valid. That the similarity axiom continues to hold follows from the observation that two angles are congruent in the old measure if and only if they are congruent in the new measure. However, the continuity axiom is obviously violated because sometimes $\phi(b)=3b$.

Many variants of the axioms are possible. Birkhoff's system consisted essentially of the following axioms: D1–D4, L1, L2, A1, the similarity and continuity axioms, and the following two:

AXIOM L. *On each line l there is a function x which is a one-one correspondence between the points P on l and the real numbers, such that* $d(PQ)=|x(P)-x(Q)|$ *for all P and Q on l.*

AXIOM A. *For each point O there is a function θ which is a one-one correspondence between the rays r from O and the real numbers modulo* 360 *such that* $L(rs)=\theta(s)-\theta(r)$ *for all rays r and s from O.*

These axioms are exactly Birkhoff's four postulates (he used D1–D4, L1, and A1 without numbering them) with the unessential changes that his angles are measured mod 2π, and that his primitive for angles is $\angle AOB$ and not $\angle rs$

(while his rays are defined as half-lines). Since both axiom systems give the (cartesian) plane, his system is equivalent to ours; however it is not until the Theorem of Pythagoras is proved that it follows that his line AB contains all points metrically between A and B.

In our system the line need not be taken as a primitive concept, since one can define Line AB as the set of all points C such that $C=A$, $C=B$, or one of A, B, and C is between the other two. This definition does not seem to allow us to reduce the number of axioms.

Another variant of our system without lines as primitive notion is the following. Take, point, distance, and angle measure $\angle rs$ as primitive, with rays defined as above. As axioms take D1–D4, A1–A4, and D5 (the present Theorem 1) and D6 (the present L4). Then define a line to be the union of a point O with two rays r, r', from O such that $\angle rr'=180°$. It can be shown that these axioms also yield a pre-Euclidean plane.

Still other variants can be set up. For example, one may replace the angle between two rays r and s by the angle $\angle AOB$ given for any three points $A\neq O\neq B$. The ray OA can then be defined as the set of all points $B\neq O$ with $\angle AOB=0°$. As axioms for a pre-Euclidean plane, one may then take D1-D4, L4 (with uniqueness required for the point B) and the following:

A1′. *If $A\neq O\neq B$, then $\angle AOB$ is a real number modulo* 360.

A2′. *If A, B, and C are all different from O, then $\angle AOB+\angle BOC=\angle AOC$.*

A3′. *If $A\neq O$, and c is a real number, there is a point $B\neq O$ with $\angle AOB=c°$.*

A4′. *If $A\neq O\neq B$, then $\angle AOB=180°$ if and only if $d(AB)=d(AO)+d(OB)$.*

A5. *If $A\neq O\neq B$, then $\angle AOB=0°$ if and only if $d(AB)=|d(AO)-d(BO)|$.*

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AN INTEGER CONSTRUCTION PROBLEM

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1. The problem. "With the digits 1, 2, 3, 4 express the consecutive integers from 1 upwards as far as possible: each of the four digits being used once, and only once, in the expression of each number."

The puzzle quoted above from Ball* is illustrative of a common type of mathematical recreation: a numerical construction problem. A set of numbers is given (with possibly a fixed number of repetitions for each of the numbers); certain numerical operations are allowed to be performed on these numbers; and some or all of the resulting numbers are required to be found. In the puzzle quoted above, an answer is that, if only the operations of addition, subtraction, and multiplication on the integers 1, 2, 3, 4 (each integer being used exactly once) are allowed, then all integers from 1 through 28, as well as 30, 32, and 36, but no others, can be constructed. A proof of this statement is found in Section 6. If other operations are allowed, then it may be expected that different sets of numbers may be constructed; Ball gives several examples of this.

In this paper, an explicit expression is determined for a specific integer, namely the largest possible integer, constructible from a given set of positive integers, using only the operations of addition, subtraction, and multiplication. Use of the specific properties of a system of notation for integers is not allowed.

2. Solution. At first, only addition and multiplication are considered to be allowed operations. It will be shown in Theorem 2 that subtraction is never necessary.

THEOREM 1.† *Let r, s, and t be nonnegative integers, not all zero. Let r 1's, s 2's, and t integers greater than 2 be given, where the larger numbers are given in nondecreasing order*:

$$3 \leqq a_1 \leqq \cdots \leqq a_t.$$

Then the largest number constructible out of the $r+s+t$ given numbers, using addition and multiplication only, each of the $r+s+t$ integers being used exactly once, is

(a) $1,$ *if* $r = 1$ *and* $s = t = 0$;

(b) $(a_1 + 1)\cdot \prod_{j=2}^{t} a_j$ *if* $r = 1$, $s = 0$, *and* $t \geqq 1$;

(c) $2^{s-r}\cdot(2+1)^r\cdot \prod_{j=1}^{t} a_j$ *if* $r \leqq s$;

* W. W. R. Ball, Mathematical Recreations and Essays, New York, 1947, p. 15.

† I wish to thank the referee for a strong brevity-inducing transformation of the proof of Theorem 1.

(d) $(1 + 1)\cdot 2\cdot (2 + 1)^{s-1}\cdot \prod_{j=1}^{t} a_j$ *if* $r - s = 1$ *and* $s \geqq 1$;

(e) $(1 + 1)^2\cdot (1 + 1 + 1)^{(r-s-4)/3}\cdot (2 + 1)^s\cdot \prod_{j=1}^{t} a_j$

if $r - s \geqq 4$ *and* $r - s \equiv 1 \pmod 3$;

(f) $(1 + 1 + 1)^{(r-s)/3}\cdot (2 + 1)^s\cdot \prod_{j=1}^{t} a_j$ *if* $r > s$ *and* $r - s \equiv 0 \pmod 3$;

(g) $(1 + 1)\cdot (1 + 1 + 1)^{(r-s-2)/3}\cdot (2 + 1)^s\cdot \prod_{j=1}^{t} a_j$ *if* $r > s$ *and* $r - s \equiv 2 \pmod 3$.

If q *denotes the remainder on division of* $s-r$ *by* 3, *i.e., if*

$$q = s - r - 3\cdot\left[\frac{s - r}{3}\right],$$

then forms (e), (f), *and* (g) *may be written uniformly as*

(h) $(1 + 1)^q\cdot (1 + 1 + 1)^{(r-s-2\cdot q)/3}\cdot (2 + 1)^s\cdot \prod_{j=1}^{t} a_j$ *if* $r \geqq s + 2$.

In the above formulas, empty products, such as $\prod_{j=2}^{1} a_j$, are considered to have the value 1. Each of the above forms indicates the explicit method of constructing a maximizing combination. The maximum value is a product whose factors are the numbers $a_1, \cdots, a_t$ and as many 3's as possible (made up out of 1's and 2's), together with any leftover 2's. A single remaining 1 is added to the smallest factor.

The forms (a) through (g) (or (a) through (d), and (h)) are mutually exclusive and exhaustive.

3. Definitions. A *primitive symbol* is a symbol chosen from the r 1's, or the s 2's, or from $a_1, \cdots, a_t$.

A *subcombination* is a nonempty sequence of symbols containing a subset of the $r+s+t$ primitive symbols once each, in any order, interspersed with $+$, $\cdot$, (, and) in such a way that the symbol sequence makes sense and can be evaluated arithmetically. The result of the evaluation is called the *value* of the subcombination. A *combination* is a subcombination using all $r+s+t$ of the primitive symbols.

4. Proof of Theorem 1.

LEMMA 1 (the product-sum form). *If* X *is any combination, then there is a combination* Y *of not smaller value such that* Y *is in product-sum form, i.e.,* Y *is a product of subcombinations each of which is a sum of primitive symbols.*

In order that no errors of association shall be made, it is assumed that suffi-

ciently many parentheses are used. This can be assured by writing the result of each binary operation in parentheses.

Proof. The proof is by mathematical induction. Now each primitive symbol alone forms a subcombination in product-sum form. Suppose, then, that every subcombination using fewer than k of the primitive symbols may be replaced by a subcombination in product-sum form using exactly the same primitive symbols and having a value at least as large. Let now V be a subcombination using k of the primitive symbols, where $k>1$. Then either $V=(V_1+V_2)$ or $V=(V_1\cdot V_2)$, where each of V_1, V_2 is a subcombination using fewer than k primitive symbols. But, by the induction assumption, V_1 and V_2 are replaceable by W_1 and W_2, where W_1 and W_2 are subcombinations in product-sum form using the same primitive symbols as V_1 and V_2, respectively, and such that $V_1\leqq W_1$ and $V_2\leqq W_2$.

If $V=(V_1\cdot V_2)$, then $(W_1\cdot W_2)$ uses the same primitive symbols as V; furthermore, obviously, $V\leqq(W_1\cdot W_2)$. Hence $(W_1\cdot W_2)$ is the required subcombination in product-sum form.

Let now $V=(V_1+V_2)$. If both W_1 and W_2 have values at least 2, then $V\leqq(W_1+W_2)\leqq(W_1\cdot W_2)$, so that $(W_1\cdot W_2)$ is the required subcombination. But if, for example, W_1 has value 1, then $W_1=(X_1\cdot 1)$, where X_1 is a (possibly empty) product of 1's. Since W_2 is in product-sum form, let X_2 be one of its factors (a sum of primitive symbols), and let X_3 be the (possibly empty) product of the remaining factors of W_2. Then $V\leqq(W_1+W_2)=((X_1\cdot 1)+(X_2\cdot X_3))\leqq(X_1\cdot((1+X_2)\cdot X_3))$, since X_1 and X_3 have values at least 1. But the last expression uses exactly the same primitive symbols as V, and is also in product-sum form, since no multiplication is involved in the factor $(1+X_2)$. Thus V may be replaced by a subcombination in product-sum form if W_1 has value 1. A similar proof holds if W_2 has value 1.

The proof is completed by letting $k=r+s+t$.

LEMMA 2. *If X is a subcombination in product-sum form, then a subcombination in product-sum form can be found which has a not smaller value, which uses the same primitive symbols as X, and whose only factors are subcombinations of the form* 1, $(1+1)$, $(1+1+1)$, 2, $(2+1)$, a_j, *or* (a_j+1) *for* $1\leqq j\leqq t$.

Proof. If any factor of X is expressible as a sum of two or more subcombinations each with value not less than 2, then such a sum may be replaced by a product. This procedure is carried out as far as possible. The process must end, since each such replacement increases the number of factors, and this cannot be larger than $r+s+t$. The only factors which cannot be replaced in this way are those listed in the conclusion of the lemma.

We return to the proof of Theorem 1. It is sufficient to assume that a combination satisfying the conclusion of Lemma 2 is given. Let g_1, g_{11}, g_{111}, g_2, g_{21}, g_0, and g_{01} be, respectively, the number of factors in the combination which are of the form 1, $(1+1)$, $(1+1+1)$, 2, $(2+1)$, a_j, or (a_j+1) for $1\leqq j\leqq t$.

First, if $g_{01}\geqq 2$ and, after rearrangement of factors, $1\leqq j<k\leqq t$, then a larger combination can be obtained by use of the inequality $(a_j+1)\cdot(a_k+1)=a_j\cdot a_k+a_j+a_k+1<a_j\cdot a_k+3\cdot a_k\leqq(1+1)\cdot a_j\cdot a_k$.

If, however, $g_{01}=1$, then one of the inequalities $1\cdot(a_j+1)<(1+1)\cdot a_j$, $(1+1)\cdot(a_j+1)<(1+1+1)\cdot a_j$, $(1+1+1)\cdot(a_j+1)\leqq(1+1)^2\cdot a_j$, $2\cdot(a_j+1)<(2+1)\cdot a_j$, $(2+1)\cdot(a_j+1)\leqq(1+1)\cdot 2\cdot a_j$ for $1\leqq j\leqq t$ can be applied to give a combination of not smaller value with $g_{01}=0$, unless both $r=1$ and $s=0$. (Note that r cannot be 0, since $g_{01}\leqq r$.)

If $g_{01}=1$, $r=1$, and $s=0$, then use of the inequality $a_1\cdot(a_j+1)\leqq(a_1+1)\cdot a_j$ with $1<j\leqq t$ yields form (b).

If $g_{01}=0$, $r=1$, and $s=0$, then either $t=0$, yielding form (a), or $t\geqq 1$, in which case use of the inequality $1\cdot a_1<a_1+1$ yields a combination of larger value (which is then in form (b)).

The remaining cases then have $g_{01}=0$ and either $r\neq 1$ or $s>0$.

If, now, $g_1\geqq 2$, then a combination of larger value is obtained by use of the inequality $1\cdot 1<1+1$.

If $g_1=1$, then $r\geqq 2$ (since $g_1\leqq r\neq 1$). Hence $g_1<r$, so that at least one of the numbers g_{11}, g_{111}, g_{21} is positive (since $g_{01}=0$). Hence use of one of the inequalities $1\cdot(1+1)<1+1+1$, $1\cdot(1+1+1)<(1+1)^2$, $1\cdot(2+1)<(1+1)\cdot 2$ yields a combination of larger value.

Thus it is now necessary to consider only those cases in which $g_1=g_{01}=0$. But then $g_0=t$, and a count of 1's and 2's shows that

$$2\cdot g_{11}+3\cdot g_{111}+g_{21}=r, \tag{1}$$

$$g_2+g_{21}=s. \tag{2}$$

Hence

$$2\cdot g_{11}+3\cdot g_{111}=g_2+r-s. \tag{3}$$

If, now, $r\leqq s$, then (3) yields the inequalities

$$g_{111}\leqq 2\cdot g_{11}+3\cdot g_{111}\leqq g_2. \tag{4}$$

Sufficient use of the identity $(1+1+1)\cdot 2=(1+1)\cdot(2+1)$ yields a combination with $g_{111}=0$. Then (4) shows that the new combination has $2\cdot g_{11}\leqq g_2$.

If $g_{11}>0$, then $g_2\geqq 2$, so that the inequality $(1+1)\cdot 2^2<(2+1)^2$ can be applied to yield a combination with larger value. But if $g_{11}=0$, then $g_{21}=r$ and $g_2=s-r$, so that the combination is in form (c).

Therefore, it is now necessary to consider only the case $r>s$ (with the conditions $g_1=g_{01}=0$, $g_0=t$, and either $r\neq 1$ or $s>0$). Then one of the strict inequalities

$$\begin{aligned}(1+1)\cdot 2^2 &< (2+1)^2,\\ (1+1)^2\cdot 2 &< (1+1+1)\cdot(2+1),\\ (1+1)^3 &< (1+1+1)^2,\end{aligned} \tag{5}$$

yielding a combination of larger value, is applicable unless

$$g_2 \geqq 2 \quad \text{and} \quad g_{11} = 0, \text{ or} \tag{6}$$

$$g_2 = 1 \quad \text{and} \quad g_{11} \leqq 1, \text{ or} \tag{7}$$

$$g_2 = 0 \quad \text{and} \quad g_{11} \leqq 2. \tag{8}$$

Now sufficient use of the identity $(1+1+1)\cdot 2=(1+1)\cdot(2+1)$ yields a combination with either

$$g_{111} = 0 \quad \text{or} \quad g_2 = 0. \tag{9}$$

One of the inequalities (5) may now be applicable, yielding a combination with larger value; if (5) is not applicable, then (3), together with the condition $r>s$, yields

$$2\cdot g_{11} + 3\cdot g_{111} > g_2. \tag{10}$$

If, now, $g_2 \geqq 2$, then $g_{11}=g_{111}=0$ (from (6) and (9)). But this contradicts (10). Hence $g_2 \leqq 1$.

If $g_2=1$, then $g_{11}\leqq 1$ and $g_{111}=0$ (from (7) and (9)). But this contradicts (10) unless $g_{11}=1$, in which case (1) and (2) imply that $g_{21}=r-2=s-1$, so that the combination is in form (d).

If $g_2=0$, then (2) implies that $g_{21}=s$, so that $3\cdot g_{111}=r-s-2\cdot g_{11}$ (from (1)). But then (8) implies that one of the following holds:

$$g_{11} = 0 \quad \text{and} \quad g_{111} = (r-s)/3, \text{ or}$$
$$g_{11} = 1 \quad \text{and} \quad g_{111} = (r-s-2)/3, \text{ or}$$
$$g_{11} = 2 \quad \text{and} \quad g_{111} = (r-s-4)/3.$$

Hence the combination is in form (f), (g), or (e), respectively (or in form (h)).

The proof of Theorem 1 is now complete, since it has been shown that every combination either is of the same value as one of the forms (a)–(g) or can be replaced by a combination of larger value. Since there are only finitely many combinations, the replacement process must end.

5. Extensions. In this section two additional operations, namely subtraction and negation, will be allowed. For example, the combination (-3) involves negation but not subtraction.

THEOREM 2. *With the same hypotheses as in Theorem* 1, *except that subtraction and negation, as well as addition and multiplication, are considered to be allowed operations, the same conclusions follow. In other words, subtraction and negation are never necessary.*

Proof. Let V be any combination in which addition, subtraction, multiplication, and negation may be used. By use of the associative, commutative, and distributive laws and the laws of signs, V is multiplied out without simplification. An expression in sum-product form (which need not be a combination) is

obtained. A new expression is formed by making all signs positive. This new expression has a value which is not smaller than the value of V. But this new expression is easily seen to be equal in value to the combination obtained from V by changing subtraction to addition and omitting negation. Hence V can be replaced by a combination not involving subtraction or negation and having a not smaller value.

It is conjectured that if division and reciprocation, as well as subtraction and negation, are also made allowed operations, then they are never necessary to achieve a maximizing combination.

6. Application. It is easy to construct combinations (allowing addition, subtraction, multiplication, and negation) using the digits 1, 2, 3, 4 once each, and having any integer value from 1 through 28, or the value 30, 32, or 36. By means of Theorem 2 it is possible to show that no other values are obtainable.

For example, it will be shown that 29 cannot be obtained as value of a combination; the proof for any other excluded value is similar. Now if 29 were obtainable using 1, 2, 3, 4, then it would be obtainable using ten 1's, since 2, 3, and 4 can be replaced by the proper sums of 1's in any combination.

Let $M(n)$ denote the largest number obtainable using exactly n 1's (and no other numbers). The table below lists a few convenient values of $M(n)$ and $M(10-n)$.

n	1	2	3	4	5	6	7	8	9	10
$M(n)$	1	2	3	4	6	9	12	18	27	36
$M(10-n)$	27	18	12	9	6	4	3	2	1	—

In evaluating a combination with value 29, the last ("outermost") operation performed must be an addition, subtraction, or multiplication of two *proper* subcombinations. It may be assumed that each of these proper subcombinations has nonnegative value; otherwise a different combination having these properties could be obtained by an obvious change of signs. Each of the subcombinations must involve not more than 9 1's. If subtraction is the last operation, then the minuend must have value at least 29 (since the subtraend is nonnegative). If multiplication is the last operation, then one of the factors must have value 29. But both of these are impossible, since $M(9)<29$. If, however, addition is the last operation, then 29 cannot be obtained, since $M(n)+M(10-n)<29$ if $n \leqq 9$. Hence 29 is unobtainable.

A later paper will contain results concerning constructions using numbers which need not be integers.

THE INTEGRABILITY OF CERTAIN FUNCTIONS AND RELATED SUMMABILITY METHODS II

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I return to my previous paper (this MONTHLY, vol. 66, 1959, pp. 361–375) to add the following three remarks which are independent of each other.

Remark 1. Professor A. S. Besicovitch communicated to me the following beautiful direct proof of the property (1.5) of the fractions (1.4): We are to prove that *to any* $\delta > 0$ *there corresponds an* N *such that if* $n > N$ *then one of the* ϕ *fractions* (1.4) *falls in any subinterval* I_δ, *of* $[0, 1]$, *of length* δ. To prove this, denote by $p_1, \cdots, p_k$ all distinct prime factors of n. Given $\delta > 0$ there corresponds an N such that if $n > N$ then at least one of the inequalities

$$\text{(i)} \qquad (p_1 \cdots p_k)/n < \delta, \qquad\qquad \text{(ii)} \qquad 2/p_k < \delta$$

holds. If (i) holds then any I_δ contains an irreducible fraction of the form $(qp_1 \cdots p_k + 1)/n$.

Let now (ii) hold. Any I_δ contains at least $[n\delta]$ fractions of the form m/n; but $n\delta > 2n/p_k \geqq 2p_1 \cdots p_{k-1}$. Hence an I_δ contains at least $2p_1 \cdots p_{k-1}$ consecutive fractions m/n. However, any $2p_1 \cdots p_{k-1}$ consecutive integers m contain a pair of numbers of the form

$$q \cdot p_1 \cdots p_{k-1} + 1, \qquad (q + 1)p_1 \cdots p_{k-1} + 1$$

and at least one of these, call it m, is prime to n. But then m/n is in I_δ and the proof is complete.

Remark 2. In his paper *On the limits of Riemann approximating sums*, Quart. J. Math., Oxford Ser., vol. 18, 1947, pp. 124–127, P. Hartman establishes an interesting result for vector-valued functions which, stated in two dimensions, is as follows: *Let* $f(x)$ *be real- or complex-valued and bounded in* $[0, 1]$. *An expression of the form*

$$S = \sum_{\nu=1}^{n} f(\xi_\nu)(x_\nu - x_{\nu-1}), \qquad (x_0 = 0 < x_1 < \cdots < x_n = 1, \qquad x_{\nu-1} \leqq \xi_\nu \leqq x_\nu),$$

is called, as usual, a Riemann sum. Let $\{S_n\}$ *be an infinite sequence of R-sums with the following two properties*: (i) *The maximal subinterval of* S_n *converges to zero as* $n \to \infty$, (ii) *The limit* $\lim_{n\to\infty} S_n = \sigma$ *exists. Let* $R = \{\sigma\}$ *denote the set of all such limits. Then* R *is a bounded and closed convex domain in the complex plane.*

It is now clear that $f(x)$ is Riemann integrable if and only if the convex set R reduces to a point, *the* value of the integral.

Let us now take a second look at our special function (2.1), where λ and the elements of the sequence $\{\gamma_n\}$ are now real- or complex-valued. The following theorem generalizes our conditions (2.2) for the R-integrability of $f(x)$:

THEOREM 6. *The Riemann-Hartman set R for the function $f(x)$, defined by* (2.1), *is described by*

$$R = K(\lambda \cup \{\lim \gamma_n\}),$$

where the right side is the closed convex hull of the union of the point λ with the set of limit points of the sequence $\{\gamma_n\}$.

Proof. Let the set $\Gamma = \lambda \cup \{\lim \gamma_n\}$ be contained in the open half-plane H: $u \cos \theta + v \sin \theta - h < 0$ of the complex plane $u + iv$. Now Riemann sums over the range $[0, 1]$ are of the nature of centroids; if $\{S_n\}$ is a sequence of R-sums satisfying Hartman's conditions (i) and (ii), then it is clear from (2.1) that also the point $S_n = u_n + iv_n$ will be in the half-plane H, provided that n is sufficiently large. This proves that $R \subseteq K(\Gamma)$.

In order to prove the opposite inclusion, let $\sigma \in K(\Gamma)$ and let us show that $\sigma \in R$. Since Γ is a closed set, σ may be represented as $\sigma = \alpha a + \beta b + \gamma c$, where $a, b, c \in \Gamma$, $\alpha, \beta, \gamma \geqq 0$, $\alpha + \beta + \gamma = 1$. To fix the ideas let α, β, γ, be all positive. Given $\epsilon > 0$ we can evidently construct for our function $f(x)$ three R-sums S_a, S_b, S_c, all three corresponding to the same division $x_0 = 0 < x_1 < \cdots < x_n = 1$, differing only in the ξ's, and such that

$$|S_a - a| < \epsilon, \quad |S_b - b| < \epsilon, \quad |S_c - c| < \epsilon, \quad x_\nu - x_{\nu-1} < \epsilon, \quad (\nu = 1, \cdots, n).$$

Let now i and j be integers defined by the inequalities

$$x_{i-1} \leqq \alpha < x_i, \qquad x_{j-1} \leqq \alpha + \beta < x_j.$$

If we now form a new R-sum S^* obtained by taking the ξ_ν of S_a if $\nu = 1, \cdots, i$, the ξ_ν of S_b if $\nu = i+1, \cdots, j$, and finally those of S_c if $\nu = j+1, \cdots, n$, then evidently S^* will be close to $\sigma = \alpha a + \beta b + \gamma c$ and in fact as close as we wish provided that ϵ is sufficiently small. This shows that $\sigma \in R$ and completes the proof.

Theorem 6 immediately yields the following *converse* of Hartman's theorem:

THEOREM 7. *Any given bounded and closed convex domain D in the complex plane may serve as the Riemann-Hartman domain R of an appropriate bounded function $f(x)$ in $[0, 1]$, in fact functions of the type* (2.1) *suffice.*

Proof. Indeed, select at will $\lambda \in D$ and also the sequence $\{\gamma_n\}$ so as to be everywhere dense on the boundary of D. Theorem 6 now shows that for the function (2.1) we have $R = D$.

Remark 3. I was kindly informed by Professor G. Szegö that the remark of Section 6 was already made long ago by the then very young Hungarian mathematician, F. Lukács, and published by him in Math. Annalen, vol. 70, 1911, pp. 561–562.

A SPECIAL VANDERMONDIAN DETERMINANT

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The primary problem under consideration here is that of the evaluation of a particular determinant whose general term is of the form

$$a_{ij} = x_i^j, \qquad\qquad i, j = 0, 1, \cdots, m,$$

where $x_i = \cos\theta_i$, $\theta_i = \pi i/m$, $m \geqq 1$, and a_{i0} is taken to be one if x_i equals zero. This particular case might arise if one attempted to solve the problem of passing a polynomial of degree m through $m+1$ points, whose (Chebyshev) abscissas are as given by the above cosine formula, by the use of determinants. The resulting coefficient determinant is the one that interests us here.

We will show that it has a value D such that

$$D^2 = m^{m+1}/2^{m^2-3}.$$

Thus, D is relatively small for moderately large m. For example, for $m = 20$, we find that D is approximately $8(10)^{-47}$, and hence a reasonable amount of care would have to be taken to obtain a numerical solution of any significance.

In the course of the proof, various identities involving binomial coefficients arise which are of perhaps more interest than the basic evaluation referred to initially. For example, in the first part of the proof, we require the identities indicated by the numbers appearing on and above the principal diagonal in the product of the following infinite matrices:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \cdots \\ 1 & 2^{-1}\binom{1}{0} & 2^{-3}\binom{3}{1} & 2^{-5}\binom{5}{2} & 2^{-7}\binom{7}{3} & \cdots \\ 2^{-1}\binom{1}{0} & 2^{-3}\binom{3}{1} & 2^{-5}\binom{5}{2} & 2^{-7}\binom{7}{3} & 2^{-9}\binom{9}{4} & \cdots \\ 2^{-3}\binom{3}{1} & 2^{-5}\binom{5}{2} & 2^{-7}\binom{7}{3} & 2^{-9}\binom{9}{4} & 2^{-11}\binom{11}{5} & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 & -1 & 1 & \cdots \\ 0 & 2^2 & -2^2\binom{3}{1} & 2^2\binom{4}{2} & -2^2\binom{5}{3} & \cdots \\ 0 & 0 & 2^4 & -2^4\binom{5}{1} & 2^4\binom{6}{2} & \cdots \\ 0 & 0 & 0 & 2^6 & -2^6\binom{7}{1} & \cdots \\ 0 & 0 & 0 & 0 & 2^8 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdots \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 3 & 5 & 7 & 9\cdots \\ & 1 & 1 & 1 & 1\cdots \\ & & 1 & 1 & 1\cdots \\ & & & 1 & 1\cdots \\ & & & & 1\cdots \end{pmatrix}.$$

Such identities are established through the use of formal manipulations with power series.

Since it seems that these identities, as well as the method of proof, are interesting, the procedure used will be outlined. However, as most of the elements of the proof involve mathematical induction, and are rather long and tedious, details will be omitted.

We first obtain the determinant for D^2 by multiplying the ith column of D by the jth column of D. This gives a determinant whose general term is of the form

$$b_{ij} = s_{i+j} \qquad\qquad i, j = 0, 1, \cdots, m,$$

where $s_{i+j} = x_0^{i+j} + x_1^{i+j} + \cdots + x_m^{i+j}$.

Since the x's are symmetric with respect to the origin, $s_{i+j} = 0$ for $i+j$ odd. Hence

$$D^2 = \begin{vmatrix} s_0 & 0 & s_2 & 0 & s_4 \cdots \\ 0 & s_2 & 0 & s_4 & 0 \cdots \\ s_2 & 0 & s_4 & 0 & s_6 \cdots \\ 0 & s_4 & 0 & s_6 & 0 \cdots \\ s_4 & 0 & s_6 & 0 & s_8 \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \end{vmatrix},$$

or, upon rearranging the columns from the order $1, 3, 5, \cdots, 2, 4, \cdots$, to the order $1, 2, 3, 4, 5, \cdots$, followed by a corresponding row rearrangement, we obtain

$$D^2 = -\begin{vmatrix} s_0 & s_2 & s_4 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & s_2 & s_4 \\ s_2 & s_4 & s_6 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & s_4 & s_6 \\ s_4 & s_6 & s_8 & \cdots & 0 & 0 \\ \cdot & \cdot & \cdot & \cdots & & \cdot \\ \cdot & \cdot & \cdot & \cdots & & \cdot \end{vmatrix} = \begin{vmatrix} s_0 & s_2 & s_4 & \cdots & 0 & 0 & \cdots \\ s_2 & s_4 & s_6 & \cdots & 0 & 0 & \cdots \\ s_4 & s_6 & s_8 & \cdots & 0 & 0 & \cdots \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots \\ 0 & 0 & 0 & \cdots & s_2 & s_4 & \cdots \\ 0 & 0 & 0 & \cdots & s_4 & s_6 & \cdots \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots \end{vmatrix} = \begin{vmatrix} s_0 & s_2 & s_4 & \cdots \\ s_2 & s_4 & s_6 & \cdots \\ s_4 & s_6 & s_8 & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{vmatrix} \cdot \begin{vmatrix} s_2 & s_4 & \cdots \\ s_4 & s_6 & \cdots \\ \cdot & \cdot & \cdots \\ \cdot & \cdot & \cdots \end{vmatrix}.$$

To obtain a simple expression for s_{i+j}, we establish next that

$$2^{2n-1} \cos^{2n} x = a_0 + a_2 \cos 2x + \cdots + a_{2n} \cos 2nx, \tag{1}$$

where $a_i = a_i(n)$ and

$$a_0 = \binom{2n-1}{n-1}, \qquad a_{2n} = 1,$$

$$a_{2i} = \binom{2n-1}{n-1-i} + \binom{2n-1}{n-1+i}, \qquad\qquad i = 1, \cdots, n-1.$$

Using (1) and the standard identity

$$\cos 0y + \cos 1y + \cos 2y + \cdots + \cos my = \tfrac{1}{2}[1 + \sin (m + \tfrac{1}{2})y/\sin \tfrac{1}{2}y]$$

with $y=2\pi/m,\ 4\pi/m,\ \cdots,\ 2(m-1)\pi/m$, we find that

$$\begin{aligned} s_0 &= m + 1, \\ s_{2j} &= 1 + \binom{2j-1}{j-1} m/2^{2j-1}, \qquad j = 1, \cdots, m-1, \\ s_{2m} &= 1 + \left[\binom{2m-1}{m-1} + 1\right] m/2^{2m-1}. \end{aligned}$$

Noting the product expression for D^2 and the form of the elements involved therein, we consider first the sequence of determinants

$$\left| t_0 \right|, \qquad \begin{vmatrix} t_0 & t_2 \\ t_2 & t_4 \end{vmatrix}, \qquad \begin{vmatrix} t_0 & t_2 & t_4 \\ t_2 & t_4 & t_6 \\ t_4 & t_6 & t_8 \end{vmatrix}, \cdots,$$

where $t_0=1+x$, $t_{2j}=1+\binom{2j-1}{j-1}x/2^{2j-1}$, $j=1, 2, \cdots$. Since the pth determinant in this sequence has elements which reduce to 1 when $x=0$, its expression has a factor x^{p-1}.

We next find a set of constant multipliers $a_i=a_i(p)$ such that

$$(2) \qquad a_0t_q + a_2t_{q+2} + \cdots + a_{2p-2}t_{q+2p-2} = x + 2p - 1, \qquad q = 0, 2, \cdots, 2p-2,$$

proving that $x+2p-1$ is also a factor of the expression just referred to.

For each p, we show that the $a_i(p)$ are the first p numbers in the pth column of the infinite matrix appearing below:

$$\begin{pmatrix} (1+z)^{-1} \\ (4z)(1+z)^{-3} \\ (4z)^2(1+z)^{-5} \\ (4z)^3(1+z)^{-7} \\ \cdots\cdots \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 & -1 & \cdots \\ 0 & 4 & -\binom{3}{1}4 & \binom{4}{2}4 & \cdots \\ 0 & 0 & 4^2 & -\binom{5}{1}4^2 & \cdots \\ 0 & 0 & 0 & 4^3 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \end{pmatrix} \begin{pmatrix} 1 \\ z \\ z^2 \\ z^3 \\ \cdot \end{pmatrix}.$$

For it follows from this identity and the relation

$$(1-u)^{-1/2} = 1 + \binom{1}{0}u/2 + \binom{3}{1}u^2/2^3 + \binom{5}{2}u^3/2^5 + \cdots,$$

that

$$(1+z)^{-1} + \binom{1}{0}(4z)(1+z)^{-3}/2 + \binom{3}{1}(4z)^2(1+z)^{-5}/2^3 + \cdots$$

$$= [1 - (4z)(1 + z)^{-2}]^{-1/2}/(1 + z) = 1 + z + z^2 + \cdots$$

$$= 1 + \left[-1\cdot 1 + 4\cdot \binom{1}{0} \Big/ 2\right]z$$

$$+ \left[1\cdot 1 - \binom{3}{1} 4\cdot \binom{1}{0} \Big/ 2 + 4^2\cdot \binom{3}{1} \Big/ 2^3\right] z^2 + \cdots .$$

Comparing coefficients, we find that

$$a_0(1)t_0' = 1, \qquad a_0(2)t_0' + a_2(2)t_2' = 1,$$
$$a_0(3)t_0' + a_2(3)t_2' + a_4(3)t_4' = 1, \cdots,$$

where the prime denotes differentiation.

Similarly, using the same expansion for $(1-u)^{-1/2}-1$, we have

$$\binom{1}{0}(1 + z)^{-1}/2 + \binom{3}{1}(4z)(1 + z)^{-3}/2^3 + \cdots$$

$$= [(1 + z)/(4z)][\{1 - (4z)/(1 + z^2)\}^{-1/2} - 1] = \tfrac{1}{2} + z + z^2 + \cdots$$

$$= \binom{1}{0}\Big/ 2 + \left[-1\cdot \binom{1}{0}\Big/ 2 + 4\cdot \binom{3}{1}\Big/ 2^3\right] z + \cdots ;$$

or, beginning with the second coefficient,

$$a_0(2)t_2' + a_2(2)t_4' = 1, \qquad a_0(3)t_2' + a_2(3)t_4' + a_4(3)t_6' = 1, \cdots .$$

Continuing this process, using the expansion of $(1-u)^{-1/2}$ with the first q terms transposed as indicated above, we are led to an expression in which all coefficients after the first q are ones. This, in turn, leads to identities involving $a(q+1)$, $a(q+2)$, $\cdots$. To conclude the present part of the argument, we have

$$(1 + z)^{-1} + (4z)(1 + z)^{-3} + \cdots$$

$$= (1 + z)^{-1}/[1 - (4z)(1 + z)^{-2}] = \sum_{=1}^{\infty} (2j - 1)z^{j-1}$$

$$= 1 + (-1\cdot 1 + 4\cdot 1)z + \cdots ,$$

implying

$$a_0(1)t_0(0) = 1, \qquad a_0(2)t_0(0) + a_2(2)t_2(0) = 3,$$
$$a_0(3)t_0(0) + a_2(3)t_2(0) + a_4(3)t_4(0) = 5, \cdots .$$

Since $t_{2j}(0) = 1$, these relations are sufficient to prove (2).

Thus, the pth determinant in our sequence is of the form $C_p x^{p-1}(x+2p-1)$, since it is obviously a pth degree polynomial in x. To determine C_p, each determinant can be evaluated in terms of the one preceding it. For example, consider the third determinant. Multiplying the rows by the appropriate a's, and adding

to the last row, we obtain

$$C_3x^2(x+5) = \begin{vmatrix} t_0 & t_2 & t_4 \\ t_2 & t_4 & t_6 \\ t_4 & t_6 & t_8 \end{vmatrix} = (1/4^2)\begin{vmatrix} t_0 & t_2 & t_4 \\ t_2 & t_4 & t_6 \\ x+5 & x+5 & x+5 \end{vmatrix},$$

or

$$C_3x^2 = (1/4^2)\begin{vmatrix} t_0 & t_2 & t_4 \\ t_2 & t_4 & t_6 \\ 1 & 1 & 1 \end{vmatrix}.$$

Repeating the process using the columns, we find that

$$C_3x^2 = (1/4^2)^2\begin{vmatrix} t_0 & t_2 & x+5 \\ t_2 & t_4 & x+5 \\ 1 & 1 & 5 \end{vmatrix}$$

or that $C_3(-5)^2=(5)C_2(-5)^1(-5+3)/4^4$. For the general term, proceeding similarly, we obtain

$$C_p = 1/2^a, \quad \text{where} \begin{cases} a = 0, & p = 1, \\ a = \binom{2p-1}{2}, & p > 1. \end{cases}$$

The sequence

$$|t_2|, \quad \begin{vmatrix} t_2 & t_4 \\ t_4 & t_6 \end{vmatrix}, \cdots$$

is treated similarly. Multipliers, leading to a factor $x+2p$, appear in the matrix below:

$$\begin{Bmatrix} 2(1+z)^{-2} \\ 2^3z(1+z)^{-4} \\ 2^5z^2(1+z)^{-6} \\ 2^7z^3(1+z)^{-8} \\ \cdot \end{Bmatrix} = \begin{Bmatrix} 1 & -2\cdot 2 & 2.3 & -2.4 & \cdots \\ 0 & 2^3 & -2^3\binom{4}{1} & 2^3\binom{5}{2} & \cdots \\ 0 & 0 & 2^5 & -2^5\binom{6}{1} & \cdots \\ 0 & 0 & 0 & 2^7 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots \end{Bmatrix}\begin{Bmatrix} 1 \\ z \\ z^2 \\ z^3 \\ \cdot \end{Bmatrix}.$$

Following an identical procedure, using the series for $(1-u)^{-1/2}$, we find that the pth determinant is equal to $D_px^{p-1}(x+2p)$, where

$$D_p = 1/2^b, \qquad b = \binom{2p}{2}.$$

Combining these results, we have

$$T_{2p} = \begin{vmatrix} t_0 & 0 & t_2 & 0 \cdots \\ 0 & t_2 & 0 & t_4 \cdots \\ t_2 & 0 & t_4 & 0 \cdots \\ 0 & t_4 & 0 & t_6 \cdots \\ \cdot & \cdot & \cdot & \cdot \cdots \end{vmatrix} = x^{p-1}(x+p)(x+p+1)/2^e,$$

$p \geqq 1$, for a determinant of order $p+1$, where $e=p^2$.

Returning now to D^2 for general m and letting $x=m$, we find that D^2 can be decomposed, since all t's and s's are identical except that $s_{2m}=t_{2m}+x/2^{2m-1}$, into a sum of two T-determinants. Thus, assuming that $t_k=0$ for k odd,

$$\begin{aligned} D^2 &= \begin{vmatrix} t_0 \cdots t_m & \\ \cdot \cdots & \cdot \\ t_m \cdots t_{2m} + x/2^{2m+1} & \end{vmatrix} \\ &= \begin{vmatrix} t_0 \cdots t_m \\ \cdot \cdots\cdot \\ t_m \cdots t_{2m} \end{vmatrix} + \begin{vmatrix} t_0 \cdots t_{m-1} & 0 \\ \cdot \cdots \cdot & \cdot \\ t_m \cdots t_{2m-1} & x/2^{2m-1} \end{vmatrix} \\ &= T_{2m} + (x/2^{2m-1})T_{2m-2} = x^{m-1}(x+m)^2/2^{m^2-1} \\ &= m^{m+1}/2^{m^2-3}. \end{aligned}$$

For the Chebyshev subdivision $\theta_i=(2i-1)\pi/(2m)$, $i=1, \cdots, m$, using

$$\cos y + \cos 3y + \cdots + \cos (2m-1)y = (\sin 2my)/(2 \sin y)$$

with $y=\pi/m, 2\pi/m, \cdots, (m-1)\pi/m$, we can show that

$$s_0 = m, \qquad s_2 = \binom{1}{0} m/2, \cdots, s_{2m-2} = \binom{2m-3}{m-2} m/2^{2m-3}.$$

If we now divide each term in T_{2p} by x and then let $x \to \infty$, we obtain, associating $p+1$ with m,

$$\begin{vmatrix} 1 & 0 & \binom{1}{0}\Big/2 & \cdots \\ 0 & \binom{1}{0}\Big/2 & 0 & \cdots \\ \binom{1}{0}\Big/2 & 0 & \binom{3}{1}\Big/2 & \cdots \\ \cdot & \cdot & \cdot & \cdots \end{vmatrix} = 1/2^{(m-1)^2}.$$

Comparing these numbers with the formulas for the s's, we conclude that, for this case, $D^2=m^m/2^{(m-1)^2}$.

THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

L. E. BUSH, Kent State University

The following results of the nineteenth William Lowell Putnam Mathematical Competition held on November 22, 1958, have been determined in accordance with the constitution of the competition. This competition is supported by the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband and is held under the auspices of the Mathematical Association of America.

The first prize, five hundred dollars, is awarded to the Department of Mathematics of Harvard University, Cambridge, Massachusetts. The members of the team were Richard M. Dudley, Robert C. Hartshorne, and Stephen Lichtenbaum; to each of these a prize of fifty dollars is awarded.

The second prize, four hundred dollars, is awarded to the Department of Mathematics of the University of Toronto, Toronto, Ontario. The members of the team were David Brillinger, John Hewett, and Joseph Lipman; to each of these a prize of forty dollars is awarded.

The third prize, three hundred dollars, is awarded to the Department of Mathematics of the California Institute of Technology, Pasadena, California. The members of the team were Alfred W. Hales, Robert Jewett, and Thomas Morton; to each of these a prize of thirty dollars is awarded.

The fourth prize, two hundred dollars, is awarded to the Department of Mathematics of Cornell University, Ithaca, New York. The members of the team were Jim Bennett, Jack Newman, and Barbara Osofsky; to each of these a prize of twenty dollars is awarded.

The fifth prize, one hundred dollars, is awarded to the Department of Mathematics of the Polytechnic Institute or Brooklyn, Brooklyn, New York. The members of the team were George Glauberman, Martin Isaacs, and Donald Passman; to each of these a prize of ten dollars is awarded.

The five persons ranking highest in the examination, named in alphabetical order, are Alfred W. Hales, California Institute of Technology; Robert C. Hartshorne, Harvard University; John Hewett, University of Toronto; Joseph Lipman, University of Toronto; and Alan Gaisford Waterman, San Diego State College. Each of these will receive a prize of seventy-five dollars.

The five succeeding persons ranking highest in the examination, named in alphabetical order, are Stephen Adler, Harvard University; Richard M. Dudley, Harvard University; Samuel Klein, College of the City of New York; Stephen Lichtenbaum, Harvard University; and Gerald Stoller, Polytechnic Institute of Brooklyn. Each of these will receive a prize of thirty-five dollars.

The following teams, named in alphabetical order, won honorable mention: Case Institute of Technology, Cleveland, Ohio, the members of the team being Donald E. Knuth, William C. Lynch, and George W. Petznick; Columbia

College, New York City, the members of the team being Joseph D'Atri, Benjamin Bennett, and Paul B. Kantor; Massachusetts Institute of Technology, Cambridge, Massachusetts, the members of the team being David Carey, John Drinks, and Steven Scheinberg; University of California, Berkeley, California, the members of the team being Stefan A. Burr, Philip Read, and Floris Y. Tsang; and Yale University, New Haven, Connecticut, the members of the team being R. W. Beals, David Book, and Lee D. Goldberg.

Eleven individuals were given honorable mention. Their names, alphabetically arranged, are: R. W. Beals, Yale University; David Handel, California Institute of Technology; Robert Jewett, California Institute of Technology; John F. Kennison, Queens College (New York); Donald E. Knuth, Case Institute of Technology; Eugene Luks, College of the City of New York; Donald Olivier, Carleton College; Barbara Osofsky, Cornell University; Donald Passman, Polytechnic Institute of Brooklyn; Thomas Morton, California Institute of Technology; and Steven Scheinberg, Massachusetts Institute of Technology.

A total of six hundred and twenty-nine contestants from one hundred and nineteen institutions entered the competition this year. Of this number, one hundred and twenty-three individuals and four institutions were unable to compete, due to various reasons. Therefore, a total of five hundred and six undergraduates from one hundred and fifteen institutions actually took part in the competition.

The following is a list, alphabetically arranged, of all colleges and universities which entered teams in the competitition:

Agricultural and Mechanical College of Texas, Agricultural and Technical College of North Carolina, American University, Anna Maria College, Arizona State University, Atlanta University System, Belhaven College, Brooklyn College, Brown University, California Institute of Technology, Carleton College, Case Institute of Technology, Centenary College, Central Michigan College, College of Saint Catherine, College of Saint Thomas, College of the City of New York, College of the Holy Cross, Columbia College, Cornell University, Dartmouth College, Drew University, Eastern Baptist College, Eastern New Mexico University, Georgetown University, Georgia Institute of Technology, Harvard University, Humboldt State College, Iowa State College, Jamestown College, Kenyon College, Knox College, Lafayette College, Lebanon Valley College, Mankato State College, Massachusetts Institute of Technology, McGill University, Memphis State University, Mississippi State University, New Mexico College of Agriculture and Mechanics, Norwich University, Oberlin College, Oklahoma State University, Oregon State College, Pasadena College, Polytechnic Institute of Brooklyn, Pomona College, Princeton University, Purdue University, Queens College (Flushing, N. Y.), Queen's University (Kingston, Ontario), Radcliffe College, Reed College, Rutgers University, Saint Francis Xavier University, Saint Martins College, San Jose State College, Santa Clara University, Siena College, South Dakota School of Mines, Southwestern at Memphis, Stanford University, State College of Washington, Stevens Institute of Technology, Swarthmore College, The Cardinal Stritch College, The Ohio State University, The Rice Institute, Tufts University, United States Naval Aacdemy, University of Alberta, University of Arizona, University of British Columbia, University of California (Berkeley), University of California (Davis), University of California (Los Angeles), University of Colorado, University of Idaho, University of Illinois, University of Kansas, University of Manitoba, University of Michigan, University of Minnesota, University of Notre Dame, University of Oregon, University of Rochester, University of Santa Clara, University of the South, University of Texas, University of

Toronto, University of Washington, University of Western Ontario, University of Wisconsin, Wake Forest College, Washington University (St. Louis), Wayne State University, Wesleyan University, Yale University, and Yeshiva College.

The following colleges and universities, alphabetically arranged, entered individual contestants only:

Aurora College, Brandeis University, Colorado College, Duquesne University, Haverford College, Kent State University, Lincoln Memorial University, Northwestern University, Rocky Mountain College, Sacramento State College, Saint Mary-of-the-Wasatch, Saint Olaf College, San Diego State College, University of Chicago, University of Cincinnati, University of Florida, University of Houston, University of Kentucky, University of New Hampshire, University of Ottawa, and Wofford College.

The individual ranks of contestants (except for the relative ranks of the first five) may be obtained by any department of mathematics for the purpose of selecting graduate students.

The problems given to those participating in the competition, together with a write-up of the solutions to the problems, will appear in a later issue of the MONTHLY.

MATHEMATICAL NOTES

EDITED BY ROY DUBISCH, Fresno State College

Material for this department should be sent to Roy Dubisch, Department of Mathematics, University of California, Berkeley 4, California.

EXPONENTIATION OF PERMUTATION GROUPS*

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1. In order to formulate a problem in the enumeration of linear graphs [1], we have defined the "exponentiation" $\mathfrak{B}^{\mathfrak{A}}$ and the "cartesian product" $\mathfrak{A}\times\mathfrak{B}$ of two permutation groups $\mathfrak{A}$ and $\mathfrak{B}$. Pólya [2] introduced the "Gruppenkranz" $\mathfrak{A}[\mathfrak{B}]$, or the "composition." The "direct sum" $\mathfrak{A}+\mathfrak{B}$, often also called the "direct product," is classical. Our object is to set forth some of the rules which govern the interaction of these operations with each other.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be permutation groups with disjoint object sets X and Y, degrees a and b, and orders m and n, respectively. We say that $\mathfrak{A}$ and $\mathfrak{B}$ are *abstractly isomorphic* and write $\mathfrak{A}\cong\mathfrak{B}$ when they are isomorphic regarded as abstract groups. They are *permutationally equivalent* (or *permutationally isomorphic*) if they are abstractly isomorphic and there is a 1-1 correspondence $f: X\leftrightarrow Y$ such that if θ is the abstract isomorphism between $\mathfrak{A}$ and $\mathfrak{B}$, then for all $x\in X$, $\alpha\in\mathfrak{A}$, we have $f(\alpha x)=(\theta\alpha)f(x)$. In this case we write $\mathfrak{A}=\mathfrak{B}$. Thus in particular $a=b$ whenever $\mathfrak{A}=\mathfrak{B}$.

* This work was supported by a grant from the Office of Naval Research.

The *direct sum* $\mathfrak{A}+\mathfrak{B}$ has $X\cup Y$ as its object set and its elements are all the permutation products $\alpha\beta$ obtained by juxtaposition of the permutations in $\mathfrak{A}$ with those in $\mathfrak{B}$.

The *cartesian product* $\mathfrak{A}\times\mathfrak{B}$ has $X\times Y$ as its object set and its elements (α, β) act on its objects (x, y) in accordance with the equation $(\alpha, \beta)(x, y) = (\alpha x, \beta y)$. Thus $\mathfrak{A}+\mathfrak{B}\cong\mathfrak{A}\times\mathfrak{B}$ but $\mathfrak{A}+\mathfrak{B}\neq\mathfrak{A}\times\mathfrak{B}$.

The *composition* $\mathfrak{A}[\mathfrak{B}]$ also has $X\times Y$ as its object set, but it is more convenient to regard the objects as the entries in an a by b matrix (x_{ij}). The elements of $\mathfrak{A}[\mathfrak{B}]$ are constructed by first permuting the matrix rows by an element of $\mathfrak{A}$ and then permuting the column indices in each of the a rows separately by an element of $\mathfrak{B}$, repetitions permitted.

The *exponentiation* $\mathfrak{B}^{\mathfrak{A}}$ has as object set Y^X, the collection of all functions from X into Y. The elements of $\mathfrak{B}^{\mathfrak{A}}$ are constructed by first permuting the domain X using a permutation in $\mathfrak{A}$ and then permuting the image objects for each domain object by elements of $\mathfrak{B}$, not necessarily distinct. It follows at once that $\mathfrak{B}^{\mathfrak{A}}\cong\mathfrak{A}[\mathfrak{B}]$ but in general $\mathfrak{B}^{\mathfrak{A}}\neq\mathfrak{A}[\mathfrak{B}]$.

More precisely, let $X=\{x_1, \cdots, x_a\}$, f be any function from X into Y, $\alpha\in\mathfrak{A}$, and $\beta_1, \cdots, \beta_a\in\mathfrak{B}$ (repetitions permitted). Then the following permutation γ of the exponentiation group $\mathfrak{B}^{\mathfrak{A}}$ is induced: $\gamma f=f^*$, where $f^*(x_i)=\beta_i f(\alpha x_i)$.

These definitions are summarized in Table 1.

TABLE 1

			Sum	Product	Composition	Exponentiation
Group	$\mathfrak{A}$	$\mathfrak{B}$	$\mathfrak{A}+\mathfrak{B}$	$\mathfrak{A}\times\mathfrak{B}$	$\mathfrak{A}[\mathfrak{B}]$	$\mathfrak{B}^{\mathfrak{A}}$
Object Set	X	Y	$X\cup Y$	$X\times Y$	$X\times Y$	Y^X
Degree	a	b	$a+b$	ab	ab	b^a
Order	m	n	mn	mn	mn^a	mn^a

2. We now find some permutational equivalences between certain combinations of these operations. Let $\mathfrak{S}_n$ be the symmetric group of degree n. It is convenient to consider not only $\mathfrak{S}_1$, the group of degree 1 and order 1, but also $\mathfrak{S}_0$, the group of degree 0 and order 1. Let $\mathcal{P}$ be the collection of all permutation groups. We omit the proof of the following theorem, which is immediate.

THEOREM 1. (a) *$\mathcal{P}$ is closed under sum, product, composition, and exponentiation.*

(b) *For all permutation groups $\mathfrak{A}$, the following identity rules hold*:

$$\mathfrak{A}+\mathfrak{S}_0=\mathfrak{A}, \qquad \mathfrak{A}\times\mathfrak{S}_1=\mathfrak{A}, \tag{1}$$
$$\mathfrak{A}[\mathfrak{S}_1]=\mathfrak{A}, \qquad \mathfrak{S}_1[\mathfrak{A}]=\mathfrak{A}, \qquad \mathfrak{A}^{\mathfrak{S}_1}=\mathfrak{A}.$$

(c) *The following commutativity laws hold*:

$$\mathfrak{A}+\mathfrak{B}=\mathfrak{B}+\mathfrak{A}, \qquad \mathfrak{A}\times\mathfrak{B}=\mathfrak{B}\times\mathfrak{A}. \tag{2}$$

(d) *The following associativity laws hold:*

$$\mathfrak{A} + (\mathfrak{B} + \mathfrak{C}) = (\mathfrak{A} + \mathfrak{B}) + \mathfrak{C}, \qquad \mathfrak{A} \times (\mathfrak{B} \times \mathfrak{C}) = (\mathfrak{A} \times \mathfrak{B}) \times \mathfrak{C},$$
$$\mathfrak{A}[\mathfrak{B}[\mathfrak{C}]] = (\mathfrak{A}[\mathfrak{B}])[\mathfrak{C}]. \tag{3}$$

We note, using Table 1, *that the orders of* $\mathfrak{A}[\mathfrak{B}]$ *and* $\mathfrak{B}[\mathfrak{A}]$ *are different so that* $\mathfrak{A}[\mathfrak{B}] \not\simeq \mathfrak{B}[\mathfrak{A}]$.

THEOREM 2. *Among the operations of sum, product, and composition, the only distributive law which holds is:*

$$(\mathfrak{A} + \mathfrak{B})[\mathfrak{C}] = \mathfrak{A}[\mathfrak{C}] + \mathfrak{B}[\mathfrak{C}]. \tag{4}$$

Proof. The fact that none of the other possible combinations yields permutational equivalence follows readily from Table 1 by either an order argument or a degree argument. For example, if c is the degree of $\mathfrak{C}$, then the degree of $(\mathfrak{A} \times \mathfrak{B})[\mathfrak{C}]$ is abc while the degree of $\mathfrak{A}[\mathfrak{C}] \times \mathfrak{B}[\mathfrak{C}]$ is abc^2.

To prove the assertion, consider the partitioned matrix:

$$M = \begin{bmatrix} M_1 \\ M_2 \end{bmatrix},$$

where

$$M_1 = \begin{bmatrix} x_{11} & \cdots & x_{1c} \\ \vdots & \cdots & \vdots \\ x_{a1} & \cdots & x_{ac} \end{bmatrix}, \qquad M_2 = \begin{bmatrix} y_{11} & \cdots & y_{1c} \\ \vdots & \cdots & \vdots \\ y_{b1} & \cdots & y_{bc} \end{bmatrix}.$$

Not only are the orders and the degrees of the two sides of (4) equal, but the above matrix M gives the 1-1 correspondence between their object sets under which they are permutationally equivalent. For the permutations in $\mathfrak{A}[\mathfrak{C}]$ act on the entries of submatrix M_1 while those in $\mathfrak{B}[\mathfrak{C}]$ act on M_2, and these transformations are independent.

We next turn to combinations of exponentiation with the other operations, and find the following two rules which are believed to be the only ones.

THEOREM 3. *The following two rules involving exponentiation hold:*

$$\mathfrak{A}^{\mathfrak{B}+\mathfrak{C}} = \mathfrak{A}^{\mathfrak{B}} \times \mathfrak{A}^{\mathfrak{C}}, \tag{5}$$

$$(\mathfrak{A}^{\mathfrak{B}})^{\mathfrak{C}} = \mathfrak{A}^{\mathfrak{C}[\mathfrak{B}]}. \tag{6}$$

It is immediately seen that if t is the order of $\mathfrak{C}$, then

$$ord\ \mathfrak{A}^{\mathfrak{B}+\mathfrak{C}} = ord\ \mathfrak{A}^{\mathfrak{B}} \times \mathfrak{A}^{\mathfrak{C}} = ntm^{b+c},$$

and

$$ord\ (\mathfrak{A}^{\mathfrak{B}})^{\mathfrak{C}} = ord\ \mathfrak{A}^{\mathfrak{C}[\mathfrak{B}]} = tn^c m^{bc}.$$

The permutational equivalences are readily verified.

The rules (1)–(6) lend plausibility to the notations used here for these group operations.

3. A special case of the operation of exponentiation of two permutation groups has already appeared in the literature in connection with the number of types of Boolean functions. For Pólya [**3**] notes that these types are representable as the equivalence classes of collections of vertices of the n-cube with regard to its automorphism group $\mathfrak{Q}_n$. In fact Pólya ([**3**] footnote 7) observes that $\mathfrak{Q}_n \cong \mathfrak{S}_n[\mathfrak{S}_2]$. But deg $\mathfrak{Q}_n = 2^n$ while deg $\mathfrak{S}_n[\mathfrak{S}_2] = 2n$. Slepian [**4**] uses the fact that $\mathfrak{Q}_n = \mathfrak{S}_2^{\mathfrak{S}_n}$ to obtain an explicit solution for this interesting enumeration problem. The only other application of exponentiation known to us occurs in [**1**] where the group $\mathfrak{S}_n^{\mathfrak{S}_2}$ occurs in connection with the enumeration of a certain class of bicolored graphs.

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3. ———, Sur les types des propositions composées, J. Symb. Logic, vol. 5, 1940, pp. 98–103.

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VOLUME IN VECTOR SPACES

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This note is concerned with the determinantal formula for the volume of a parallelepiped in a finite-dimensional real vector space. We shall show that the formula can be deduced from some very simple axioms concerning "volume." In fact we consider a nonnegative set-function μ, defined on a suitable class of sets, and we assume only that μ is additive and translation-invariant: these properties determine the value $\mu(P)$, apart from a multiplicative constant, for any parallelepiped P. The proof is quite straightforward, and involves only the affine structure of the space (the notion of length is not required). It can be used to ease a traditional difficulty in expounding the theory of Lebesgue measure for a general Euclidean space; namely the difficulty of showing that congruent sets have equal measure.

In an n-dimensional real vector space, a parallelepiped with a vertex at the origin is determined by n linearly independent vectors $x_1, \cdots, x_n$. These vectors being given, we consider the set P of all vectors $\lambda_1 x_1 + \cdots + \lambda_n x_n$, where $0 \leqq \lambda_i < 1$ for $i = 1, \cdots, n$. We denote by $\delta(P)$ the determinant of the $n \times n$ matrix (ξ_j^i), where $\xi_j^1, \cdots, \xi_j^n$ are the coordinates of x_j, for $j = 1, \cdots, n$, relative to a given basis (which we keep fixed throughout the discussion).

Suppose that $\mathcal{R}$ is a class of sets having the following properties:

(1) P belongs to $\mathcal{R}$, for any choice of $x_1, \cdots, x_n$, including the degenerate cases in which one of these vectors is zero;

(2) if $\mathcal{R}$ contains sets E and H then it contains the union $E\cup H$ and the relative complement $E\backslash H$, and hence also the intersection $E\cap H$;

(3) if $\mathcal{R}$ contains E then it contains all translates of E (that is, all sets of the type $E+x$, where x is an arbitrary vector).

Let μ be a nonnegative function defined on $\mathcal{R}$ with the property that if E and H belong to $\mathcal{R}$ and have empty intersection then

$$\mu(E\cup H) = \mu(E) + \mu(H).$$

Suppose also that if E belongs to $\mathcal{R}$ and x is any vector then $\mu(E+x)=\mu(E)$. (For example, $\mathcal{R}$ could consist of all the Borel sets, and μ could be Lebesgue measure.) We shall show that there is a number κ, independent of P, such that $\mu(P)=\kappa|\delta(P)|$.

When $i\neq j$, let P_{ij} be the set obtained by replacing x_i, in the definition of P, by x_i+x_j; this is equivalent to replacing λ_j by $\lambda_i+\lambda_j$. Let $A=P\backslash P_{ij}$, $B=P\cap P_{ij}$, $C=P_{ij}\backslash P$. According to our assumptions, these sets belong to $\mathcal{R}$ and

$$\mu(P) = \mu(A\cup B) = \mu(A) + \mu(B),$$
$$\mu(P_{ij}) = \mu(B\cup C) = \mu(B) + \mu(C).$$

It is easy to see that A is the set obtained from the definition of P by imposing the restriction that $\lambda_i>\lambda_j$ (since λ_j can be expressed as $\lambda_i+\lambda_j'$, with $0\leqq\lambda_j'<1$, if and only if $\lambda_i<\lambda_j$). Similarly, C is the set obtained from the definition of P by replacing λ_j by $\lambda_i+\lambda_j$ and imposing the restriction that $\lambda_i+\lambda_j\geqq 1$; this is equivalent to replacing λ_j by $\lambda_j'+1$, where $0\leqq\lambda_j'<\lambda_i$. Thus $C=A+x_j$. Hence $\mu(C)=\mu(A)$, and therefore $\mu(P_{ij})=\mu(P)$.

Next, for each i and each real number λ, let $P_i^{(\lambda)}$ be the set obtained by replacing x_i, in the definition of P, by λx_i. Evidently, $P_i^{(\lambda)}$ belongs to $\mathcal{R}$ if $\lambda\geqq 0$. If p and q are positive integers then

$$P_i^{(p/q)} = P_i^{(1/q)} \cup \left(P_i^{(1/q)} + \frac{1}{q}x_i\right) \cup \cdots \cup \left(P_i^{(1/q)} + \frac{p-1}{q}x_i\right),$$

so that $\mu(P_i^{(p/q)})=p\mu(P_i^{(1/q)})$. In particular,

$$\mu(P) = \mu(P_i^{(1)}) = q\mu(P_i^{(1/q)}).$$

Hence $\mu(P_i^{(p/q)})=(p/q)\mu(P)$. From this it follows in an obvious way (since μ is monotonic) that $\mu(P_i^{(\lambda)})=\lambda\mu(P)$ for any $\lambda\geqq 0$. We also have

$$(P_i^{(-1)} + x_i) \cup P_i^{(0)} = P \cup (P_i^{(0)} + x_i),$$

so that $P_i^{(-1)}$ belongs to $\mathcal{R}$, and $\mu(P_i^{(-1)})=\mu(P)$. Hence $P_i^{(\lambda)}$ belongs to $\mathcal{R}$ for every value of λ, and $\mu(P_i^{(\lambda)})=|\lambda|\mu(P)$.

It is now clear that $\mu(P)$, regarded as a function of the matrix (ξ_j^i), has the following properties: (i) its value is unaffected if any one column is added to

another; (ii) if any column is multiplied by λ then the value of the function is multiplied by $|\lambda|$. Hence if

$$f(P) = \begin{cases} \mu(P) & \text{when} \quad \delta(P) \geqq 0, \\ -\mu(P) & \text{when} \quad \delta(P) < 0, \end{cases}$$

then $f(P)$ has the property (i) and also the property that its value is multiplied by λ if any column of the matrix (ξ_j^i) is multiplied by λ. Now it is well known (and easy to prove*) that the only functions with these two properties are the constant multiples of $\delta(P)$. Thus, since $\mu(P) = |f(P)|$,

$$\mu(P) = \kappa|\delta(P)|,$$

where κ is the value of $\mu(P)$ when (ξ_j^i) is the unit matrix (so that $\kappa = 1$ in the case of Lebesgue measure). Since the determinant of a product of square matrices is the product of their determinants, a change of basis can only affect the value of κ.

Lebesgue measure in a Euclidean space is usually defined in terms of coverings by parallelepipeds whose edges are parallel to the vectors of a fixed orthogonal basis. It is then obvious that the measure is invariant under translations. In the light of the result we have just proved, it is almost equally obvious that the measure is invariant under general rigid displacements; or, equivalently, that it is independent of the particular orthogonal basis which is chosen for the definition. In fact it is easy to see that, more generally, if a set is subjected to any linear transformation then its Lebesgue outer measure is multiplied by the modulus of the determinant of the transformation.

*See, for example, L. Mirsky, An Introduction to Linear Algebra, Oxford, 1955, pp. 190–191.

CONVEX SETS WITH DENSE EXTREME POINTS

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Let L denote a linear space and let C be a convex subset of L. A point x_0 is called an extreme point of C if

(1) $x_0 \in C$ and

(2) no segment $\{x \mid x = x_0 + ty,\ -1 \leqq t \leqq 1\}$ with $0 \neq y \in L$ is contained in C.

If L has a suitable topology (*e.g.* if L is a Banach space) a celebrated theorem by Krein-Milman [1] states that if C is a compact convex set, then C is the closed convex hull of the set of its extreme points. (The closed convex hull of a set S is the smallest closed convex set containing S.)

The following problem was proposed to the author by W. G. Bade: Does there exist a convex set (other than a single point) which is the *closure* of the set of its extreme points. It appears to be widely known that the answer is affirmative, but apparently no example has been given in the literature, and therefore it seems to be worthwhile to publish the following simple construction.

THEOREM. *Let l^2 be the real Hilbert space of sequences $x=(x_1, \cdots, x_i, \cdots)$ with $\sum_{i=1}^{\infty} x_i^2 < \infty$. Let C denote the subset of l^2 determined by $\sum_{i=1}^{\infty}(2^i x_i)^2 \leqq 1$. Then C is a convex set, and C is the closure of its set of extreme points.*

Proof. Let E^n denote the subspace of l^2 determined by $x_{n+1}=x_{n+2}= \cdots =0$ and let P_n denote the projection on E^n, *i.e.*, if $x = (x_1, x_2, \cdots)$, then $P_n x=(x_1, \cdots, x_n, 0, 0, \cdots)$. Then we clearly have $C\cap E^n=P_n(C)=$ the ellipsoid determined by the inequality

$$\frac{x_1^2}{(2^{-1})^2}+\frac{x_2^2}{(2^{-2})^2}+\cdots+\frac{x_n^2}{(2^{-n})^2} \leqq 1.$$

Define $C_n=C\cap E^n$; it is then clear that when we consider C_n as a subset of E^n its extreme points are the points on the boundary B_n of C_n. Furthermore all points of B_n are also extreme points of C, for if $x_0 \in B_n$ and $0\neq y \in l^2$ there exists an $m \geqq n$ such that $P_m y \neq 0$. Now $x_0 \in B_n \subset B_m$; hence x_0 is an extreme point of C_m and $x_0+t\cdot P_m y=P_m(x_0+ty)$ does not belong to $C_m=P_m(C)$ for all t with $-1 \leqq t \leqq 1$. Consequently x_0+ty does not belong to C for all t with $-1 \leqq t \leqq 1$, and, since y is arbitrary, x_0 is an extreme point of C.

Define $B=\bigcup_{n=1}^{\infty} B_n$; then B is contained in the set of extreme points of C, and since C is obviously closed the theorem will be proved when we show that B is dense in C. Let $x\in C$; then $P_n x \in C_n$ and $\|x-P_n x\| \leqq 2^{-n-1}+2^{-n-2}+ \cdots =2^{-n}$. Since the smallest semi-axis of the ellipsoid C_n has length 2^{-n}, every point of C_n has distance at most 2^{-n} from B_n. Therefore every point of C has distance at most $2^{-n}+2^{-n}=2^{-n+1}$ from B_n, and the theorem follows.

It should be noted that the above construction can be carried out in any Banach space with a basis [2]. Also note that although the set C constructed above is compact this is by no means necessary—the paraboloid $x_1 \geqq \sum_{i=2}^{\infty} (2^i x_i)^2$ is another example which is even unbounded.

A somewhat more complicated example of a convex set which is the closure of the set of its vertices [3] can also be constructed. To do this, we construct an ascending sequence of convex polyhedra $H_n \subset E^n$ with properties similar to the properties of the C_n above and such that $\bigcup_{n=1}^{\infty} V_n$ is dense in the closure of $\bigcup_{n=1}^{\infty} H_n$, where V_n denotes the set of vertices of H_n.

Added in proof. V. L. Klee, Jr., has proved that in every infinite-dimensional Banach space, "most" compact convex sets are the closure of their set of extreme points. It is planned to publish this result in a paper entitled *Some new results on smoothness and rotundity in normed linear spaces.*

References

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3. H. F. Bohnenblust and S. Karlin, Geometrical properties of the unit sphere of Banach algebras, Ann. of Math., vol. 62, 1955, pp. 217–229.

CLASSROOM NOTES

EDITED BY C. O. OAKLEY, Haverford College

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THE LAW OF THE MEAN

R. C. YATES, The College of William and Mary

In discussing the Law of the Mean, we consider the function

$$\phi(x) = f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a).$$

This is a formidable expression whose origin is puzzling until it is pointed out as the difference PQ of the ordinate of a point on the graph of $f(x)$ and the ordinate to the secant line for the same x.

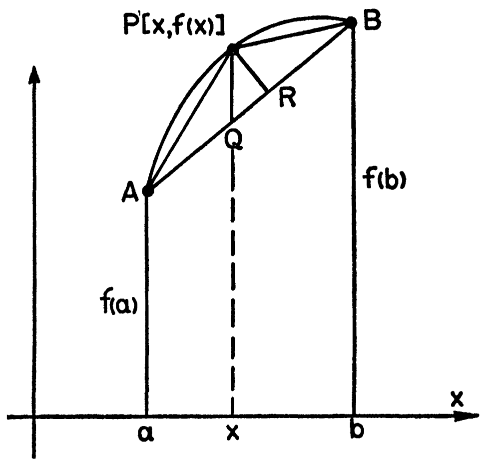


FIG. 1

A different approach results from selecting PR, the perpendicular from P to the secant (Fig. 1). Now $\sin \sphericalangle PQR = PR/PQ$ and, since $\sphericalangle PQR$ is a constant angle for all values x in the interval, then PR is proportional to PQ throughout and thus has all the desired properties of $\phi(x)$. But this variable quantity PR is the altitude of the triangle APB whose base is the chord AB. Thus $\phi(x)$ is also proportional to the area K of triangle APB. Accordingly, instead of $\phi(x)$, we might as well consider:

$$2K = \begin{vmatrix} x & f(x) & 1 \\ a & f(a) & 1 \\ b & f(b) & 1 \end{vmatrix}.$$

The derivative of this variable-area function must vanish by Rolle's theorem at some $x=X$ between a and b. That is,

$$\begin{vmatrix} 1 & f'(X) & 0 \\ a & f(a) & 1 \\ b & f(b) & 1 \end{vmatrix} = 0,$$

or

$$f(a) - f(b) - f'(X)(a - b) = 0.$$

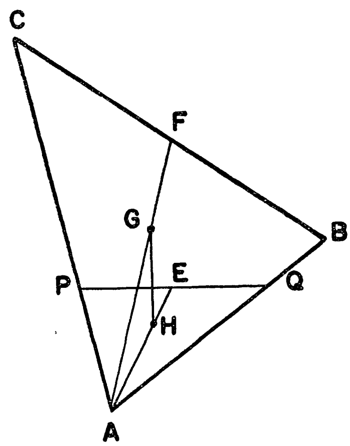
Thus

$$(b - a)f'(X) = f(b) - f(a).$$

ON EQUILIBRIUM OF A TRIANGULAR PRISM FLOATING FREELY WITH ITS EDGES HORIZONTAL

R. S. L. SRIVASTAVA, Lucknow Christian College, Lucknow, India

This classical problem has been discussed by Besant and Ramsey [1] and their investigation is based on the methods of ordinary algebra and geometry. In the present note the problem has been treated by the method of vector algebra. It will be noted that the method given here can be adopted with success in solving many a classical problem on equilibrium of floating bodies which fulfill the well-known conditions of flotation, *viz.*, (i) the centre of gravity of the body and that of the liquid displaced by it should be in the same vertical line and (ii) the weight of the body should equal the weight of the liquid displaced.



We shall adopt the nomenclature and the symbols as used by Besant and Ramsey.

Let ABC represent the section of the triangular prism by a vertical plane

through its centre of gravity G, PQ the line of flotation, *i.e.*, the line of intersection of the vertical plane and the horizontal plane surface of the liquid. Also, let H be the centre of gravity of the displaced liquid. Then, in view of condition (i), GH is perpendicular to PQ. Therefore the scalar product of the vectors $\overline{PQ}$ and $\overline{GH}$ is zero, *i.e.*,

$$\overline{PQ}\cdot\overline{GH} = 0. \tag{1}$$

Again, since the volumes of the prism and the liquid displaced are proportional to the areas ABC and AQP respectively, from condition (ii) we have

$$(\overline{AQ} \times \overline{AP})\rho = (\overline{AB} \times \overline{AC})\sigma, \tag{2}$$

where ρ and σ are the densities of the liquid and the prism respectively.

The vector equations (1) and (2) determine completely the different positions of equilibrium of the floating prism.

Since $\overline{PQ}=\overline{AQ}-\overline{AP}$ and $\overline{GH}=\overline{AH}-\overline{AG}$, we have from (1)

$$(\overline{AQ} - \overline{AP})\cdot(\overline{AH} - \overline{AG}) = 0$$

or,

$$(\overline{AQ} - \overline{AP})\cdot\left\{\tfrac{1}{3}(\overline{AQ} + \overline{AP}) - \tfrac{1}{3}(\overline{AB} + \overline{AC})\right\} = 0$$

or,

$$AQ^2 - AQ\cdot AB - AQ\cdot AC\cos\theta - AP^2 + AP\cdot AB\cos\theta + AP\cdot AC = 0,$$

where $\angle BAC=\theta$.

On taking $AB=2a$, $AC=2b$, $AQ=2x$, $AP=2y$, the above relation becomes

$$x^2 - y^2 - (a + b\cos\theta)x + (b + a\cos\theta)y = 0, \tag{3}$$

which is the equation of a rectangular hyperbola referred to conjugate diameters parallel to AB and AC as axes, (x, y) being the coordinates of E, the middle point of PQ. Thus the point E lies on the curve (3) when the prism floats freely in the liquid.

Also, equation (2) reduces to

$$xy = ab\sigma/\rho = c^2 \tag{4}$$

(say), which is also the equation of a rectangular hyperbola satisfied by the coördinates of the point E.

Equations (3) and (4) are the cartesian equations deduced from the vector equations (1) and (2) and are the same as obtained by Besant and Ramsey. Taken together they determine all the positions of equilibrium of the prism.

Reference

1. W. H. Besant and A. S. Ramsey, A Treatise on Hydromechanics, Pt. 1 (Hydrostatics), London, 1904, p. 45.

SOME PROPERTIES OF ARITHMETIC PROGRESSIONS

K. SUBBA RAO, Maharajah's College, Vizianagram, India

It has been stated [1] that "the product of four consecutive terms of an arithmetic progression of integers plus the fourth power of the common difference is a perfect square but in no case a perfect fourth power." It has been shown [2] by me and several others that the first part of the problem follows immediately and that the second part is false. In fact, if $(a-3d)$ be the first term and $2d$ the common difference of the A.P. of integers, we have

$$(a-3d)(a-d)(a+d)(a+3d)+16d^4=(a^2-5d^2)^2. \tag{1}$$

Now (a^2-5d^2) can be made equal to a perfect nth power, by the method of Euler and Lagrange [3]. Hence (1) can be made equal to a perfect $2n$th power.

Some analogous results are known [4] but results of a general nature on this topic seem to be new. I consider the solution of some Diophantine equations involving integers in A.P., whose first term is not an integral multiple of the common difference.

THEOREM 1. *The Diophantine equation*

$$x_1^2+\cdots+x_{2m+1}^2=(2m+1)z^n,$$

where $x_1, \cdots, x_{2m+1}$ *are integers in A.P., and* z, n *are integers, has infinitely many solutions.*

Proof. Setting $x_1=(a-3md)$, $x_2=\{a-3(m-1)d\}$, $\cdots$ we have

$$x_1^2+\cdots+x_{2m+1}^2=(2m+1)(a^2+kd^2), \quad \text{where} \quad k=3m(m+1).$$

Now, by the method of Euler and Lagrange cited above, (a^2+kd^2) can be made a perfect nth power and the result follows.

COROLLARY. *If* $(2m+1)$ *is a perfect square* $=q^2$ *say, then*

$$x_1^2+\cdots+x_{2m+1}^2=q^2(a^2+kd^2)=x^2+ky^2, \quad \textit{where} \quad x=qa,\ y=qd;$$

and x^2+ky^2 *can be made a perfect nth power.*

THEOREM 2. *The Diophantine equation*

$$x_1^2+x_2^2+\cdots+x_{2m}^2=2mz^n,$$

where $x_1, \cdots, x_{2m}$ *are integers in A.P. and* z, n *are integers, has infinitely many solutions.*

Proof. Setting $x_1=\{a-3(2m-1)d\}$, $x_2=\{a-3(2m-3)d\}$, $\cdots$, and proceeding as in 1 above, we prove the result.

COROLLARY. *If 2m is a perfect square, then* $\sum_1^{2m} x_i^2$ *can be made a perfect nth power.*

THEOREM 3. *The Diophantine equation*

$$x_1^2 + \cdots + x_m^2 = z^{2n+1},$$

where $x_1, \cdots, x_m$ *are integers in A.P. and* z, n *are integers, has infinitely many solutions.*

Proof. I consider two cases.

Case I. Let m be even and $=2p$.

Let $x_1 = \{a-(2p-1)3d\}$, $x_2 = \{a-(2p-3)3d\}$ $\cdots$. Then $x_1^2 + \cdots + x_m^2 = 2p(a^2+kd^2)$, where

$$k = 3(4p^2 - 1).$$

Set $a=(2p)^n x(x^2+ky^2)^n$, $d=(2p)^n y(x^2+ky^2)^n$, where x and y are integers and $(x, y)=1$. Then $\sum_1^m x_i^2 = z^{2n+1}$, where $z=2p(x^2+ky^2)$, and the theorem is proved.

Case II. Let m be odd and $=(2p+1)$. By proceeding as in Case I, the result follows.

THEOREM 4. *The Diophantine equation*

$$x_1^3 + \cdots + x_m^3 = z^n,$$

where $x_1, \cdots, x_m$ *are integers in A.P., z an integer and n is an integer of the form* $(3s+1)$ *or* $(3s+2)$, *has infinitely many solutions.*

Proof. If m be even and $=2p$, by setting

$$x_1 = \{a - (2p - 1)d\}, \qquad x_2 = \{a - (2p - 3)d\}, \cdots,$$

we have $x_1^3 + \cdots + x_m^3 = 2p \cdot a(a^2+kd^2)$, where $k=4p^2-1$. If we put $a = x(x^2+ky^2)^s$, $d=y(x^2+ky^2)^s$, where x, y, s are integers and $(x, y)=1$, we have

$$\sum_1^m x_i^3 = 2px(x^2 + ky^2)^{3s+1}.$$

On replacing x by $(2p)^{3s}x^{3s+1}$, we can make

$$\sum_1^m x_i^3 = \text{the } (3s + 1)\text{th power of an integer.}$$

Again, setting $a=x(x^2+ky^2)^{2s+1}$, $d=y(x^2+ky^2)^{2s+1}$ and proceeding as above, we can make

$$\sum x_i^3 \text{ equal to the } (3s + 2)\text{th power of an integer.}$$

The method is similar, when m is odd.

Note. Some special cases of this result, when $n=2$, 3, 4, are to be found in Chapters XXI and XXII of [**4**].

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1. E876, this MONTHLY, vol. 56, 1949, p. 473.
2. E876, this MONTHLY, vol. 57, 1950, pp. 186–187.
3. L. E. Dickson, Introduction to the Theory of Numbers, University of Chicago Press, 1929, 96–97.
4. ———, History of the Theory of Numbers, Washington, 1919, pp. 320–323.

MATHEMATICAL EDUCATIONAL NOTES

EDITED BY JOHN A. BROWN, University of Delaware, AND
JOHN R. MAYOR, AAAS and University of Maryland

All material for this department should be sent to John R. Mayor, 1515 Massachusetts Avenue, N.W., Washington 5, D. C.

WHAT TO DO ABOUT A NEW KIND OF FRESHMAN*

J. H. NEELLEY, Carnegie Institute of Technology

At Carnegie Tech last fall we had our first freshmen who had had a calculus course in high school. They were generally bright young people, but tests showed that they merely had some formulas. They knew not the why of calculus. However, some were permitted to enter second semester courses in analytics and calculus. This we did not like to do. It was bad, very bad, to put these bright ambitious students in classes with the failures of the previous year. They had to get a completely wrong idea of college work from these flunkers.

Others were not permitted to omit the first semester. Such decisions were made after the students were given individual examinations. These too were unhappy as they had been told in high school that they would have advanced credit. The fact is that no such student gets any college credit at Carnegie Tech. Their high school courses are of the simplest possible types, due to teachers of little experience in higher mathematics and to the choices of the most innocuous texts available. Hence no credit can be given. Those who start in the second semester courses merely have to take something to replace the hours toward a degree which were omitted.

At first those who were not given advanced standing were sure they had not gotten a square deal. Then they came to me later in the semester with statements

* Presented to Allegheny Mountain Section of M.A.A., May, 1958.

like, "I now know why and not merely how," or "I would have missed a lot if I had been allowed to start with the second semester." It seems to me that such statements are very significant. They definitely point to the possible futility of high school-taught analytics and calculus.

Another problem which we of the colleges face is the question of where to put such students when they offer the College Board examinations as reason for advanced standing. This is because such examinations cut across college courses rather than cover one or two courses. Any place will mean loss of what can be very important topics and ideas.

Since many colleges and universities have shifted total responsibility to high schools for all algebra and trigonometry, we are assuming that the students acquire sufficient knowledge of these subjects to permit them to handle college courses in analytics and calculus—a rather violent assumption. I say this because from 12% to 20% of our freshmen fail their first course in college mathematics, and this might be much higher if we, like many, did not use fellows, graduate students and low-ranking members of the staff as freshmen instructors. Some such instructors pass as many as they possibly can to make their records look good. However, this practice merely makes the percentage of failures in the sophomore courses higher.

The college administrations seem to think that approximately 12% is as high as should be failed. This, of course, is based on the care with which students are admitted. Really there is too little care in choice of entrants. Most institutions have more apply than they should take, but they accept them until the quotas are filled. This is done because delay in accepting a student means that he had been accepted at one of the other places of application. Hence we do not pick—the quota must be filled. So, we take them and fail them and everybody is disturbed about it, but cannot change the situation.

A study of our courses and failures over the past three years has pointed particularly to poor algebra and trigonometry as the fundamental reasons. Hence I have worked out a list of topics in mathematics for high school students who plan to go to college.* A study of this list will certainly point out that the high schools have enough to do without attempting to teach college subjects.

My sincere conclusion on this problem is that the colleges and universities should test any student presenting analytics or calculus and be very critical of his right to advanced standing. I urge that no credit be given and that advanced standing be granted very rarely and that in such cases a close check be kept on such students. I urge you to discourage the public schools in your areas in teaching analytics and calculus. Point out to them the high casualty list of their students as of now; propose more algebra and analytic trigonometry, instead of that deadening drudgery of solutions of triangles by all means and methods, ad infinitum and ad nauseam.

I propose and urge you of the secondary schools to take this advice and come

* The Mathematics Teacher, December, 1958.

up with a better prepared group than you have been able to do up to now. Perhaps you do not know that many colleges and universities must give pre-freshman courses in mathematics to many entering students. Some even give pre-pre-freshman courses.

Finally, I quote some figures given out recently by one of our state universities. Only one third of the applicants were able to pass tests given at the university to decide whether to place them in college mathematics or not. *One fifth* had to take pre-college courses. *One fourth* had to take pre-pre-college courses. This is appalling, and here is what a paper put out by the university has to say about it. "Without affixing blame anywhere, let's regard some of today's children as non-educable on the college level and create junior colleges or institutions where they can either learn less expensive and lower-grade skills and cultures, or work their way up to the universities. . . . With a doubling of the university's enrollment by 1970, the 'spoon feeding of illiterates' will be a luxury we can no longer afford."

I sincerely hope that all levels of education can come to agree with me on this situation and bring about a more happy situation in the near future.

CHEMISTRY ON FILM

JOHN F. BAXTER, University of Florida

A project aimed at putting a complete course in high school chemistry on film has just been completed at the University of Florida. The author of this report, who is the "film teacher," is Professor of Chemistry and Head of the Division of General Chemistry at the University of Florida. Following the success of the course in Physics, taught by Dr. Harvey White of the University of California, as a "live" TV course in Pittsburgh during the school year 1956–57, and filmed in color by Encyclopaedia Britannica Films, Inc., plans were begun to film a course in chemistry. The Fund for the Advancement of Education, which supported the physics project, approached the American Chemical Society (ACS) about a course in chemistry. I was selected to plan and teach the course, with the cooperation and advice of a seven-member committee representing the ACS, the ACS Division of Chemical Education, and the National Science Teachers Association. Like the physics course, the chemistry films are being produced and distributed by the Encyclopaedia Britannica Films.

The original plans called for about 160 lessons of 30 minutes each, to cover a year's work. After the project was begun it was decided to make possible greater flexibility in the use of the course by building a core of 130–135 sequential lessons. The remainder of the lessons can be used to supplement and enrich the presentation, need not be used in sequence, and, if necessary because of time limitations, one or more may be omitted without destroying the continuity of the course. Experience with the physics films had shown that the inroads of special holidays, assemblies, football rallies, and the other distractions of what seems to have become the normal high school program, often left fewer than

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1376. *Proposed by V. F. Ivanoff, San Carlos, California*

Show that if A is the area of a quadrilateral having sides a, b, c, d and diagonals e, f then

$$16A^2 = 4e^2f^2 - (a^2 - b^2 + c^2 - d^2)^2.$$

E 1377. *Proposed by D. J. Newman, AVCO Research and Development, Lawrence, Mass.*

Solve the system of equations

$$u + v = a, \qquad ux + vy = b, \qquad ux^2 + vy^2 = c, \qquad ux^3 + vy^3 = d$$

for u, v, x, y.

E 1378. *Proposed by Paul Pargas, Washington, D. C.*

Draw a circle by tracing around a given circular disk D and mark an arbitrary point A on the circle. By drawing circles only with the use of the disk D, where one is permitted to place the rim of D on any two distinct given points, find the point B on the original circle which is diametrically opposite the point A.

E 1379. *Proposed by A. A. Mullin, University of Illinois*

Let S be the set of all real numbers and let $a * b$ denote $\max(a, b)$, where $a \in S$, $b \in S$. Show that the system $\{S; *\}$ is an abelian semigroup.

E 1380. *Proposed by D. S. Passman, Polytechnic Institute of Brooklyn*

Let A and B be $n \times n$ nonsingular matrices with $A + B = kE$, where k is a scalar and E is a matrix all of whose elements are 1's. If $S(C)$ denotes the sum of the elements of C, show that

$$\{1 - kS(A^{-1})\}\{1 - kS(B^{-1})\} = 1.$$

SOLUTIONS

A Mechanical Quadrature Formula

E 1345 [1958, 775]. *Proposed by I. J. Schoenberg, University of Pennsylvania*

There are given $2n$ points on the x-axis: $x_1 < \cdots < x_{2n}$, $(n \geqq 1)$. For convenience we also write $x_0 = x_1$, $x_{2n} = x_{2n+1}$ and determine, for each $k = 1, \cdots, n$,

a point ξ_k between x_{2k-1} and x_{2k} satisfying the linear equation

$$(\xi_k - x_{2k-2})(\xi_k - x_{2k-1}) = (\xi_k - x_{2k})(\xi_k - x_{2k+1}),$$

thus obtaining n points $\xi_1 < \cdots < \xi_n$. Finally, let

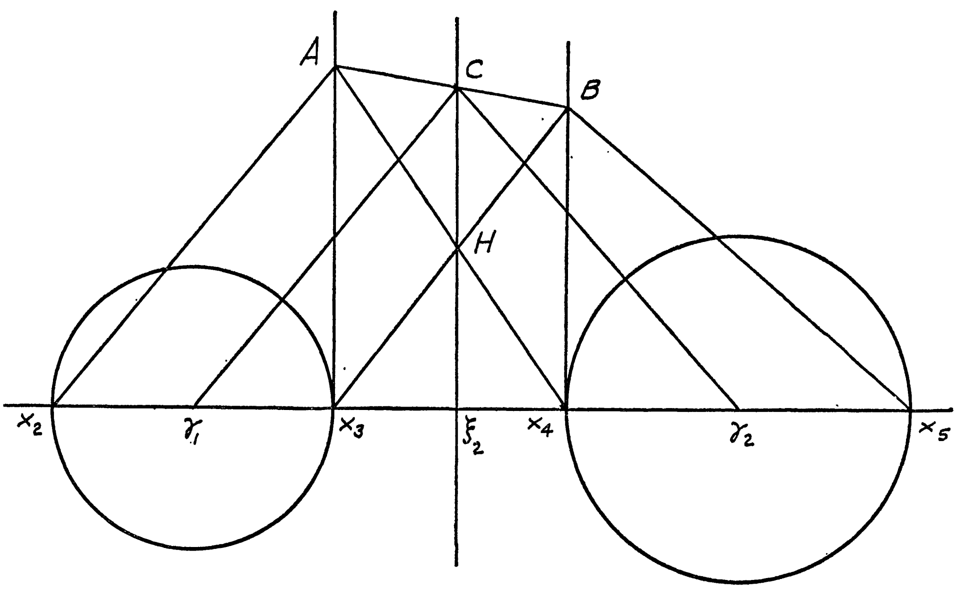
$$\gamma_k = (x_{2k} + x_{2k+1})/2, \qquad k = 0, 1, \cdots, n.$$

Show that the relation

$$\sum_{k=1}^{n} (\gamma_k - \gamma_{k-1}) f(\xi_k) = \int_{x_1}^{x_{2n}} f(x)dx \tag{1}$$

holds for any continuous function $f(x)$ which is linear in each of the intervals (x_k, x_{k+1}), $k=1, \cdots 2n-1$.

Solution by the proposer. It suffices to prove that (1) holds for each of the $2n$ "roof-functions" obtained by putting $f(x_k)=0$ for all k, with the exception of just *one* such value. For indeed, an arbitrary broken line $f(x)$ may be represented as an appropriate linear combination of such roof-functions. The case of the two roof-functions corresponding to $f(x_3)>0$ and $f(x_4)>0$, respectively, is typical. The relation (1) for each of these functions follows from a theorem in elementary geometry illustrated by the accompanying figure.



The line $\xi_2 H$ is the radical axis of the circles having centers γ_1 and γ_2. H being a point of this axis, produce the lines x_3H, x_4H, and let them intersect at A and B the tangents to the circles at the points x_3 and x_4 respectively. Let AB intersect the axis at C.

THEOREM. *The three triangles x_2Ax_4, $\gamma_1C\gamma_2$, and x_3Bx_5 all have the same area.*

We omit the simple proof, which can be carried out in several ways. Observing that $\xi_2H=HC$, the equality of the areas of the first two triangles may be written as

$$(\gamma_2 - \gamma_1)(\xi_2 H) = \text{area of triangle } x_2Ax_4,$$

and this is precisely the formula (1) for the roof-function whose graph is the polygonal line $x_1x_2Ax_4x_{2n}$. The case of the first two roof-functions ($f(x_1)>0$ and $f(x_2)>0$) is also covered by the geometric theorem if the left circle is allowed to shrink to a point.

Formula (1) shares with Gauss's quadrature formula the property of giving the integral of each function of a $2n$-parameter family from its ordinates at only n fixed abscissae. Noteworthy is the case of equidistant points $x_i=i-1$, $i=1, \cdots, 2n$, when (1) becomes

$$(3/2)f(2/3) + 2\sum_{k=1}^{n-2} f(2k + 1/2) + (3/2)f(2n - 5/3) = \int_0^{2n-1} f(x)dx.$$

This formula holds for any broken line function with vertices at integral values of x.

Also solved by D. C. B. Marsh and David Zeitlin.

A Congruence

E 1346 [1959, 61]. *Proposed by J. M. Gandhi, Belgaum, India*

Prove that if p is a prime then $\binom{2p}{p}\equiv 2$, mod p.

I. *Solution by David Zeitlin, Remington Rand Univac.* Since

$$\binom{2p}{p} = \sum_{k=0}^{p}\binom{p}{k}^2 = 2 + \sum_{k=1}^{p-1}\binom{p}{k}^2,$$

the result follows upon noting that for p prime, $\binom{p}{k}/p$ is an integer, $k=1, \cdots, p-1$.

II. *Solution by J. H. Hodges, Cornell Aeronautical Laboratory, Inc.* Let k be an arbitrary positive integer. Then

$$\binom{kp}{p} \equiv \frac{(kp)!}{p!(kp-p)!} \equiv \frac{k(kp-1)\cdots(kp-p+1)}{(p-1)!} \equiv \frac{k(p-1)!}{(p-1)!} \equiv k, \text{ mod } p.$$

Taking $k=2$ gives the desired result.

III. *Solution by T. M. Little, University of California.* In the expansion of $(1+1)^{2p}=2^{2p}=4^p$, every term except the first, middle, and last is divisible by p. Therefore $4^p\equiv 2+\binom{2p}{p}$, mod p. By Fermat's theorem, $4^p\equiv 4$, mod p. So $\binom{2p}{p}\equiv 2$, mod p.

IV. *Solution by Leo Moser, University of Alberta.* Consider the $\binom{2p}{p}$ arrangements of p white and p black beads in a circular necklace, with the necklace fixed in position. There are two arrangements in which the white and black beads are interlaced and since p is a prime the remaining arrangements come in sets of $2p$ by rotation. Hence $\binom{2p}{p}\equiv 2 \pmod{2p}$, a slightly stronger result than the one required.

A similar argument reveals that, more generally, for p and q primes,

$$(qp)!/(p!)^q \equiv q! \pmod{pq}.$$

V. *Solution by L. D. Goldstone, New York State Public Works Lab.* In 4322 [1950, 347] it is shown that if p is prime, then $\binom{n}{p}\equiv[n/p] \pmod{p}$, where $[n/p]$ indicates the largest integer contained in n/p. Since $[2p/p]=2$, it then follows that $\binom{2p}{p}\equiv 2 \pmod{p}$.

VI. *Solution by N. G. Gunderson, University of Rochester.* Actually, if $p>3$, $a>0$, then $\binom{ap^n}{p}\equiv ap^{n-1} \pmod{p^{2n+1}}$. Thus for $p>3$, $\binom{2p}{p}\equiv 2 \pmod{p^3}$. For $p=2$, $\binom{4}{2}\equiv 2 \pmod{2^2}$.

Letting S_j represent the sum of the products of $1, \cdots, p-1$ taken j at a time, and letting T be defined by

$$T(p-1)! \equiv 1 \pmod{p^{2n+1}},$$

we have

$$\begin{aligned}\binom{ap^n}{p} &= (ap^n)(ap^n-1)\cdots(ap^n-p+1)/p!\\ &\equiv ap^{n-1}\{1 - TS_{p-2}ap^n + TS_{p-3}a^2p^{2n}\} \pmod{p^{2n+1}}.\end{aligned}$$

However,

$$\begin{aligned}(p-1)! &= \{p-(p-1)\}\{p-(p-2)\}\cdots\{p-1\}\\ &\equiv (p-1)! - pS_{p-2} + p^2S_{p-3} \pmod{p^3},\end{aligned}$$

and so

$$pS_{p-2} \equiv p^2S_{p-3} \pmod{p^3}.$$

But $S_{p-3}\equiv 0 \pmod{p}$ from $(x-1)\cdots(x-p+1)\equiv x^{p-1}-1\equiv 0 \pmod{p}$, and so $S_{p-2}\equiv 0 \pmod{p^2}$, and the first stated result follows.

Also solved by A. N. Aheart, R. G. Albert, J. L. de M. Arenal, Anders Bager, H. F. Bechtell, G. E. Bloch, J. L. Botsford, D. A. Breault, R. F. Brown and Anna Endelman and Joel Levy (jointly), J. M. Calloway, Bomshik Chang, G. B. Charlesworth, P. L. Chessin, A. E. Danese, F. J. Duarte, E. S. Eby, N. J. Fine, Brother Louis Francis, D. A. Freedman, Rheba Galloway, Michael Goldberg, Sidney Glusman, Mildred Gross, Joseph Hammer, L. H. Hauer, S. A. Hoffman, J. Hooley, J. M. Howell, E. M. Horadam, J. T. Humphrey, M. S. Itzkowitz, M. I. Knopp and E. M. Scheuer (jointly), Sidney Kravitz, K. L. Loewen, D. C. B. Marsh, Helen Marston, C. E. Miller, Franklin Mohr, D. L. Muench, J. B. Muskat, Stewart Nagler, K. K. Norton, C. S. Ogilvy, C. Oster, F. D. Parker, Erna Pearson, D. J. Persico, Stanton Philipp, J. L. Pietenpol, C. F. Pinzka,

E. J. F. Primrose, Susan Pyeatt, T. L. Reynolds, L. A. Ringenberg, M. J. Saadaldin, Benjamin Sapolsky, Y. S. Sathe, L. J. Schneider, R. E. Shafer, Robert Spira, R. H. Sprague, E. P. Starke, Gerald Stoller, W. R. Talbot, R. G. Thompson and Mrs. R. G. Thompson (jointly), W. F. Trench, W. A. Veech, Harry Weingarten, Charles Wexler, Clement Winston, Dale Woods, and the proposer.

The problem is a special case of Prob. 4, p. 157, *Elementary Number Theory*, Uspensky and Heaslet, 1st ed. (1939). It is also a consequence of E 435 [1941, 269]. For numerous allied references see Dickson, *History of the Theory of Numbers*, vol. 1, pp. 270–278, in particular the reference under A. Cunningham on p. 274.

Fibonacci Numbers

E 1347 [1959, 61]. *Proposed by V. F. Ivanoff, San Carlos, California*

Prove that $\sum_{i=0}^{n}\binom{n}{i}F_{n-i}=F_{2n}$, where $F_0, F_1, \cdots, F_n$ is any set of $n+1$ consecutive Fibonacci numbers.

I. *Solution by E. P. Starke, Rutgers University.* By the difference relation of the Fibonacci numbers we have $F_{2n}=F_{2n-1}+F_{2n-2}$, which is the case $k=1$ of

$$F_{2n} = F_{2n-k} + \binom{k}{1}F_{2n-k-1} + \binom{k}{2}F_{2n-k-2} + \cdots + F_{2n-2k}.$$

For an induction hypothesis assume this equation, replace each F_j on the right by $F_{j-1}+F_{j-2}$ and replace $\binom{k}{j}+\binom{k}{j-1}$ by $\binom{k+1}{j}$. The result, after obvious reductions, is

$$F_{2n} = F_{2n-(k+1)} + \binom{k+1}{1}F_{2n-(k+1)-1} + \cdots + F_{2n-2(k+1)},$$

and the induction is complete. The proposed formula is the case $k=n$.

II. *Solution by N. J. Fine, Institute for Advanced Study.* Let f_j be the jth Fibonacci number, and let the operator E be defined by $Ef_j=f_{j+1}$. The required result is

$$(1 + E)^n f_j = f_{j+2n}.$$

By definition, it holds for $n=1$ and all j. Assuming that it holds for $n=k$, we have

$$(1 + E)^{k+1}f_j = (1 + E)(1 + E)^k f_j = (1 + E)f_{j+2k} = f_{j+2k+2}.$$

Thus it holds for $n=k+1$ and the induction is complete.

III. *Solution by David Zeitlin, Remington Rand Univac.* Since

$$F_k = c_1[(1 + \sqrt{5})/2]^k + c_2[(1 - \sqrt{5})/2]^k, \qquad k = 0, 1, \cdots$$

we have

$$\sum_{i=0}^{n}\binom{n}{i}F_{n-i} = c_1[(3 + \sqrt{5})/2]^n + c_2[(3 - \sqrt{5})/2]^n = F_{2n}.$$

Also solved by A. N. Aheart, R. G. Albert, J. L. de M. Arenal, R. J. Beeber, R. F. Brown and Anna Endelman and Joel Levy (jointly), J. M. Calloway, Leonard Carlitz, P. L. Chessin, A. E. Danese, David DeVol, Merritt Elmore, E. A. Fay, Michael Goldberg, Cornelius Groenewoud, Leonard Hauer, J. H. Hodges, S. A. Hoffman, Vern Hoggatt, J. Hooley, A. F. Horadam, A. S. Howard, H. Jager, P. G. Kirmser, M. I. Knopp and E. M. Scheuer (jointly), A. G. Konheim, T. M. Little, D. C. B. Marsh, K. K. Norton, C. S. Ogilvy, F. R. Olson, Stanton Philipp, J. L. Pietenpol, E. J. F. Primrose, B. E. Rhoades, Benjamin Sapolsky, R. E. Shafer, Ruth G. Smith, Robert Spira, R. H. Sprague, Gerald Stoller, R. G. Thompson and Mrs. R. G. Thompson (jointly), W. F. Trench, Peter Treuenfels, W. A. Veech, Julius Vogel, Everett Walter, and the proposer.

Equilateral Triangles Inscribed in an Ellipse

E 1348 [1959, 62]. *Proposed by M. S. Klamkin and Raphael Miller, AVCO Research and Development, Lawrence, Mass.*

Find the locus of the centroids of all equilateral triangles inscribed in an ellipse.

Solution by Sister Mary Stephanie, Georgian Court College, Lakewood, New Jersey. The problem is not new; it appears on page 170 of C. Smith, *Conic Sections*, Macmillan (1937). The solution given there is essentially as follows. Let the ellipse be $x^2/a^2+y^2/b^2=1$. If the eccentric angles of the vertices of an inscribed triangle are A, B, C, the centroid of the triangle is given by

$$x = a(\cos A + \cos B + \cos C)/3,$$
$$y = b(\sin A + \sin B + \sin C)/3.$$

The circumcenter of the triangle is given by

$$x = (a^2 - b^2)\{\cos A + \cos B + \cos C + \cos (A + B + C)\}/4a,$$
$$y = (b^2 - a^2)\{\sin A + \sin B + \sin C - \sin (A + B + C)\}/4b.$$

Since in an equilateral triangle the centroid coincides with the circumcenter, in this case we have

$$4ax/(a^2 - b^2) - 3x/a = \cos (A + B + C),$$
$$4by/(b^2 - a^2) - 3y/b = - \sin (A + B + C).$$

Squaring and adding we obtain the ellipse

$$(a^2 + 3b^2)^2x^2/a^2 + (b^2 + 3a^2)^2y^2/b^2 = (a^2 - b^2)^2$$

as the required locus.

Also solved by A. E. Currier, Stanton Philipp, and P. D. Thomas.

Cos $n\theta$ as a Tridiagonal Determinant

E 1349 [1959, 62]. *Proposed by P. L. Chessin, University of Maryland*

Consider the $n\times n$ matrix $[a_{ij}]$ where $a_{11}=\cos\theta$, $a_{ii}=2\cos\theta$, $(i=2, \cdots, n)$, $a_{i,i+1}=a_{i+1,i}=1$ $(i=1, \cdots, n-1)$, and all other elements are zero. Show that $|a_{ij}|=\cos n\theta$.

Solution by D. A. Breault, Sylvania Electric Products, Inc., Somerville, Mass. Denote the $n\times n$ determinant by D_n. Clearly $D_1=\cos\theta$ and $D_2=\cos 2\theta$. Assume $D_{k-1}=\cos(k-1)\theta$ and $D_k=\cos k\theta$, and expand D_{k+1} by minors with respect to the last row to obtain

$$D_{k+1} = 2D_k\cos\theta - D_{k-1} = 2\cos\theta\cos k\theta - \cos(k-1)\theta$$
$$= \cos\theta\cos k\theta - \sin\theta\sin k\theta = \cos(k+1)\theta.$$

The desired result now follows by mathematical induction.

Also solved by R. G. Albert, J. L. de M. Arenal, R. J. Beeber, H. F. Bechtell, Marvin Blum, A. P. Boblétt, Louis Brand, J. L. Brenner, R. F. Brown and Anna Endelman and Joel Levy (jointly), A. W. Brunson, Leonard Carlitz, G. B. Charlesworth, George Cherlin, A. E. Danese, E. Desautels, E. S. Eby, Merritt Elmore, H. E. Fettis, J. F. Foley, D. A. Freedman, D. P. Giesy, Michael Goldberg, Bernard Greenspan, D. S. Greenstein, J. D. Haggard, Franklin Haimo, J. H. Hodges, S. A. Hoffman, A. F. Horadan, A. R. Hyde, P. G. Kirmser, A. G. Konheim, M. M. Levine and P. F. Zweifel (jointly), K. L. Loewen, W. R. McEwen, D. C. B. Marsh, D. S. Mitrinović, J. B. Muskat, C. S. Ogilvy, C. Oster, N. T. Peck, J. M. Perry, D. J. Persico, Stanton Philipp, E. J. F. Primrose, Susan Pyeatt, T. L. Reynolds, B. E. Rhoades, R. A. Rosenbaum, H. D. Ruderman, Benjamin Sapolsky, E. M. Scheuer, R. E. Shafer, R. H. Sprague, E. P. Starke, T. J. Syson, R. C. Thompson, Peter Treuenfels, W. A. Veech, Dale Woods, Chih-yi Wang, David Zeitlin, and the proposer.

Rosenbaum showed that the conclusion follows if the assumption $a_{i,i+1}=a_{i+1,i}=1$ is replaced by the weaker condition $a_{i,i+1}a_{i+1,i}=1$. The given problem occurs as Prob. 6, page 84, in Rev. E. M. Radford's *Mathematical Problem Papers*, Cambridge University Press (1923), and as Prob. 12, page 191, in D. S. Mitrinović's *Zbornik matematičkih problema*, vol. 1, 2nd ed., Belgrade (1958).

A Homothecy

E 1350 [1959, 62]. *Proposed by N. A. Court, University of Oklahoma*

(a) The tangents to the ninepoint circle of a triangle T at the midpoints of the sides of T form a triangle homothetic to the orthic triangle of T. (b) The homothetic center of the two triangles is a point on the Euler line of T.

I. *Solution by H. E. Fettis, Wright-Patterson Air Force Base, Dayton, Ohio.* Let the vertices of the triangle T be A_1, A_2, A_3, the midpoints of the sides be O_1, O_2, O_3, the feet of the altitudes be H_1, H_2, H_3, and the intersections of the tangents to the ninepoint circle at the midpoints of the sides be P_1, P_2, P_3. It is known (see Johnson, *Modern Geometry*, art. 253e) that A_1O is perpendicular to H_2H_3, where O is the circumcenter of T. Also, A_1O is parallel to O_1F, where F is the ninepoint center of T (ibid, art. 308). Therefore P_2P_3 is parallel to H_2H_3, so that triangles $H_1H_2H_3$ and $P_1P_2P_3$ are homothetic. Further, H, being the incenter of $H_1H_2H_3$, is in correspondence with F, the incenter of $P_1P_2P_3$, so that the join of these two points, namely the Euler line of T, must pass through the homothetic center of $H_1H_2H_3$ and $P_1P_2P_3$.

II. *Solution by G. B. Charlesworth, Hofstra College.* The tangential triangle of T and the orthic triangle of T are homothetic, with center of homothecy on the Euler line of T (Altshiller-Court, *College Geometry*, art. 205, p. 102). Also, if T'

is the triangle whose vertices are the midpoints of the sides of T, then T and T' are homothetic, with the centroid G of T as center of homothecy. It follows that the tangential triangles of T and T' are homothetic with G as center of homothecy. We now conclude (by art. 56, p. 18, of Altshiller-Court, *Modern Pure Solid Geometry*) that the tangential triangle of T' and the orthic triangle of T are homothetic, with center of homothecy on the Euler line of T.

Also solved by J. W. Clawson, J. F. Darling, L. D. Goldstone, D. C. B. Marsh, W. R. Talbot, Victor Thébault, and the proposer.

Thébault stated that the ratio of homothecy is $4 \cos A \cos B \cos C$, where A, B, C are the angles of triangle T. Darling used complex coordinates.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4857. *Proposed by I. N. Baker, University of Alberta, and D. J. Newman, AVCO Research, Wilmington, Mass. (independently).*

What is the most general entire function which takes real values on the real axis only?

4858. *Proposed by J. A. Riley and Theodore Hatcher, Parke Mathematical Laboratories, Carlisle, Mass.*

Define the binary capacity, $C(n)$, of a positive integer n to be the exponent of the highest power of 2 which divides n.

If $\binom{n}{h}$ is the ordinary binomial coefficient, it is known that

$$C\binom{n}{h} = l(h) + l(n-h) - l(n),$$

where $l(n)$ is the number of 1's in the binary representation of n. What is

$$C\left(\sum_{j=0}^{h}\binom{n}{j}\right)?$$

4859. *Proposed by Richard Bellman, The RAND Corporation*

Consider the Fredholm integral equation

$$u(x) = v(x) + \int_a^1 k(x, y)u(y)dy, \qquad a < 1,$$

with the solution

$$u(x) = v(x) + \int_a^1 K(x, y, a)v(y)dy.$$

Under appropriate conditions on $k(x, y)$ show that the resolvent kernel satisfies the equation

$$\frac{\partial K}{\partial a}(x, y, a) = - K(x, a, a)K(a, y, a).$$

4860. *Proposed by Immanuel Marx, Purdue University*

Given the polynomial $F_n(x) = x^n - a_1x^{n-1} + a_2x^{n-2} - \cdots + (-1)^n a_n$, and integers p and q, $0 < q \leqq p \leqq n$. From each subset of the roots of F_n taken p at a time suppose formed the elementary symmetric function of degree q, *i.e.*, the sum of all products of distinct members of the subset taken q at a time. Knowing only the coefficients of F_n, devise a method to construct a polynomial whose roots are the $\binom{n}{p}$ symmetric functions described. Illustrate the method by deriving from the quartic $F_4(x) = x^4 - a_1x^3 + a_2x^2 - a_3x + a_4$, having (unknown) roots r_1, r_2, r_3, r_4, the quartic whose roots are $R_1 = r_2r_3 + r_3r_4 + r_4r_2$, $R_2 = r_1r_3 + r_3r_4 + r_4r_1$, $R_3 = r_1r_2 + r_2r_4 + r_4r_1$. $R_4 = r_1r_2 + r_2r_3 + r_3r_1$; in other words, the case $n = 4$, $p = 3$, $q = 2$.

4861. *Proposed by Joseph Lehner, Michigan State University*

Find a function $f(x)$, nonnegative and bounded on the closed interval $[0, \infty]$, such that

$$\limsup_{t \to \infty} \left| \int_0^t f(x)e^{2\pi i\lambda x}dx \right| = + \infty$$

for all real λ with the exception of a countable set.

4862. *Proposed by D. J. Newman, AVCO Research, Wilmington, Mass.*

Given a set S of distinct numbers $a_1, \cdots, a_n$. Define $k = k(S)$ to be the number of numbers of the form $a_i + 2^j$, $j = 1, 2, \cdots, n$. What is $\min_S k(S)$?

SOLUTIONS

The Jacobi Symbol

4809 [1958, 633]. *Proposed by Sylvan H. Greene, General Electric Co., Philadelphia*

Let n be an odd, square-free integer greater than 3; let A_n be the set of all a (mod n) such that $(a/n) = +1$, and B_n that of all b (mod n) such that (b/n)

$=-1$, where (b/n) is the Jacobi symbol. Prove that in a complete system of residues (mod n), $\sum a \equiv \sum b \equiv 0$ (mod n). The summations are taken over all elements of A_n and B_n respectively.

Solution by Leonard Carlitz, Duke University. Because of the hypothesis on n, there exists a number $c \not\equiv 1$ (mod n) such that $(c/n)=1$. [Proof. Let p be a prime >3 that divides n, whence $(n, n/p)=1$. Let r be an integer such that $r \not\equiv 1$ (mod p) and $(r/p)=1$. Then determine c by means of $c \equiv r$ (mod p), $c \equiv 1$ (mod n/p).] Then if $A=\sum a$, $B=\sum b$, it follows that

$$A \equiv \sum_a ca \equiv cA, \qquad B \equiv \sum_b cb \equiv cB,$$

whence $A \equiv 0$, $B \equiv 0$ (mod n).

Also solved by N. Sugunamma and by the proposer.

Convex Curves

4810 [1958, 712]. *Proposed by G. B. Dantzig and L. R. Ford, Jr., The Rand Corporation*

Suppose that C_1 and C_2 are convex curves with continuously turning tangents, and that the tangents at the left endpoints are parallel, as are the tangents at the right endpoints. Let C be the locus of points which are the midpoints of line segments joining a point on C_1 with a point on C_2 where the tangents are parallel.

Show that: (a) C is also convex; (b) the midpoint of any segment joining a point on C_1 to a point on C_2 cannot lie below C; and (c) if C_1 and C_2 are placed on parallel planes in 3-space, the lines joining points with parallel tangents generate a convex surface.

Solution by P. C. Hammer, University of Wisconsin. This problem is readily seen to be a special application of more general and well-known results. Let B_0 and B_1 be closed bounded convex sets in E_n. Treating points as vectors, let $0 \leqq t \leqq 1$ and define $B_t=(1-t)B_0+tB_1$. Here the $+$ refers to the direct sum. Then it is quickly established that B_t is a bounded closed convex set in E_n and the union of all B_t, $0 \leqq t \leqq 1$, is a closed bounded convex set in E_n. Moreover, if H_0 and H_1 are respectively supporting closed half-spaces of B_0 and B_1 such that H_1 is a translate of H_0 then the translate H_t of H_0 which is a supporting half-space of B_t may be written $H_t=(1-t)H_0+tH_1$. If x_0 is a point of B_0 in the boundary of H_0 and x_1 is a point of B_1 in the boundary of H_1 then $x_t=(1-t)x_0+tx_1$ is a point of B_t in the boundary of H_t.

To apply this to the problem, let B_0 be the convex hull of C_1 and B_1 be the convex hull of C_2. Then with $t=\frac{1}{2}$ we have that $B_{1/2}$ is the convex hull of C and C lies in its boundary; *i.e.*, C is convex. Now $x_0 \in B_0$ and $x_1 \in B_1$ implies that $\frac{1}{2}(x_0+x_1) \in B_{1/2}$ and hence no such point lies below C. Finally, place B_0 and B_1 in parallel planes in E_3. Then the union of the planar sets B_t, $0 \leqq t \leqq 1$, gives a convex solid and part of the surface of this solid is that required by part (c). If

the length of C_1 is L_1 and the length of C_2 is L_2, then the length of C is $\frac{1}{2}(L_1+L_2)$. It is not necessary that the curves have continuously turning tangents to establish these results.

Also solved by the proposers.

A Functional Equation

4811 [1958, 712]. *Proposed by Leonard Carlitz, Duke University*

Find the most general solution of the functional equation

$$2^{n/2}f_n(x\sqrt{2}) = \sum_{r=0}^{n}\binom{n}{r}f_r(x)f_{n-r}(x) \tag{1}$$

that is analytic at the origin.

Solution by A. E. Danese, Union College, Schenectady, N. Y. Let

$$g(x, y) = \sum_{n=0}^{\infty}\frac{f_n(x)y^n}{n!}, \qquad |y| < 1. \tag{2}$$

Then, there exists a region about the origin throughout which $g(x, y)$ is continuous. A simple transformation yields

$$g(\sqrt{2}x, \sqrt{2}y) = \sum_{n=0}^{\infty}\frac{2^{n/2}f_n(x\sqrt{2})y^n}{n!}$$

which, in view of (1), becomes $g(\sqrt{2}x, \sqrt{2}y) = g^2(x, y)$. The general continuous function which satisfies this functional equation is

$$g(x, y) = \exp(ax^2 + 2bxy - c^2y^2),$$

where a, b, c are arbitrary constants. From (2) we obtain

$$\exp(2bxy - c^2y^2) = \sum_{n=0}^{\infty}\frac{e^{-ax^2}f_n(x)y^n}{n!}.$$

Expanding the left side and equating the coefficients of y^n we obtain

$$f_n(x) = c^n e^{ax^2}H_n(bx/c),$$

where

$$H_n(x) = n!\sum_{i=0}^{[n/2]}\frac{(-1)^i(2x)^{n-2i}}{i!(n-2i)!}$$

are the nth degree Hermite polynomials defined by

$$e^{2xy-y^2} = \sum_{n=0}^{\infty}y^nH_n(x)/n!.$$

By analytic continuation, $f_n(x)$ is the solution of the functional equation for all x.

Also solved by J. H. Hodges, Richard Otter and Robert Weinstock, W. F. Trench, and the proposer.

Necessary and Sufficient Condition for a Polynomial

4813 [1958, 712]. *Proposed by H. D. Brunk, University of Missouri*

Given a differentiable function $f(x)$ such that to each $x \in (0, 1)$ there corresponds a positive integer $k=k(x)$ for which $f^{(n)}(x)=0$ for all $n \geqq k$. Prove f is a polynomial.

Solution by R. P. Boas, Jr., Northwestern University. More generally, it can be proved that if for each x there is a $k=k(x)$ such that $f^{(k)}(x)=0$, then f is a polynomial. (Corominas and Sunyer Balaguer, *Revista Mat. Hisp.-Amer.* (4) 14 (1954), 26–43; *Math. Reviews* 15, 942.) A proof is sketched in the review. Since the original paper is not readily accessible a (slightly different) proof is given here.

Let E_n be the set of points x for which $f^{(n)}(x)=0$. Every x is in at least one E_n. By Baire's theorem there is a subinterval I in which some E_n is everywhere dense. Since $f^{(n)}$ is continuous, $f^{(n)}(x)=0$ throughout I and f coincides in I with some polynomial. If I is not all of $(0, 1)$, repeat the reasoning with any remaining part of $(0, 1)$, and so on. We thus obtain a dense open set in each of whose component intervals f coincides with some polynomial. The complement H of this set is closed: we next show that, if not empty, it is perfect. If H is not perfect it has an isolated point, which is the common endpoint of two intervals on each of which f coincides with a polynomial. If n exceeds the degree of both polynomials, $f^{(n)}(x)=0$ for x in both intervals, so f coincides with some polynomial in the union of the two intervals, and at their common endpoint by continuity; so the point cannot belong to H after all.

Now, since H is perfect, if it is not empty we can consider it as a complete metric space and apply Baire's theorem to it. Some E_n is then dense in some neighborhood in H, that is in the part of H that is in some interval J. In other words, there is an interval J that contains points x of H with $f^{(n)}(x)=0$ for every such x (the same n for all x). J also contains intervals K complementary to H (since H is nowhere dense), and in each K, $f^{(m)}(x)=0$ for some m. If $m \leqq n$, $f^{(n)}(x)=0$ in K by differentiating. If $m>n$, we have $f^{(n)}(x)=f^{(n+1)}(x)= \cdots$ at the endpoints of K, by differentiating over H (since these endpoints are points of H). Then by integrating $f^{(m)}$ repeatedly we get $f^{(n)}(x)=0$ throughout K. The same reasoning applies to every K, so $f^{(n)}(x)=0$ throughout J. Thus J contains no points of H after all. This contradiction means that H was empty to begin with, so there was only one interval I and f coincides with a polynomial throughout $(0, 1)$.

Also solved by Robert Breusch and A. B. Willcox, J. M. Horváth, A. F. Kaupe, Jr., James Misho, and Henry Helson and the proposer.

The Greatest Integer Function

4814 [1958, 713]. *Proposed by D. J. Newman, AVCO Research, Wilmington, Mass.*

Given that $[n\alpha]+[n\beta]=[n(\alpha+\beta)]$ for all positive integers n. Prove that one of α, β is an integer.

Solution by Leonard Gewirtzman, University of Pennsylvania. Removing the integer parts of α, β in the given equation, we need consider only $0\leqq\alpha<1$, $0\leqq\beta<1$. We have $\alpha+\beta<1$, otherwise the hypothesis does not hold for $n=1$. Write $(x)'$ for $x-[x]$, and assume $\alpha>0$ and $\beta>0$. Then the hypothesis implies

$$(n(\alpha+\beta))' = (n\alpha)' + (n\beta)'. \tag{1}$$

Corresponding to any $\epsilon>0$ there exists an integer m such that

$$(m(\alpha+\beta))' < \epsilon, \tag{2}$$

since if $(\alpha+\beta)=p/q$, rational, we can take $m=q$; while if $(\alpha+\beta)$ is irrational, it is well known that the set $\{(m(\alpha+\beta))'\}$, $m=1, 2, \cdots$, is everywhere dense on $(0, 1)$. Therefore (1) implies $(m\alpha)'<\epsilon$, $(m\beta)'<\epsilon$. Choose $\epsilon<\min(\alpha, \beta)$. Then

$$\begin{gathered}[(m-1)(\alpha+\beta)] = [m(\alpha+\beta)] - 1,\\ [(m-1)\alpha] = [m\alpha] - 1, \qquad [(m-1)\beta] = [m\beta] - 1.\end{gathered} \tag{3}$$

By the original hypothesis, $[(m-1)(\alpha+\beta)]=[(m-1)\alpha]+[(m-1)\beta]$, whence by (3) we obtain $[m(\alpha+\beta)]=[m\alpha]+[m\beta]-1$, which contradicts the same hypothesis. Therefore $\alpha>0$ and $\beta>0$ is impossible and the theorem follows.

Also solved by A. C. Aitken, J. H. Hodges, D. C. B. Marsh, M. Sugunamma, and the proposer.

Editorial Note. Hodges bases his solution on a similar result of Jacobsthal. See *Norske Vid. Selsk, Forh.*, Trondheim, 30 (1957). 1–5.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

Numerical Analysis (2nd ed.). By D. R. Hartree. Oxford University Press, London, 1958. \$6.75.

As the first edition (1952) of Professor Hartree's book is firmly established on the shelf of all serious numerical analysts, we need only, in reviewing its second edition, to note some of the changes. There has been added a fuller treat-

ment of the Gaussian quadrature formula and of the solution of ordinary differential equations with two point boundary conditions. The material on hyperbolic differential equations has been slightly augmented. Three pages have been added on Whittaker's cardinal function—a theory which certainly deserves wider knowledge. The orientation of the book is toward the computer who uses a desk machine and indeed some of the material on programming which appeared in the first edition has been withdrawn. Still, it gives one a slight jar to read on page 11 that the main tools for numerical work are desk machine, tables, slide rule, and graph paper.

PHILIP J. DAVIS
National Bureau of Standards

Modern Mathematics for the Engineer. Edited by Edwin F. Beckenbach. McGraw-Hill, New York, 1956. xx+514 pp. $7.50.

It seems impossible for one man to criticize a collection of surveys such as this, written by a panel which is extremely well qualified. The problem is to ascertain a utility to the reader of the review and attempt to optimize it within these constraints.

It is stated in the preface that the committee sought to institute a series of lectures that would generate in the minds of engineers and applied scientists engaged in research, design, and administration, an awareness of the recent rapid advancement in applied mathematical thought—an advancement made possible in part by recent advances in basic mathematics and statistics and by the development of the analogue devices and digital computing machines of extremely high capacity and speed.

A prospective reader of the book should be forewarned of certain expectations. He should not expect a handbook of how to solve problems, nor even a coverage of the applicable cases, although such an undertaking should make a most profitable follow-up. He will find presented in each topic an excellent "feeling" for solving problems in the area, and when he should apply these "tools," provided he has more than a modicum of mathematical experience with the topic beforehand. With the exception of Wiener's article, which presupposes a maturity of thought concerning integration (working in the field requires a knowledge of Lebesgue integration, but gleaning the essence of the article requires only the realization that the Lebesgue integral has the desired properties), most of the articles can be appreciated with a background of ordinary differential equations and matrix theory (which is presented in the book).

The reader will not find applications of the analysis and design of experiments, regression and multivariate analysis to engineering problems nor analysis of error in interpolation problems, although these, too, might provide another volume. It seems to the reviewer that the book should prove most profitable to graduate students in physics, engineering, or applied mathematics, to industrial mathematicians or mathematical engineers, to industrial research teams,

and to administrators of both industrial and academic organizations who will consult with professional people in these fields in order to ascertain the trends in the applications of mathematics.

Part One has seven chapters: linear and nonlinear oscillations, by S. Lefschetz; stability theory, by R. Bellman; exterior ballistics, by J. W. Green; calculus of variations, by M. R. Hestenes; hyperbolic equations, by R. Courant; elliptic equations, by M. M. Schiffer; elastostatics, by I. S. Sokolnikoff. Part Two has five chapters: prediction, by N. Wiener; games, by F. Bohnenblust; operations research, by G. W. King; dynamic programming, by R. Bellman; Monte Carlo methods, by G. W. Brown. Part Three has seven chapters: matrices, by L. A. Pipes; functional transformations, by J. L. Barnes; conformal mapping, by E. S. Beckenbach; nonlinear methods, by C. B. Morrey, Jr.; relaxation methods, by G. E. Forsythe; methods of steep descent, by C. B. Tompkins; high-speed computing devices, by D. H. Lehmer. Each chapter was originally given as a lecture in an extension course at the University of California, Los Angeles, and also at the Corona Laboratories of the National Bureau of Standards.

R. B. DEAL
Oklahoma State University

Eigenfunction expansions associated with second-order differential equations, Part II. By E. C. Titchmarsh. Clarendon Press, Oxford, 1958. xi+404 pages.

This scholarly monograph is a sequel to Part I, published in 1946. The author states that "the whole work is the result of an attempt by an 'analyst' to understand those parts of quantum mechanics which can be regarded as exercises in analysis." The object of study is the partial differential equation $\Delta\psi+(\lambda-q(x))\psi=0$, where Δ is Laplace's operator, x is an n-dimensional real variable, and q is a given function of x. Expansions in a rectangle, in the whole plane, and in more than two dimensions are treated. Properties of the spectrum are studied, distribution of the eigenvalues for certain classes of $q(x)$ is determined, convergence questions are taken up, perturbation theory is described, and the special case of periodic $q(x)$ is dealt with. The last chapter is, so to say, a mathematical appendix, where many needed theorems are collected.

As the foregoing summary indicates, this is a book for experts or those who wish to become experts. Most of the results have appeared in the periodical literature, but they are here collected for the first time. Professor Titchmarsh's style is as usual very clear. He starts with simple cases and with formalities, as he calls them, which are as a rule plausibility arguments. Following these, the reader finds himself very rapidly drawn into computations almost grotesque in their complexity. In his preface, Professor Titchmarsh writes: "It seems that physicists do not object to rigorous proofs provided that they are rather short and simple . . . . Unfortunately, it has not always been possible to provide proofs of this kind." With the last sentence, the reviewer is in complete agreement.

EDWIN HEWITT
University of Washington

Physics and Mathematics, Ser. I, Vol. II. *Progress in Nuclear Energy*. Edited by D. J. Hughes, et al. Pergamon Press, New York, 1958. vii+375 pp. $14.00.

This book, like its predecessor (Volume I), aims to provide a review of recent progress in the experimental and theoretical aspects of nuclear reactor physics. Of the eight chapters, most are concerned with organizing the mass of relevant empirical data for anyone interested in making reactors. However, the eighth chapter on Monte Carlo Methods in Transport Problems by G. Goertzel and M. H. Kalos is a self-contained treatment of its subject matter which is of considerable interest to applied mathematicians. This chapter provides a compact introduction to Monte Carlo Methods, their relation to game theory and digital computer methods, and well-chosen examples of how they can be applied to obtain approximate solutions for the integral equations and partial differential equations which arise in reactor physics.

DAVID L. FALKOFF
Brandeis University

Mathematics for the Layman. By T. H. Ward Hill. Philosophical Library, New York, 1958. 339 pp. $4.75.

This is a book on what G. H. Hardy called "school" mathematics. It covers roughly the same material as Hogben's *Mathematics for the Million*. The main emphasis is on arithmetic, elementary algebra, and practical plane geometry, with briefer treatments of trigonometry, analytic geometry, and the calculus. Thus it does nothing to convey to the reader the nature of the mathematics of today. One must, however, grant an author the privilege of writing on whatever topic he pleases, and, after lamenting the fact that the book is not called "A Manual of Practical Old-Fashioned Mathematics for the Layman," pass on to its intrinsic merits and defects.

By and large, the book is clearly written with excellent explanations of the procedures used in arithmetic and elementary algebra. The author does have a talent for simplification and popularization. Naturally, he stresses examples and applications and there is little to indicate that one of the most important tasks of the mathematician is to provide rigorous proofs. In fact, in one of the rare occasions when a proof is given, it is incorrect. Thus the author (p. 27) "proves" the infinitude of primes as follows: "Multiply together all the numbers in the 'run' from 1 to (say) 10. Add 1 to the product. The result must be a prime number." And on page 125 we read that " . . . an important principle of mathematics . . . . It is that as all the 'plotted' points (and we can easily increase the number) lie upon a straight line," while on page 132 we have "The old definition [of a point] says 'a point has position but no magnitude.' It is a perfectly good definition, but rather top-heavy for such a simple idea. For a point is just a mark used to fix a particular position."

Other errors noted were in the definition of a circle (p. 139); the use of c/m in discussing $y=mx+c$ without specifying $m\neq 0$ (p. 247); the writing of $\int x^n dx$

$=(x^{n+1})/(n+1)+c$ with no restriction on n (p. 313); and the statement that "$\cdots x^3+y^3+2xy=1$ where it is not possible to calculate y directly in terms of x" (p. 257).

The sale of such books on "school" mathematics indicates the widespread desire of people to become mathematically literate. But, just as no "popular" history should be excused errors in dates just because it is for the layman, one cannot excuse gross mathematical errors in "popular" mathematics. And, while we may only lament that modern mathematics is not even touched upon, we can actively protest the attitude that elementary mathematics is just a collection of rules—no matter how clever.

ROY DUBISCH
Fresno State College

Mathematical Tables and Formulae (6th Ed.). By F. J. Camm. Philosophical Library, New York, 1958. 144 pp. $2.75.

C.R.C. Standard Mathematical Tables (11th Ed.). Chemical Rubber Publishing Company, Cleveland, 1958. 480 pp. $3.00.

Handbook of Calculus, Difference and Differential Equations. By E. S. Cogan and R. Z. Norman. Prentice-Hall, Englewood Cliffs, N. J., 1958. 263 pp. $6.00.

The first volume under review costs almost as much as the student edition of the Chemical Rubber Publishing Company's *Handbook of Physics and Chemistry* and contains about one-hundredth as much material.

The C.R.C. *Standard Mathematical Tables* are so well known that little need be said about this new edition. It has been enlarged here and there and a section on solutions of ordinary and partial differential equations has been added. It is still about the best table of its kind.

The Cogan and Norman volume contains the usual logarithmic, trigonometric, exponential and hyperbolic tables. It also contains a small table of integrals and two chapters are devoted to solutions of ordinary differential equations. A unique feature is the table of solutions of difference equations. These are well put together and should be helpful to users of this handbook.

The explanatory material, however, shows signs of having been prepared in haste. The inverse of the derivative operator D is an operator, not an indefinite integral. The set of all functions which satisfy a given functional equation is called the *general solution* of the equation (p. 12). This definition leads to error and confusion and it is small consolation to find (p. 229) that the authors did not mean it after all.

When one is talking about a function f whose values are given by $f(x)=Cx^2$, it seems undesirable to write $f=Cx^2$. Upper graph, p. 153, has a marking which should read $p>0$ instead of $p=0$. Upper graph, page 155, is incorrectly placed with respect to the y-axis. Graph (b), page 156, the three graphs on page 157 and/or the explanatory material are in error. Formula (15), page 177, is

wrong. The general form of the nonlinear equation of the first degree is *not* $F(x, f, Df)=0$ (p. 229).

The integral formula (50), page 189, is wrong but here the authors are in good company: Many calculus books and tables of integrals write

$$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\cos^{-1}\left(\frac{a}{x}\right).$$

This is false; the correct answer is $(1/a)\sec^{-1}(x/a)$. The error is elementary:

$$D\cos^{-1}\left(\frac{1}{x}\right) = \frac{1}{x^2\sqrt{\dfrac{x^2-1}{x^2}}} \neq \frac{1}{x\sqrt{x^2-1}}.$$

It is remarkable that this basic error, repeated in many similar integrals, should still exist today.

No wonder we have not yet reached the moon. Maybe our missilemen are using integral 21, 125, 128, 142, 147, 151, or 154 (each wrong) in C.R.C. *Standard Mathematical Tables.*

C. O. OAKLEY
Haverford College

College Algebra. (4th ed.) By J. B. Rosenbach, E. A. Whitman, B. E. Meserve and P. W. Whitman. Ginn, New York, 1958. xiv+579+xlvi pp. $5.25.

This seems a particularly good book on college algebra. In the Preface is found a clear statement as to how the fourth edition differs from earlier ones.

Especially to be commended are the following: historical notes, with dates given for mathematicians, etc.; warnings, in particular that on avoiding the use of a necessary condition to prove convergence (p. 542); work with complex numbers; footnotes with references to other books for further material; treatment of one-to-one correspondence, absolute value, inequalities, division by zero; a very large number of exercises; number systems with bases other than 10; the "fundamental assumptions" of algebra; problems leading up to more advanced mathematics; "measure of accuracy" of a number; problems from the Rhind Papyrus and from a Greek anthology (c. 500 A.D.); miscellaneous exercises at the ends of many chapters; the illustration of mathematical induction, using a row of dominoes; exercise 1 on p. 552; the statement of the ratio test, and the warning (p. 550) about its use.

A very few errors have been noted. On page 83, Example 2 (above exercises), numerator after first equal sign, write $9^{1/2}$, not 9^1. On page 94, Illustration 1, last sentence, write $\sqrt{-b}$ for $\sqrt{-3}$. On page 311, last line write $2k\pi$ for $2k_\pi$. On page 542, exercise 8, second term should be 4/9, *if nth* term formula holds for all terms. The form of the "check" on page 488 seems so much better than that found earlier in the book (p. 154 for example).

Statistics, matrices and infinite series could be omitted. The first does not seem to be college algebra. The second will leave some students feeling that they know more about the subject than they do. The third could be left for a more advanced text, though it is excellently treated here.

Students of this book should not get a dozen points, instead of three, for the graph of a straight line. They should end the course with an excellent idea of what a general proof is. And they should certainly not be afraid of "story problems."

MARION E. STARK
Wellesley College

Mathematics and Logic for Digital Devices. By James T. Culbertson. Van Nostrand, Princeton, N. J., 1958. x+224 pp.

This book contains a collection of elementary mathematical and logical topics which are applicable to the study of digital devices.

The central notion used is that of the nerve net (systems of receptors, central neurons, and effectors) as an exemplification of a digital device. To the reviewer this is a mistake since the necessary mathematics can be discussed with reference to much less controversial models now in wide use such as Von Neumann diagrams or, more directly, relay or electronic circuit schemata.

The topics discussed include elementary number theory, permutations and combinations and elementary probability theory, conversion of number system bases, traditional logic, algebra of classes, and 2-valued Boolean algebra.

The book is very clearly written. It would probably be most useful as a text for nontechnical students who have had a freshman mathematical-concepts type course. Also, it would probably be more useful if it contained a brief discussion of computers and computing circuits, if only to motivate the book. The meager assortment of examples of switching circuits in the last chapter is not likely to whet the student's appetite for a study of the deeper and more interesting parts of computational mathematics.

RAYMOND J. NELSON
Case Institute of Technology

Foundations of Information Theory. By Amiel Feinstein. McGraw-Hill, New York, 1958. x+135 pp. $6.50.

From the author's preface: "The intention of the author in writing this book is to provide a concise but rigorous exposition of the fundamentals of the mathematical theory of information. Since the basic work of C. E. Shannon appeared in 1948, a good deal of work has been done in this field. Nevertheless there exists at the present time (January 1957) no single reference to which one interested in this subject can turn for a presentation which is up to date and yet reasonably complete." A. I. Khinchin in one of his fundamental papers on information theory written in 1956 expressed the same opinion.

In a sense the book is an outgrowth of the author's research papers on information theory. However, if one looks for "entropy" in this book he will find it mentioned only in the titles of some of the references. The author apparently does not like to use this name borrowed from statistical mechanics for the function, $H = -\sum p_i \log p_i$.

The book will be of interest to people trained both in mathematics and in communication engineering. Although the mathematician will appreciate its high level of rigor, the nonmathematical engineer will be lost in attempting to use the book.

ROBERT E. GREENWOOD
The University of Texas and
General Analysis Corporation

Introduction to Difference Equations. By Samuel Goldberg. Wiley, New York, 1958. xii+260 pp. $6.75.

Although intended primarily for social scientists, this elegant volume on difference equations will be of great value to any student of the subject. The reader need only have facility with elementary techniques of algebra and trigonometry; other mathematical results necessary for particular topics are included in the text in a very concise and intuitive fashion. The overall impression of the book is one of completeness and rigor, excellent motivation, and ease of reading.

The introduction and first three chapters treat the calculus of finite differences, the operator notation, natural problems leading to difference equations, and a complete discussion of the general nth order linear difference equation. Many important properties of the solutions of these equations are developed and applied to problems of considerable interest, including examples from economics, psychology, sociology and probability theory. The fourth and final chapter contains a number of special topics, including stability, the notion of generating functions, characteristic-value problems and matrix methods. At the same time the analogy between difference and differential equations is illustrated. This chapter also serves the purpose of drawing the whole subject of difference equations together with other, perhaps better-known techniques, making it properly a part of the whole, rather than an isolated topic.

Each chapter has starred sections requiring slightly more advanced knowledge, which may be omitted by the beginner as they are not referred to elsewhere in the text. The careful preparation, logical motivation, and easy rigor of this book make it ideal for use in teaching a first course in difference equations at, say, the junior level.

GORDON LATTA
Stanford University

BRIEF MENTION

They Come for the Best of Reasons—College Students Today. By W. Max Wise. American Council on Education, Washington, D. C., 1958. xi+65 pp. $1.00.

This inexpensive publication of the American Council on Education should provoke debate and inquiry on your own campus concerning the nature of the students whom you are serving.

The Mathematics Student. W. W. Sawyer, Ed. National Council of Teachers of Mathematics, 1201 Sixteenth St. N.W., Washington, D. C., 1958. 8 pp. $0.30 per year (4 issues).

A journal for grades 7 through 12. Sold only in groups of five or more subscriptions to a single address. This excellent little journal should certainly be called to the attention of high school and junior high school teachers throughout the country. Our congratulations to Mr. Sawyer, the editor, on the enlarged format and the addition of color.

An Emerging Program of Secondary School Mathematics. By Max Beberman. Harvard University Press, Cambridge, Mass., 1958. 44 pp. $1.50.

This authoritative explanation of the Beberman project of the University of Illinois' Committee on School Mathematics (U.I.C.S.M.) was presented at Harvard University as the Inglis Lecture in 1958. It is certainly of considerable interest to mathematicians everywhere.

College Algebra, (2nd ed.). By Thurman S. Peterson. Harper, New York, 1958. viii+413 pp. $4.00.

A genuine revision of Peterson's earlier *College Algebra* text. In the opinion of this reviewer it pays considerable lip service to "modern mathematics" without capturing its spirit. The text is still presented in a "rule for this" and "rule for that" type of presentation, with very little explanation of *why* the rules hold.

The Algebra of Electronics. Solving Circuit Problems. By Chester H. Page. Van Nostrand, Princeton, N. J., 1958. x+258 pp. $8.75.

Not a mathematical book, but it certainly is a book which this reviewer would have enjoyed having while he was in high school. High school algebra and a touch of the very simplest ideas from calculus are used to obtain solutions to the basic problems of electronic circuits.

Fundamentals of Digital Computers. By Mathew Mandl. Prentice-Hall, New York, 1959. xi+297 pp. $5.00.

Your reviewer "learned" a number of remarkable "facts" from this little volume. For example, it states that "the (I.B.M.) 650 magnetic core memory provides storage of the 20,000 characters in such a manner that the information is available at 3,528,000 characters per minute. Each of the 20,000 memory positions can be contacted individually at any time. Hence they can be grouped to form fields of any size or records of any length."

Designing the Mathematics Classroom. By Lawrence P. Bartnick. National Council of Teachers of Mathematics, Washington, D. C., 1957. iii+40 pp. $1.00.

This publication of the National Council of Teachers of Mathematics may prove a helpful reference in these times of imminent building of classrooms. One wonders from the list of tools and equipment which are suggested as essential, desirable, or possible in

a mathematics classroom whether mathematics may be more of a hand-skill than a mental skill. Nevertheless, there are some thought-provoking ideas in the small booklet, and it is well worth the modest price.

Guide to the Literature of Mathematics and Physics, Including Related Works on Engineering Science. By Nathan Grier Parke III. Dover, New York, 1958. xviii+436 pp. $2.49.

The first 74 pages of this interesting revision of the author's 1947 work are devoted to a discussion of the why, how, and where of library research and study. Certainly this guide could be very handy for anyone wishing to browse a bit, or for intensive study of some particular field.

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor R. B. Deal, Jr., Oklahoma State University, represented the Association at the inauguration of Dr. B. G. Henneke as President of the University of Tulsa on April 16, 1959.

Professor I. L. Battin, Trenton State College, represented the Association at the inauguration of M. W. Gross as President of Rutgers, the State University, on May 6, 1959.

Associate Professor Mabel S. Barnes, Occidental College, represented the Association at the inauguration of Dr. Ralph Prator as the first President of San Fernando Valley State College on May 7, 1959.

Professor Hobart Bushey, Hunter College, represented the Association at the dedication of the Bronx Community College and the inauguration of Dr. Morris Meister as President of the college on May 11, 1959.

Professors Samuel Eilenberg, Columbia University, and K. O. Friedrichs, New York University, have been elected to membership in the National Academy of Sciences.

Professor K. W. Wegner, Carleton College, has been awarded a Fulbright Educational Exchange Grant to teach at National Taiwan University, Taipei, Taiwan, during the 1959–60 academic year.

Arizona State University: Prof. J. E. Freund has resigned as Department Chairman and has returned to full-time teaching; Professor L. L. Lowenstein has been appointed Department Chairman; Dr. Max Dengler, AiResearch Manufacturing Company, has been appointed Professor; Assistant Professor W. O. Portmann, Case Institute of Technology, has been appointed Associate Professor; Assistant Professor N. W. Savage, Polytechnic Institute of Brooklyn, has been appointed Associate Professor; Dr. L. T. Smith, Sequoia Adult School, has been appointed Assistant Professor; Adm. James Cohn

(USN Ret.) has been appointed Assistant Professor; Mrs. Wilma Thompson, University of Wyoming, Mrs. Gladdis Loehr, Baylor University, Adm. C. C. Seabury (USN Ret.) and Mr. W. R. Neal have been appointed Instructors; Instructors Albert Romano and George Peck have been promoted to Assistant Professors.

University of British Columbia: Associate Professor B. N. Moyls has been promoted to Professor; Assistant Professor Marvin Marcus has been promoted to Associate Professor; Dr. Fred Brauer, Dr. Charlotte Froese, Mr. Henryk Minc and Dr. C. A. Swanson have been promoted to Assistant Professors; Dr. N. J. Divinsky, University of Manitoba, has been appointed Associate Professor; Dr. D. W. Bressler, Tufts University, Mr. M. W. Katz, University of Illinois, and Dr. J. H. Lindsay, Jr., University of Toronto, have been appointed Instructors.

Brown University: Professor P. R. Masani, Institute of Science, Bombay, India, has accepted an appointment as Lecturer; Dr. D. J. Newman, Massachusetts Institute of Technology, has been appointed Assistant Professor; Associate Professor John Wermer has been given a leave of absence for the year 1959–60 to accept an appointment as a Visiting Lecturer at Harvard University; Mr. R. W. Rishel has been appointed Instructor for the year 1959–60.

Canisius College: Assistant Professor R. F. Tidd has been promoted to Associate Professor; Miss June McArtney, University of Buffalo, has been appointed Assistant Professor.

University of Chicago: Associate Professor A. P. Calderon, Massachusetts Institute of Technology, has been appointed Professor; Dr. K. R. Stromberg, Yale University, and Dr. L. E. Baum, National Science Foundation Fellow, have been appointed Research Lecturers; Dr. D. R. Hughes has been appointed ESSO Research Lecturer; Assistant Professors Eldon Dyer and R. K. Lashof have been promoted to Associate Professors.

Haverford College: Professor Hans Rademacher has been appointed Philips Visitor for 1959–60; Assistant Professor Louis Solomon, Bryn Mawr College, and Mr. James Brooks, University of Michigan, have been appointed Assistant Professors; Assistant David Harrison has been appointed Assistant Professor at the University of Pennsylvania; Assistant Professor R. J. Wisner has received a National Science Foundation Science Faculty Fellowship and will spend the year at the Institute for Advanced Study.

Knox College: Dr. J. R. Mayor, University of Wisconsin and AAAS, an alumnus of Knox, was given a Doctor of Laws degree at the June Commencement for his work in mathematical and scientific education; Professor Rothwell Stephens was made the first Distinguished Service Professor on the Robert J. Szold Foundation; Associate Professor A. O. Lindstrum, Jr. was promoted to Professor; Assistant Professor C. R. Ohman will be on leave to spend a year as Instructor at Princeton University; Mr. P. H. Yearout, University of Washington, has been appointed Assistant Professor.

Massachusetts Institute of Technology: Professor R. D. Schafer, University of Connecticut, has been appointed Professor and Deputy Head of the Mathematics Department; Associate Professor I. M. Singer has been promoted to Professor; Assistant Professors L. N. Howard and Hartley Rogers, Jr., have been promoted to Associate Professors; Dr. K. M. Hoffman, has been promoted to Assistant Professor.

University of Minnesota: Dr. A. M. Garsia, Massachusetts Institute of Technology, and Dr. C. D. Gorman, University of Washington, have been appointed Assistant Professors; Mr. J. T. Joichi, University of Illinois, Mr. F. B. Knight, Princeton University, Mr. C. L. Miracle, University of Kentucky, and Mr. D. A. Woodward have been appointed Instructors.

University of Nebraska: Professor Edwin Halfar has been appointed Acting Chairman of the Mathematics Department for 1959–60; Associate Professor L. K. Jackson has been promoted to Professor; Dr. G. H. Meisters, Duke University, has been appointed Assistant Professor; Professor Trevor Evans, Emory University, will be a

Visiting Professor; Professor C. C. Camp has retired with the title Professor Emeritus; Professor W. G. Leavitt will be on leave at Princeton University for the year 1959–60.

University of North Carolina: Dr. J. H. Wells has been promoted to Assistant Professor; Mr. G. W. Henderson, University of Texas, has been appointed Instructor.

Tulane University: Professor G. S. Young, University of Michigan, has been appointed Professor; Assistant Professors F. D. Quigley, Yale University, and Bernhard Banaschewski, Hamilton College, McMaster University, have been appointed Associate Professors.

University of Wisconsin, Mathematics Research Center, United States Army: The following have received appointments: Dr. Gaetano Fichera, University of Rome, Italy; Professor Haim Hanani, Israel Institute of Technology; Professor C. C. Hsuing, Lehigh University; Professor N. D. Kazarinoff, University of Michigan; Professor Cornelius Lanczos, Institute for Advanced Study, Dublin, Eire; Professor W. S. Loud, University of Minnesota; Professor A. M. Ostrowski, University of Basel, Switzerland.

Mr. D. S. Adorno, Harvard University, has accepted the position of Senior Research Mathematician at the Jet Propulsion Laboratory, California Institute of Technology.

Dr. D. O. Banks, Carnegie Institute of Technology, has been appointed Acting Assistant Professor at the University of California at Davis.

First Lt. D. R. Barr, University of Iowa, has been appointed Instructor at the U. S. Air Force Academy.

Mr. James Bercos, Lockheed Aircraft Corporation, Marietta, Georgia, has accepted the position of Research Mathematician with the RCA Service Company, Patrick Air Force Base, Florida.

Associate Professor Olivier Biberstein, Memorial University, St. John's, Newfoundland, Canada, has been appointed Professor Extraordinario at the University of Costa Rica.

Mr. J. R. Bienas, Indiana Technical College, has accepted the position of Engineer with the Philco Corporation, Philadelphia, Pennsylvania.

Professor Enrico Bompiani, University of Rome, Rome, Italy, has been appointed Andrew Mellon Professor of Mathematics at the University of Pittsburgh.

Mr. G. T. Boswell, White Sands Missile Range, has accepted the position of Engineer with Convair Astronautics, San Diego, California.

Dr. L. R. Bragg, Duke University, has been appointed Associate Professor at West Virginia University.

Mr. J. R. Brashear, R. A. Cummings, Jr., and Associates, Pittsburgh, Pennsylvania, has accepted the position of Data Processing Analyst with United States Steel, Pittsburgh, Pennsylvania.

Associate Professor Paul Brock, Purdue University and the University of Michigan, has accepted a position as Head, Industrial Dynamics Research, Hughes Aircraft Company, Culver City, California.

Captain E. E. Brown, Air Force Department, Stanford University, has been promoted to Major, United States Air Force, Offutt Air Force Base, Nebraska.

Mr. A. L. Buchman, Hutchinson-Central Technical High School, Buffalo, New York, has been appointed Associate in Mathematics with the State Education Department.

Assistant Professor W. O. Buschman, California State Polytechnic College, has been promoted to Associate Professor.

Mr. D. A. Celarier, Northwestern University, has accepted the position of Research Engineer "B" with Boeing Airplane Company, Pilotless Aircraft Division, Seattle, Washington.

Mr. D. R. Childs, Westinghouse Electric Corporation, Pittsburgh, Pennsylvania, has accepted the position of Staff Physicist with Allied Research Associates, Inc., Boston, Massachusetts.

Dr. Dorothy J. Christensen, Wellesley College, has been appointed Instructor at Reed College.

Mr. K. L. Conrad, Goodyear Aircraft Corporation, Akron, Ohio, has been promoted to Research and Development Engineer.

Assistant Professor E. H. Crisler, University of Notre Dame, has accepted the position of Principal Mathematician with the Advanced Development Laboratories, Bendix Aviation Corporation, South Bend, Indiana.

Dr. R. T. Dames, Massachusetts Institute of Technology, has accepted a position as a Member of the Technical Staff of Ramo-Wooldridge, Los Angeles, California.

Mr. L. D. Davis, University of Miami, has been appointed Part Time Instructor at the University of North Carolina.

Mr. August Deckert, Boeing Airplane Company, Renton, Washington, has accepted a position as Applied Science Representative with International Business Machines Corporation, South Gate, California.

Mr. F. H. Ditto, Northeastern University and Lincoln Laboratory, Massachusetts Institute of Technology, has accepted the position of Mathematician with the Systems Analysis Directorate, National Aviation Facility Experimental Center, Atlantic City, New Jersey.

Mr. F. A. Dunn, Jr., La Salle College, has been appointed Assistant Instructor at the University of Pennsylvania.

Mr. D. V. Easter, Sandia Corporation, Livermore, California, has accepted the position of Senior Engineer with the Martin Company, Orlando, Florida.

Mr. R. L. Edwards, Jr., Johns Hopkins University, has accepted the position of Engineer with Aircraft Armaments Inc., Cockeysville, Maryland.

Professor F. A. Ficken, University of Tennessee, has been appointed Professor at New York University.

Mr. M. V. Fiondella, University of Florida, has been appointed Instructor at the University of New Mexico.

Mr. L. C. Fletcher, Mt. Sterling High School, Mt. Sterling, Ohio, has been appointed Instructor at Eastern Kentucky State College.

Mr. J. B. Garner, Carleton College, has been appointed a Teacher for the American Board of Commissioner for Foreign Missions, Boston, Massachusetts.

Dr. H. H. Goldstine, International Business Machines Corporation, Yorktown Heights, New York, has been appointed Resident Manager of the Lamb Estate Research Center of the International Business Machines Corporation in the Town of Cortlandt, New York.

Associate Professor L. C. Graue, Coe College, has been appointed Associate Professor at Bowling Green State University.

Associate Professor Simon Green, University of South Carolina, has been appointed Associate Professor at the Assumption University of Windsor, Essex College, Windsor, Ontario, Canada.

Dr. Edison Greer, Lockheed Missiles and Space Division, Sunnyvale, California, has been appointed Professor and Head of the Mathematics Department at San Jose State College.

Dr. W. J. Hardell, Remington Rand Univac, St. Paul, Minnesota, has accepted the position of Engineer with the Radio Corporation of America, Moorestown, New Jersey.

Dr. D. L. Hartford, University of Kentucky, has been appointed Assistant Director of Testing Service, University of Kentucky

Mr. J. W. Haynes, Jr., Pearl Harbor Navy Shipyard, has accepted the position of Mathematical Physicist with the Dalmo Victor Company, California.

Mr. T. E. Hildick, Jr., Mississippi State College, has accepted the position of Engineer with the Western Electric Co., Winston-Salem, North Carolina.

Dr. T. R. Horton, International Business Machines Corporation, Washington, D. C., has been appointed Manager of 700–7000 Systems Marketing, Data Processing Division, White Plains, New York.

Mr. W. R. Hutcherson, Jr., United States Air Force, has accepted a position with the System Development Corporation, Santa Monica, California.

Professor S. B. Jackson, University of Maryland, will be on sabbatical leave during 1959–60 on an NSF Fellowship at the University of Washington

Mr. A. J. Janis, De La Salle Institute, Chicago, has been appointed Teacher at St. George High School, Evanston, Illinois.

Dr. M. P. Jarnagin, Jr., Naval Aviation Ordnance Test Station, Chincoteague, Virginia, has accepted a position as Mathematician at the Naval Proving Ground, Dahlgren, Virginia.

Mr. Raymond Kassler, Atlantic Design Company, Newark, New Jersey, has accepted the position of Analysis Engineer with the Data Processing Company, Elmhurst, New York.

Mr. W. M. Lindstrom, University of Kansas, has accepted the position of Applied Science Representative for International Business Machines Corporation, Hammond, Indiana.

Mr. J. M. Lowerre, University of Buffalo, has been promoted to Instructor.

Mr. R. D. Mason, Jr., American University, has accepted the position of Programmer with the Burroughs Corporation, Paoli, Pennsylvania.

Assistant Professor I. A. McCollum, North Carolina College at Durham, will study at the University of North Carolina on an NSF Faculty Fellowship during the year 1959–60.

Professor Norman Miller, Queen's University, Kingston, Ontario, Canada, has retired from active teaching.

Mr. D. E. Morrill, Western Electric Company, Winston-Salem, North Carolina, has accepted the position of Supervisor, Computer Applications Unit, Reaction Motors Division, Thiokol Chemical Corporation, Danville; New Jersey.

Mr. W. L. Morse, State College of Washington, has been aoppinted Instructor at Portland State College.

Assistant Professor W. O. J. Moser, University of Saskatchewan, Saskatoon, Saskatchewan, Canada has been appointed Associate Professor at the University of Manitoba, Winnipeg, Manitoba, Canada.

Dr. Z. I. Mosesson, The Prudential Insurance Company of America, Newark, New Jersey, has been promoted to Assistant Actuary.

Mr. A. M. Nagy, on sabbatical leave from the Pomfret School, Pomfret, Connecticut, has been appointed Assistant Professor at the University of Hawaii for 1959–60.

Mr. J. H. Ortman, Army Michigan Ordnance Missile Plant, Detroit, Michigan, has accepted the position of Senior Research Engineer for North American Aviation, Inc., Los Angeles, California.

Dr. Mary H. Payne, Inertial Systems Research Laboratory, Wyandanch, New York, has accepted the position of Principal Mathematician with Republic Aviation Corporation, Farmingdale, New York.

Mr. D. J. Persico, Ohio State University Research Foundation, has accepted a position as Associate Engineer in the Mathematical Analysis Section of Lockheed Aircraft Company, Burbank, California.

Associate Professor G. M. Petersen, University of New Mexico, has been appointed Lecturer at the University College of Swansea, Swansea, Great Britain.

Dr. W. N. Prentice, Denison University, has been promoted to Assistant Professor.

Assistant Professor John Raleigh, Lehigh University, has been appointed Assistant Professor at Temple University.

Professor O. J. Ramler, Catholic University of America, has retired with the title Professor Emeritus after 44 years of service, the last 8 years as Head of the Mathematics Department.

Mr. M. F. Reese, South Texas College, has accepted the position of Electronics Engineer with the Secode Corporation, San Francisco, California.

Dr. B. L. Reinhart, University of Michigan, has been appointed Assistant Professor at the University of Maryland.

Assistant Professor B. E. Rhoades, Lafayette College, has been promoted to Associate Professor.

Mr. Norman Schaumberger, The Cooper Union School of Engineering, has been appointed Assistant Professor at Bronx Community College.

Mr. T. A. Schoen, University of Cincinnati, has been appointed Instructor at the University of Dayton.

Professor J. P. Scholz, New Mexico Institute of Mining and Technology, has been appointed Professor and Head of the Department of Mathematics and Physics at the Western College for Women.

Visiting Associate Professor Seymour Schuster, Carleton College, has accepted the permanent position of Associate Professor.

Mr. R. A. Sebastian, Ballistic Research Laboratories, Aberdeen, Maryland, has accepted the position of Mathematician with Caywood-Schiller, Associates, Chicago, Illinois.

Mr. G. H. Silberberg, Bell Aircraft Corporation, Buffalo, New York, has accepted the position of Associate Engineer—Mathematical Analysis, with the Lockheed Aircraft Corporation, Burbank, California.

Mr. A. J. Silverman, Naval Ordnance Test Station, China Lake, California, has been appointed as a member of the Technical Staff, Space Technology Laboratories, Inc., Patrick Air Force Base, Florida.

Mr. D. J. Sine, Aero-Jet General, Frederick, Maryland, has accepted a position as Research Specialist for the Allegany Ballistics Laboratory, Cumberland, Maryland.

Sister Rose Marian, O.S.F., Ladycliff College, has been promoted to Assistant Professor and Chairman of the Division of the Natural Sciences.

Mr. D. J. Smith, Breeze Corporation, Inc., Newark, New Jersey, has accepted the position of Electronics Engineer for the Convair (San Diego) Division of the General Dynamics Corporation.

Mr. Irwin Stoner, Arma Division, American Bosch Arma Corporation, Garden City, New York, has accepted a position as Senior Mathematician with the Service Bureau Corporation, New York.

Associate Professor D. D. Strebe, University of South Carolina, has been promoted to Professor.

Mr. B. K. Swartz, Research Assistant, Los Alamos Scientific Laboratory, has been promoted to Staff Member.

Mr. Melvin Tainiter, American Bosch Arma Corporation, Garden City, New York, has accepted the position of Systems Engineer for International Electric Corporation, Paramus, New Jersey.

Assistant Professor Selmo Tauber, University of Kansas, has been appointed Associate Professor at Portland State College.

Mr. D. F. Templeton, Jr., University of Maryland, has accepted a position as Mathematician at the David Taylor Model Basin, Washington, D. C.

Miss Eileen J. Theisen, North American Aviation, Canoga Park, California, has accepted the position of Applied Mathematician with the Bell & Howell Company, Chicago, Illinois.

Mr. P. H. Thrower, International Business Machines Corporation, Dallas, Texas,

has been appointed Education Center Manager, IBM Corporation, Seattle, Washington.

Assistant Professor D. B. Tillotson, Northwest Nazarene College, is at the University of Kansas during 1959–60 on a NSF Science Faculty Fellowship.

Mr. R. A. Wonderly, Univac, Remington Rand Division, Sperry Rand Corporation, St. Paul, Minnesota, has been appointed Research Assistant at the University of North Carolina.

Mr. B. A. Yale, Cornell Aeronautical Laboratories, Buffalo, New York, has accepted a position as Mathematician with the Bendix Pacific Division, Bendix Aviation Corporation, North Hollywood, California.

Dr. William Zlot, City College of New York, has been appointed Associate Professor at Paterson State College.

Dean E. A. Bailey, Lagrange College, died on May 5, 1959. He had been a member of the Association for twenty-five years.

Professor E. T. Browne, University of North Carolina, died on March 31, 1959. He had been a member of the Association for thirty-eight years.

Dr. James A. Bullard, Professor Emeritus, University of Vermont and State Agricultural College, died on April 10, 1959. He was a charter member of the Association.

Professor H. P. Evans, University of Wisconsin, died on June 2, 1959. He had been a member of the Association for thirty years and was a member of the Board of Governors.

Professor T. E. Gravatt, Pennsylvania State University, died on February 12, 1959. He was a charter member of the Association.

Professor G. W. Hess, Howard College, died on March 7, 1959. He was a charter member of the Association.

Dr. D. G. Humm, Director, Humm Personnel Consultants, Los Angeles, California, died on May 17, 1959. He had been a member of the Association for seven years.

Professor J. J. Knox, Dakota Wesleyan University, died on April 10, 1959. He had been a member of the Association for thirty-eight years.

Associate Professor Emeritus L. T. Moore, Brooklyn College, died on April 16, 1959. He had been a member of the Association for thirty-six years.

Mr. G. N. Robinson, New Bedford, Massachusetts, died on January 28, 1959. He had been a member of the Association for thirteen years.

Assistant Professor C. A. Rogers, Colorado State University, died on May 30, 1959. He had been a member of the Association for five years.

Professor Emeritus E. I. Yowell, University of Cincinnati, died on March 12, 1959. He was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 294 persons have been elected to membership by the Board of Governors on applications duly certified.

PAUL ABRAMSON, M.A. (Columbia) Grad. Asst., Syracuse University

WADE L. ALLEN, Ph.D. (Florida) Senior Research Scientist, Litton Industries.

NORMAN L. ALLING, Ph.D. (Columbia) Asst. Professor, Purdue University.

GEORGE R. ANDERSON, Ed.D. (Pennsylvania S.U.) Professor, State Teachers College, Millersville, Pennsylvania.

PHILIP H. ANDERSON, M.A. (Notre Dame) Part-time Instr., Aquinas College; Teacher, Muskegon Catholic High School, Michigan.

MONA RAE ARMSTRONG, Student Humboldt State College.

ALFONSO G. AZPEITIA, Ph.D. (Madrid) Asst. Professor, University of Massachusetts.
MRS. CAROLYN RUTH R. BAHOUS, M.A. (North Carolina) Asso. Professor, Lynchburg College.
MARJORIE J. BAKIRAKIS, A.M. (Radcliffe) Instr., Hood College.
ROBERT W. BASS, Ph.D. (Johns Hopkins) Member, Research Institute for Advanced Study.
LEWIS E. BATSON, M.S. (Louisiana S.U.) Lecturer, University of Texas.
JAN R. BAUER, B.S. (Kent S.U.) Grad. Student, Kent State University.
JAMES N. BAUSCH, Student, University of Oklahoma.
GEORGE L. BAYER, M.A. (Loyola, Illinois) Teacher, Chicago Board of Education, Illinois.
ARMOND W. BEAR, M.S. (Marquette) Chairman of Dept., Rufus King High School, Milwaukee, Wisconsin.
HOWARD F. BECKSFORT, Ph.D. (Syracuse) Asso. Professor, Carroll College.
MICHAEL E. BENNETT, Student, DeWitt Clinton High School, New York, New York.
GUY M. BENSON, M.S. (Kansas City) Project Mathematician, I.B.M. Corp.
GEORGE W. BEST, B.S. (Union) Instr., Phillips Academy.
JAMES R. BOEN, Grad. Asst., University of Illinois.
WILLIAM J. BONINI, M.S. (Wyoming) Instr., Idaho State College.
DAVID M. BOODMAN, Ph.D. (Pittsburgh) Operations Analyst, Massachusetts Institute of Technology.
PAUL E. BOUDREAU, M.S. (Rochester) Programmer, I.B.M. Corp.
DONALD W. BOWAN, Student, Millikin University.
PAUL J. BOWLBY, M.S. in Ed. (Indiana) Teacher, Bellflower Unified School District, California.
JAMES W. BRADLEY, B.S. (Loyola, Maryland) Mathematician, Ballistic Research Labs.
DAVID F. BRAUCH, Student, University of Wisconsin.
LAWRENCE M. BREED, Student, Stanford University.
MRS. LUCILE C. BRIGGS, Computer, Boeing Airplane Co.
MILTON BRIZEL, M.A. (Columbia) Teacher, Fallsburgh Central School, New York.
HARRY BRODINE, M.A. (Fordham) Instr., State University of New York, Albany
JAMES T. BROGAN, Student, Mississippi State University.
WILLIAM A. BROWN, A.B. (Bowdoin) Grad. Student, University of Maine.
THOMAS J. BRUGGEMAN, B.S. (Dayton) Instr., Xavier University, Ohio.
PETER H. BUCKLEY, Student, University of Alberta.
GEORGE T. BUCKWALTER, B.S. in Ed. (West Chester S.T.C.). Instr., Valley Forge Military Academy.
GLENN A. BURDICK, B.S. (Georgia Inst. Tech.) Grad. Student, Georgia Institute of Technology.
VICTOR J. BUZZANCA, B.S. (Queens) Reliability and Quality Control Engineer, Arma Corp.
GLENN D. CAREY, A.B. (Wilkes) Engineer, Western Electric Co.
KENNETH E. CARLSON, B.S. (Texas Christian) Instr., North Texas State College.
CAPT. GEORGE C. CARLSTEDT, M.S. (Purdue) Instr., Bradley University.
KENNETH S. CARMAN, M.A. (Tennessee) Asso. Professor, Kansas Wesleyan University.
JOHN H. CASE, Student, Whittier College.
C. RONALD CASSITY, Ph.D. (Illinois) Senior Mathematician, New Mexico Institute of Mining and Technology.
VLADIMIR B. CERVIN, Ph.D. (Prague) Research Psychologist, Imperial Oil Limited.
JOSEPH T. CHELL, B.A. (British Columbia) Teacher, Burnaby South High School, British Columbia.
WILLIAM G. CHINN, A.B. (California, Berkeley) Teacher, San Francisco Unified School District, California.
ROBERT E. CLARK, Ph.D. (North Carolina) Asst. Professor, Virginia Military Institute.
ELEANOR M. CLIFFORD, Student, Anna Maria College.
N. DONALD COHEN, Student, Los Angeles State College.
MIRIAM V. N. COLLINS, M.A. (Columbia) Teacher, Haverford Township Junior High School, Pennsylvania.
ANNABELLE T. COMFORT, Student, University of Oklahoma.
SAM S. COMO, Student, Cooper Union; Asst. Office Manager, Bandt, Inc.
ARTHUR H. COPELAND, JR., Ph.D. (M.I.T.) Asst. Professor, Purdue University.
ALBERT G. COX, Student, Agricultural and Mechanical College of Texas.
KEITH J. CRASWELL, A.S. (Olympic) Grad. Student, University of Washington.
GORDON R. DAHLSTROM, B.A. (Chicago) Lithographic Technician, Rand McNally & Co.
COLDWELL DANIEL, III, Ph.D. (Virginia) Chairman of Econ. Dept., Mississippi Southern College.
WILLIAM H. DANNACHER, M.A. (Villanova) Asst. Professor, Villanova University.
DONALD B. DAVIS, B.S. (Carnegie Inst. Tech.) Grad. Student, Stanford University.
FRANK DEAN, Student, University of Redlands.
ANTHONY J. DEGENNARO, M.Ed. (Pennsylvania S.U.) Instr., Firelands High School, Oberlin, Ohio.
MARION E. DENINNO, A.B. (Washington & Jefferson) Chairman of Dept., Charleroi High School, Pennsylvania.
EDOUARD J. DESAUTELS, Student, University of Manitoba.
MORTON DEUTSCH, Ph.D. (M.I.T.) M.T.S., Bell Telephone Labs.
WILLIAM D. DEVITT, Ed.M. (Harvard) Teacher, Salem Classical and High School, Massachusetts.
ROY A. DOBYNS, M.A. (Vanderbilt) Asst. Professor, McNeese State College.
GERALD C. DODDS, B.S. (St. Vincent) Mathematician, Haller, Raymond & Brown.
PETER DORATO, M.S. in E.E. (Columbia) Instr., Polytechnic Institute of Brooklyn.
BRYAN M. EAGLE, Student, Southwestern College.
ROY M. ELLEDGE, Student, University of California.
LOWELL EMERLING, M.A. (Michigan) Senior Scientist, Lockheed Aircraft Corp.
HOWARD B. EMERSON, Student, State College of Washington.
GARRET J. ETGEN, Student, College of William and Mary.
WILLIAM D. ETLING, B.S. in Ed. (Kent S.U.) Temporary Instr., Kent State University.
SARAH R. EVANGELISTA, B.S. (Temple) Instr., Temple University.
W. BUELL EVANS, PH.D. (Illinois) Asso. Professor, Georgia Institute of Technology.
ROBERT P. EVERETT, A.B. (Georgia) 180 Westview Drive, Athens, Georgia.
EDWARD L. FADER, B.S. (Wyoming) Grad. Student, University of Wyoming.
MRS. ETTA Z. FALCONER, M.S. (Wisconsin) Head of Dept., Okolona College.
BARBARA A. FLORES, Student, Baylor University.
GUY M. FORMICHELLA, B.S. (C.C.N.Y.) Electronic Scientist, Rome Air Development Center.
MAURICE A. FORTIER, Clerk, National Defence, Ottawa, Ontario.
ALFRED P. FOX, B.A. (Temple) Mathematician, Burroughs Research Center.
GEOFFREY T. FOX, Student, University of Santa Clara.
PAUL O. FREDERICKSON, B.A. (Maryland) Grad. Student, University of Maryland; Asso. Mathematician, Applied Physics Lab., Johns Hopkins University.
MARVIN I. FREEDMAN, Student, Massachusetts Institute of Technology.
HAROLD G. FROEHLICH, M.A. (Fordham) Chairman of Dept., John Adams High School, Ozone Park, New York.
RICHARD E. FULLER, B.S. (Westminster, Pennsylvania) Head of Dept., Technical-Vocational High School, Hammond, Indiana.
SOPHRONIA A. GAMBLE, Student, William Smith College.
MRS. OLIVE R. GARDNER, M.A. (Minnesota) Teacher, Lyons Township High School and Junior College.
GEORGE C. GASTL, Student, University of Kansas.
PAUL C. GILMORE, Ph.D. (Amsterdam) Staff Mathematician, I.B.M. Research Centre.
ANTON GLASER, B.A. (Temple) Teacher, Upper Moreland High School, Willow Grove, Pennsylvania.
MRS. EVA W. GRAY, Ph.D. (Zurich) Instr., Douglass College.
JOSEPH GREENBAUM, M.S. (Columbia) Chairman of Dept., Long Beach High School, New York.
JOEL E. GREENBERG, Student, City College of New York.
MARY L. GRIGGS, M.A. (Georgia) Teacher, Piedmont College.
ROBERT L. GUDERJOHN, Student, Idaho State College.
PETER H. GUM, B.A. (Oberlin) Engineer, Melpar, Inc.

HADI HADDAD, M.A. (Texas) Grad. Student, University of Texas.
DOROTHEA C. HADSELL, M.A. (N.Y.S.C. for Teachers) Supervisor, Secondary Math., Albany, New York.
HAROLD W. HAGER, A.M. (Missouri) Instr., Southeast Missouri State College.
RAOUL HAILPERN, B.A. (London) Instr., University of Buffalo, Park School of Buffalo.
EDWARD M. HARRIS, B.S. (Colorado) Instr., San Francisco State College.
DAVID E. HARVEY, SR., Instr., Chicago Technical College.
FRANK HARWAYNE, B A. (Brooklyn) Consulting Actuary, Frank Harwayne & Associates.
SIMON W. HECHT, Student, Trinity College.
KENNETH B. HENDERSON, Ph.D. (Ohio S.U.) Professor, University of Illinois.
HANS-HEINRICH W. HERDA, Student, Wayne State University.
MRS. GLORIA C. HEWITT, B.A. (Fisk) Grad. Student, University of Washington.
KENNETH M. HEWITT, Student, College of St. Thomas.
CHARLES HODES, A.M. (Columbia) Teacher, Polytechnic Institute of Brooklyn.
MRS. AIKO M. HORMANN, A.M. (Missouri) Mathematician, Westinghouse Research Labs.
KENT P. HOWE, B.S. (St. John's, Minnesota) Grad. Student, University of Illinois.
DOROTHY H. HOY, M.Ed. (Pennsylvania S.U.) Teacher, Harrisburg School District, Pennsylvania.
ANNE HUGHES, M.A. (Fordham) Teacher, Convent of Sacred Heart, New York, New York.
NORMAN L. HUGHES, M.S. (Wisconsin) Instr., Beloit College.
JOHN R. HUNTER, Student, Stanford University.
OSSIE J. HUVAL, Ph.D. (Louisiana S.U.) Asst. Professor, Southwestern Louisiana Institute.
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J. ROBERT JOHNSON, JR., Ph.D. (Duke) Asst. Professor, Wake Forest College.
STANFORD H. JOHNSON, M.A. (Geo. Peabody) Asst. Professor, East Tennessee State College.
CONSTANCE J. JONES, M.A. (Texas) Instr., San Antonio College.
ROBERT L. JONES, A.B. (Hiram) Teacher, Chagrin Falls High School, Ohio.
WILLIAM E. JONES, A.M. (Chicago) Teacher, Belmont High School, Dayton, Ohio.
ROBERT V. KESTER, A.B. (Whittier) Acting Instr., University of Idaho.
ROBERT W. KING, M.A. (Mississippi Southern) Instr., Vanderbilt University.
CHARLES W. KINNEY, A.B. (Akron) Teacher, Buchtel High School, Akron, Ohio.
JOHN R. KINZER, Ph.D. (Geo. Peabody) Human Factors Scientist, System Development Corp.
V. D. KIRKLAND, M.A. (Georgia) The Citadel.
WILLIAM E. KIRWAN, II, Asst., University of Kentucky.
ISRAEL KLEINER, Student, McGill University.
BARRON G. KNECHTEL, M.A. (Northwestern) Instr., Orange Coast Junior College.
WILLIAM F. KOEHMSTEDT, Student, University of Santa Clara.
MRS. RUTH D. KOIDAN, B.S. (Geo. Washington) Grad. Student, George Washington University; Mathematician, NASA, Washington, D. C.
ROBERT W. KOLTS, B.A. (Utah) Teacher, West High School, Salt Lake City, Utah; Grad. Student, University of Chicago.
PAUL G. KUMPEL, JR., B.S. (Trenton S.C.) Instr., Lafayette College.
IRA A. KWEITKO, Student, Los Angeles State College.
FRANK LAMANTIA, M.S. (Omaha) Teacher, Thomas Jefferson High School, Council Bluffs, Iowa.
REV. JEAN-PAUL LANDRY, M.S. (Laval) Teacher, Seminaire de Saint-Jean, Quebec.
FRANCIS G. LASAK, M.S. (Michigan) Asst. Professor, New York City Community College of Applied Arts & Sciences.
CHARLES J. LAVIOS, Student, San Jose State College.
LEON LEBLANC, M.A. (Chicago) Asst. Professor, University of Montreal.
PIERRE LEDUC, B.S. (Montreal) Grad. Student, University of Montreal.
L. DAVID LEWIS, B.A. (Colorado) Teaching Asst., Stanford University.
RALPH D. LEWIS, B.S. (New York) Teacher, Malverne Jr. High School.
ERIC H. LINDBERG, B.S. (Gustavus Adolphus) Teacher, West High School, Minneapolis, Minnesota.
LEONARD J. LIPKIN, Student, Oberlin College.
BENJAMIN LIPSTEIN, Ph.D. (Columbia) Asso. Research Director, Benton & Bowles, Inc.
WILLIAM G. LISTER, Ph.D. (Yale) Professor, State University College on Long Island.
DONALD M. LONG, B.S. (Columbia) Teacher, Barnard School for Boys, Bronx, New York.
JAMES J. LOUGHNEY, M.S. (Bucknell) Applied Science Representative, I.B.M. Corp.
JAMES M. LOWERRE, B.S. (U. S. Military Acad.) Instructor, University of Buffalo.
WILLIAM L. LYNN, JR., M.A. (Duke) Principal, Laurel High School, Maryland.
THOMAS H. MACGREGOR, A.M. (Pennsylvania) Instr., Rutgers University.
EDWIN S. MACK, B.S. (Ursinus) Grad. Student, Duke University.
TRAVIS E. MADOLE, Student, Texas A. & M.
KENNETH D. MAGILL, JR., B.S. (Shippensburg S.T.C.) Grad. Asst., Pennsylvania State University.
HARRY MALISOFF, Ph.D. (Columbia) Asso. Professor, Brooklyn College.
MITZI L. MALLARIAN, Student, North Dakota Agricultural College.
L. JEROME MANDEVILLE, Asso. in Arts (Holyoke J.C.) Engraver, Gravure Engraving Corp.
LAWRENCE D. MARCOTTE, M.S. (Ft. Hays, Kansas S.C.) Instr., Victoria Rural High School, Kansas.
FRANK J. MASSEY, JR., Ph.D. (California, Berkeley) Asso. Professor, University of California.
ROBERT MCFADDEN, B.A. (Queen's, Belfast) Teacher, Grosvenor High School, Belfast, Ireland.
J. PATRICK MCGAY, Student, University of Tulsa.
JAMES G. MCGREW, M.A. (Colorado S.C.) Teacher, Denver Public High Schools, Colorado.
BRENDA C. MCKEON, B.S. (Marymount) Grad. Asst., Georgetown University.
THOMAS G. MCLAUGHLIN, Student, University of California.
LOUIS D. MELNICK, Student, North Central College.
JOHN W. MEUX, M.A. (Arkansas) Grad. Asst., University of Florida.
ARNOLD L. MILNER, M.A. (Alabama) Instr., University of Alabama.
LOIS E. MINNING, Student, Oberlin College.
VESPER D. MOORE, Ed.D. (Michigan) Professor, Indiana State Teachers College.
FRED L. MORRISON, Student, University of Kansas.
JOHN C. MORRISON, Student, University of Santa Clara.
EDITH MOSS, M.A. (Maryland) Instr., Vassar College.
MICHAEL R. MULLEN, M.A. (Fordham) Instr., Fordham College.
JOSEPH P. NATHANSON, M.Ed. (Boston) Teacher, Weeks Junior High School, Massachusetts.
WARREN E. NUTE, Student, Stanford University.
GLADYS A. O'BRIEN, M.A. (Loyola, Illinois) Teacher, South Shore High School, Chicago, Illinois.
FRANCIS J. O'MEARA, B.S. (Creighton) Jr. Mathematician, Westinghouse Research Labs.
STANLEY S. PAGE, Student, University of Oregon.
THEODORE W. PALMER, M.A. (Johns Hopkins) Grad. Student, Harvard University.
JAMES T. PARENT, A.B. (Ricker) Grad. Student, University of Maine.
NANCY L. PARR, A.B. (Hanover) Teaching Fellow, University of Kentucky.
EMANUEL PARZEN, Ph.D. (California) Asso. Professor, Stanford University.
RICHARD H. PATTERSON, Student, University of British Columbia.
DIANE M. PAWLAK, Student, University of Wisconsin.
C. FREDERICK PETERS, A.B. (Vanderbilt) Mathematician, Naval Proving Ground.
ROBERT L. PEXTON, B.A. (California) Mathematician, Lawrence Radiation Lab.
ISAAC R. PFEIFFER, Student, Brooklyn College.

MARY L. POTTS, Student, University of Detroit.
WILLIAM D. POWERS, M.A. (Cornell) Teaching Asst., University of Tennessee.
JOHN B. PRATER, M.A. (Missouri) Instr., Southwest Missouri State College.
EUGENE V. PRICE, Student, North Dakota Agricultural College.
RICHARD H. PROSL, Student, College of William & Mary.
CALVIN M. PULLIAS, B.A. (Geo. Pepperdine) Grad. Student, University of Southern California.
SEYMOUR W. PUSTILNIK, M.A. (Bowling Green S.U.) Instr., New York City Community College of Applied Arts & Sciences.
DANIEL G. QUILLEN, Student, Harvard University.
DENNIS RADER, Student, Cooper Union.
PETER L. RENZ, Student, Reed College.
CARL E. RIECK, JR., B.A. (Yale) Grad. Student, Tufts University.
DOCK S. RIM, Ph.D. (Indiana) Research Asso., Columbia University.
MELVIN B. RINGEL, B.S. (M.I.T.) Jr. Instr., Johns Hopkins University.
CHARLES E. ROBE, Electronics Instr., Naval School of Interior Communications.
CHARLES D. ROBINSON, B.A. (Hardin-Simmons) Instr., Hardin-Simmons University.
R. LEWIS ROFMAN, M.S. in Ed. (Temple) Teacher, Olney High School, Philadelphia.
EDWARD ROSENBERG, M.A. (Illinois) Instr., East Los Angeles Junior College.
GEORGE M. ROSENSTEIN, JR., Student, Oberlin College.
CAPT. ROBERT C. ROUNDING, M.S. (Oklahoma S.U.) Instr., U. S. Air Force Academy.
MRS. MELBA L. ROY, M.S. (Howard) Mathematician, Army Map Service.
MRS. LAUREL R. RUCH, Student, Colorado College.
CHARLES T. RYAN, JR., B.S. (Canisius) Mathematician, Cornell Aeronautical Lab.
ABDEL S. SAID, M.S. (M.I.T.) Research Chemist, Columbia University.
SOCRATES W. SAUNDERS, Ph.D. (Pittsburgh) Professor, Morgan State College.
JAMES W. SAUVE, S.J., B.S. (Spring Hill) Grad. Student, Johns Hopkins University.
BERNADETTE SAVIGNAC, Student, Anna Maria College.
NATHAN S. SCARRITT, JR., M.A. (Oklahoma) Special Instr., University of Oklahoma.
WILLIAM H. SCHMIDT, B.A. (Columbia) Vice President & Chief Actuary, Mutual Life Ins. Co. of New York.
ALVIN L. SCHREIBER, B.S. (Johns Hopkins) Research Asso., Human Sciences Research, Inc.
CAROLE E. SCOTT, Student, Butler University.
JACK SEGAL, M.S. (Miami) Grad. Asst., University of Georgia.
STEPHEN B. SEIDE, Student, James Madison High School, Brooklyn, New York.
GEORGE SENGE, Student, University of California.
GEORGE C. SETHARES, B.Mus. (Boston) Grad. Student, University of Massachusetts.
DANIEL M. SHEEHAN, Student, College of William & Mary.
JAMES L. SIEBER, B.S. (Shippensburg S.T.C.) Grad. Asst., Pennsylvania State University.
NICHOLAS T. SIMOPOULOS, M.S. (Ohio S.U.) Vice President, Dayton Electronic Products Co.
DALE D. SIMS, Teacher, Nevada Union High School District, California.
SISTER M. DENIS, O.S.F., M.A. (New York) Chairman of Dept., St. Joseph's Girls High School, West New York, New Jersey.
SISTER MARY DE PAZZI, O.S.F., Ph.D. (Notre Dame) Head of Dept., Briar Cliff College.
SISTER MARY KENNETH, R.S.M., M.A. (Loyola) Head of Dept., Mother McAuley Liberal Arts High School, Chicago, Illinois.
SISTER MARY LAWRENCE, M.A. (Canisius) Teacher, Mt. St. Joseph Teachers College.
SISTER ROSE MARIAN, O.S.F., Instr., Ladycliff College.
MICHAEL S. SKAFF, B.S. (Michigan) Grad. Asst., University of Illinois.
WILLIAM E. SLESNICK, M.A. (Oxford) Instr., Dartmouth College.
GEORGE R. SMITH, A.B. (Princeton) Head of Dept., St. Paul's School, Concord, New Hampshire.
MARTIN R. SMITH, Stationary Engineer, International Breweries, Inc.
NATHANIEL B. SMITH, B.S.E. (Princeton) Teacher, Taft School, Watertown, Connecticut.
IAN N. SNEDDON, D.Sc. (Glasgow) Simson Professor, University, Glasgow, Scotland.
VIRGINIA A. SNOW, B.A. (Houghton) Teaching Fellow, University of Buffalo.
MAX A. SOBEL, Ph.D. (Columbia) Asso. Professor, Montclair State College.
DAVID C. SOMMER, Student, University of Minnesota.
JESSE C. SOWELL, B.S. (Carson-Newman) Instr., Elon College.
MRS. KATYE O. SOWELL, M.S. (South Carolina) Asst. Professor, Elon College.
WARREN G. SPARKS, A.M. (Boston) Instr., St. Michael's College, Vermont.
HARLAN C. STAMM, M.S. (Northern Illinois) Instr., Cerritos Junior College.
JERRY D. ST. CLAIR, B.S. (Central Michigan) Grad. Student, Michigan State University.
EDWIN I. STEIN, B.S. (Pennsylvania) Author, Philadelphia.
VINCENT STELMACH, B.A. (Connecticut) Mathematician, Daystrom Electric.
JOHN W. SUMMERS, M.A. (San Jose S.C.) Grad. Student, University of California.
COL. DANIEL N. SUNDT, M.A. (Pennsylvania) Professor, University of Delaware.
BARRY H. TALSKY, Student, Syracuse University.
MRS. OLGA TAUSSKY, Ph.D. (Vienna) Research Asso., California Institute of Technology.
WILLIAM C. TEACHOUT, JR., Student, Memphis State University.
EDWARD O. THORP, Ph.D. (U.C.L.A.) Moore Instr., Massachusetts Institute of Technology.
GEORGE C. TIBBETTS, S.B. (Harvard) Director of Research, Tibbetts Industries, Inc.
JOHN L. TILLEY, M.Ed. (Florida) Grad. Asst., University of Florida.
LEON TISDALE, M.S. (Oklahoma S.U.) Asst. Professor, Southern Methodist University.
BURNETT R. TOSKEY, Ph.D. (Washington) Instr., Seattle University.
CATHERINE B. TURIANO, B.A. (Good Counsel) Programmer, Sperry Gyroscope Co.
CAROL S. UMBREIT, B.S. (Pennsylvania S.U.) Grad. Student, University of Michigan.
KEENE C. VAN ORDEN, M.A. (Minnesota) Chairman of Dept., Central High School, San Angelo, Texas.
MELVIN N. VESELY, M.Litt. (Pittsburgh) Senior Math. Teacher, Carrick High School, Pittsburgh, Pennsylvania.
MARVIN E. WALDEN, Student, Wayne State University.
ROY F. WALLER, B.S. (East Central S.C.) Engineer Asso., Western Electric Co.
CDR. WILLIAM R. WALLIS, M.A. (Texas) United States Navy, Retired, Austin, Texas.
LETTY MAY WALSH, B.A. (Augustana, Illinois) Instr., University of Hawaii.
EVERETT L. WALTER, M.S. (New Mexico S.U.) Instr., New Mexico State University.
JOHN W. WARNER, M.S. (S.U. of Iowa) Instr., College of Wooster.
ROBERT H. WASSERMAN, Ph.D. (Michigan) Asst. Professor, Michigan State University.
BARTON WASSERMANN, A.B. (Temple) Engineer, Philco Corp.
GARTH H. WEBBER, A.B. (Indiana Central) Programmer, System Development Corp.
RONALD C. WEGER, Student, William Jewell College.
ALEXANDER WEINER, M.A. (Columbia) Instr., Hofstra College.
MRS. CAROLE H. WEISS, Student, Vassar College.
MARY W. WELTY, Student, Ohio Wesleyan University.
MRS. HELEN C. WHEELER, M.Ed. (Trinity) Asst. Professor, San Antonio College.
PATRICK J. WHELAN, Student, St. Francis Xavier University, Nova Scotia.
JOHN T. WHITE, M.A. (Texas) Lecturer, University of Texas.
ROBERT J. WHITLEY, A.A. (San Diego J.C.) Student, San Diego State College.
FRED H. WHITLOCK, Student, Agricultural and Technical College of North Carolina.
H. HOLLIS WICKMAN, Student, University of Omaha.
ERIC L. WILSON, B.S. (Westminister, Pennsylvania) Vanderbilt University.
CHARLES T. WOLF, B.S. (Millersville S.T.C.) Asst. Professor, State Teachers College, Shippensburg, Pennsylvania.

C. RAYMOND WOODROW, B.A. (Greenville) Grad. Asst., Oklahoma State University.
LAWRENCE E. WRIGHT, Student, Massachusetts Institute of Technology.
WALTER ZAYACHOWSKI, M.A. (Saskatchewan) Grad. Student, University of Alberta.
MARVIN ZELEN, Ph.D. (American) Mathematician, National Bureau of Standards.

NEW SECTIONAL GOVERNORS OF THE ASSOCIATION

The following have been elected Governors of the Association for a three-year term beginning July 1, 1959 by a mail vote of the membership of the Association in the Sections indicated:

Illinois	Rothwell Stephens, Knox College
Iowa	H. T. Muhly, State University of Iowa
Louisiana-Mississippi	Arthur Ollivier, Mississippi State College
Maryland-D. C.-Virginia	R. C. Yates, College of William and Mary
Michigan	R. M. Thrall, University of Michigan
Minnesota	J. M. H. Olmsted, University of Minnesota
Philadelphia	Albert Wilansky, Lehigh University
Southern California	P. B. Johnson, Occidental College
Texas	W. T. Guy, Jr., University of Texas

In two sections, votes were received from more than 50% of the membership. These sections were Iowa with 58% and Louisiana-Mississippi with 52%.

L. J. MONTZINGO, *Associate Secretary*

THE NEW SECRETARY OF THE ASSOCIATION

Professor Henry L. Alder of the University of California, Davis, has been elected by the Board of Governors to the office of Secretary of the Association for the five-year term 1960–1964. He will begin to assume some of the responsibilities of his office after September 1, 1959, and it is expected that the transfer of all secretarial duties will be completed by January 1, 1960.

Professor Alder has been a member of the Association since 1950. He has served as a member of the Committee on High School Contests. In the Northern California Section of the MAA he has served as Vice-Chairman and Chairman as well as the first Chairman of the Section's Committee on Contests and Awards. He has been active in organizing the Program of Visiting Lecturers to Secondary Schools in Northern California and has himself served as a Visiting Lecturer. He has been President of Mu Alpha Theta since its organization in 1957; his term as President expires on July 31, 1959. He is Vice-Chairman of the Board of Governors of the Pacific Journal of Mathematics for a three-year term beginning June 1, 1957.

The By-Laws of the MAA were amended last January at the Philadelphia meeting to provide for the separation of the offices of Secretary and Treasurer. Since then I have been serving as Acting Secretary as well as Treasurer. It is with great pleasure that I am turning over some of my duties to Dr. Alder. A statement will appear in one of the early issues of the MONTHLY regarding the division of responsibilities between the Secretary and the Treasurer, in so far as this concerns the membership of the Association.

HARRY M. GEHMAN
Acting Secretary

THE TWENTIETH ANNUAL WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

The twentieth annual William Lowell Putnam Mathematical Competition will be held on Saturday, November 21, 1959. This competition, made possible by the trustees of the William Lowell Putnam Intercollegiate Memorial Fund left by Mrs. Putnam in memory of her husband, is under the sponsorship of the Mathematical Association of

America and is open to regularly enrolled undergraduate students in universities and colleges of the United States and Canada who have not yet received a college degree.

Application blanks will be mailed about October first to the regular mailing list. If an application blank is not received by October 15, you may secure one by writing the director, Professor L. E. Bush, 301 Merrill Hall, Kent State University, Kent, Ohio. Your application must be filed with the director not later than November 1, 1959. For further details of the examination and the list of prizes (including the $3000 scholarship at Harvard), see the announcement which will be mailed out along with the application blank.

Reports of the previous competitions and the examinations will be found in this MONTHLY for May 1938, 1939, 1940, 1941, 1942; October 1946; August-September 1947; December 1948; August-September 1949, 1950, 1951; October 1952, 1953, 1954, 1955; December 1956; August-September (announcement of winners) and November (questions and solutions) 1957; August-September, 1958; and this issue.

CARUS MONOGRAPH NUMBER 12

A new Carus Monograph is now in press. Publication is expected in November. Its title is:

MONOGRAPH 12: *Statistical Independence in Probability, Analysis and Number Theory*, by Mark Kac.

Each member of the Association may purchase one copy of this Carus Monograph at the special price of $1.75. Orders accompanied by payment should be addressed to: Mathematical Association of America, University of Buffalo, Buffalo 14, New York.

Additional copies of Monograph 12 for members and copies for nonmembers may be purchased at $3.00 from John Wiley and Sons, 440 Fourth Avenue, New York 16, New York.

Professor Kac's monograph is an expanded version of the three Hedrick Lectures he delivered at the 1955 Summer Meeting of the Mathematical Association of America. The monograph is designed to illustrate how simple observations can be made the starting point of rich and fruitful theories and how the same theme recurs in seemingly unrelated disciplines.

An elementary but thorough discussion of the game of "heads or tails," including the normal law and the laws of large numbers, is presented in a setting in which a variety of purely analytic results appear natural and inevitable. The chapter "Primes play a game of chance" uses the same setting in dealing with problems of the distribution of values of arithmetic functions. The final chapter "From kinetic theory to continued fractions" deals with a spectacular application of the ergodic theorem to continued fractions. A novel feature is the inclusion of a large number of problems.

The author is professor of mathematics at Cornell University and is the winner of the 1950 Chauvenet Prize of the Association.

THE FEBRUARY MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

The thirty-sixth annual meeting of the Louisiana-Mississippi Section of the Mathematical Association of America was held at the Buena Vista Hotel, Biloxi, Mississippi, on February 13–14, 1959, with Delta State College acting as host institution. The Friday afternoon meeting was held in two concurrent sessions. The Rev. T. F. Mulcrone, Louisiana Vice-Chairman, and Professor A. A. Ritchie presided. Professor C. G. Killen, Chairman of the Section, presided at the Friday evening and Saturday morning sessions. There were 153 persons registered, including 61 members of the Association.

The following officers were elected for the coming year: Chairman, Professor S. B. Murray, Mississippi State University; Vice-Chairman for Louisiana, Professor T. K.

Maddox, Southeastern Louisiana College; Vice-Chairman for Mississippi, Professor W. M. Sanders, Mississippi Southern College; Secretary-Treasurer, Professor T. L. Reynolds, Millsaps College.

At the business meeting a report on the progress of the high school mathematics contests was given by Professor Noel Childress, Chairman of the Committee on Contests. Professor Z. L. Loflin, Sectional Governor, reported on the recent meeting of the Sectional Officers.

The invited speaker for the meeting was Professor W. V. Parker of Alabama Polytechnic Institute. His lecture on Friday evening was entitled *Matrices and Polynomials* and his Saturday morning address was on *Commutative Matrices.*

The following papers were presented:

1. *The LSU year-long Mathematics Institute sponsored by NSF*, by Professor H. T. Karnes, Louisiana State University.

Beginning with the fall semester of 1959, Louisiana State University will have a year-long Mathematics Institute in mathematics. The program is sponsored by the National Science Foundation. This program will include the academic year, plus the summer semester of 1960. With this arrangement, the participants will be able to complete a program leading to the Master's degree. Forty participants will be in the program.

A specially designed curriculum of twenty-one semester hours has been prepared for the participants. This curriculum is designed to give the participants a broad background for the field of mathematics, and to consider topics which are likely to appear in the secondary school mathematics program in the near future. Those desiring a Master's degree will have to complete an additional nine semester hours of study in mathematics or related fields.

2. *Cyclo subpedal curves*, by Professor V. B. Temple, Millsaps College.

If $H(X_1, Y_1)$ and $E(X_2, Y_2)$ are two points on the hypocycloid and epicycloid respectively, corresponding to any given angle θ, then $P_1(X_1, Y_2)$ and $P_2(X_2, Y_1)$ are points on two cyclo subpedal curves. Since H and E are pedal curves, P_1 and P_2 are subpedals. In like manner the subpedals of the hypo- and the epirose curves are defined. (See this MONTLHY, vol. 59, p. 67.) All of this family of eight curves are governed by a single parametric angle θ; that is, if θ changes, all of the eight points move in harmony, each tracing its own distinct curve. A chart showing the curves P_1 and P_2 of the four cusped hypo- and epicycloids was presented.

3. *An experiment in readiness for logical thinking and demonstrative geometry*, by Professor G. J. Corley, Northwestern State College of Louisiana.

A discussion of an experiment in teaching some lessons in methods of reaching conclusions; intuition, inductive reasoning, and deductive reasoning; and an introductory unit in demonstrative geometry to sections of students from the sixth grade to the tenth grade.

4. *A set of postulates for trigonometry*, by Professor Arthur Ollivier, Mississippi State University.

The object of this paper is to exhibit a set of postulates from which all the properties of analytical trigonometry may be derived. The sine and cosine functions are defined in terms of a real variable without introducing the concept of an angle. Numerical aspects of the subject form an example of the theory.

5. *Remarks on inverse trigonometric functions and integration by trigonometric substitution*, by Professor W. E. Koss, Louisiana Polytechnic Institute.

The procedure in integration by trigonometric substitution as described in many calculus texts is fallacious, or at the least highly misleading. To rectify this it is essential to introduce into trigonometry meanings for, say, $\sin x = a$ and $\sin^{-1} a$, where $|a| \leq 1$, which are analogous to meanings

assigned in algebra to say, $x^n = a$ and $\sqrt[n]{a}$, where $a>0$. To accomplish this it is proposed that symbols for the inverse trigonometric functions be defined to insure first, single-valuedness and secondly, if possible, continuity.

Furthermore $\cos x = \sqrt{1-\sin^2 x}$ is not an identity unless the domain is restricted properly. In most calculus texts it is used as an identity without any mention of any restrictions. With the inverse functions properly defined and inverse trigonometric substitution used these errors may be avoided.

6. *On a certain property of infinite series*, by Professor R. D. Boswell, Jr., Mississippi State University.

The question answered in this paper arose in the teaching of infinite series in the calculus course. Consider the series $\sum_{k=1}^{\infty} u_k$. It is pointed out that a necessary and sufficient condition that a series converge is that (a) $\lim_{k\to\infty} u_k = 0$ and (b) there exists a sequence of positive integers $n_1, \cdots, n_i, \cdots$ such that $0 < n_i - n_{i-1} \leq M$ where M is a positive real number and such that the subsequence $S_{n_1}, \cdots, S_{n_i}, \cdots$ of the sequence of partial sums converges. An example is exhibited to show that the bound M on the differences $n_i - n_{i-1}$ is necessary.

7. *On algebra, trigonometry, and analytic geometry*, by Professor B. E. Mitchell, Louisiana State University.

Mathematics is supposed to be precise. Elementary textbooks in algebra, trigonometry, and analytic geometry are usually not precise, at times. This statement is clarified by illustrations from polynomials, factors, nth roots, equations, identities, and polar coordinates.

8. *G-spaces with the Barilian metric*, by Professor W. M. Sanders, Mississippi Southern College.

A Barilian space is defined to be the interior of a simple, closed curve J in E^2 remetrized with the Barilian metric $\delta(a, b) = \log\ (\max_{p\in J} ap/bp) + \log\ (\max_{q\in J} bq/aq)$. If the defining curve J is convex, then the Barilian space is a straight G-space.

9. *A characterization of the symmetric group*, by Professor J. D. Gilbert, Louisiana Polytechnic Institute.

After a preliminary theorem, the following result is presented:

Let G be a group transitive on $M = (a_1, \cdots, a_n)$, and let H_i be the subgroup of G which leaves $a_1, \cdots, a_i$ fixed for $i = 1, \cdots, n$. If $H_i \neq H_j$ for all pairs i, j such that $i \neq j$ and $i < n-1$, then G is the symmetric group on n elements. Conversely, if G is the symmetric group on n elements, then $H_i \neq H_j$ for all pairs $i \neq j$ and $i < n-1$.

10. *A note on certain approximations and their corresponding curves*, by Professor T. A. Bickerstaff, University of Mississippi.

Certain approximations were developed for distances, and their curves were described. Using the arithmetic mean of a geometric predecessor and successor of value to be approximated, the following result was obtained:

$$\sqrt{x^2 + y^2} \sim (8x^4 + 8x^2y^2 + y^4)/4x(2x^2 + y^2), \qquad x \geqq y.$$

Another adaptation led to the following interesting formula, with x^2 equal to the largest square in N, $D = |N - x^2|$: $\sqrt{N} \sim (8x^2N + D^2)/4x(x^2 + N)$. Curiously this represents a good approximation whether x^2 is greater than N or less than N so long as it is relatively near.

11. *An integration formula for the hyperbolic partial differential equation with region of integration exterior to the characteristic cone*, by Professor Paul Sanders, Louisiana Polytechnic Institute.

At the annual meeting of the American Mathematical Society in Pittsburgh in December

1954, R. B. Deal and E. W. Titt presented a method of deriving an integration formula for the hyperbolic equation with region of integration interior to the characteristic cone. Their method is applied to the problem of deriving an integration formula for the hyperbolic equation with region exterior to the characteristic cone.

12. *A characterization of compactness for regular spaces*, by Professor R. W. Bagley, Mississippi Southern College.

Let β be a family of finite character contained in 2^α where α is the collection of open sets in a topological space. We consider the following conditions, where $M(\beta)$ denotes the set of maximal elements of β. (1) If $\Omega \in M(\beta)$ then $U \notin \Omega$ and $V \notin \Omega$ implies that $U \cap V \notin \Omega$. (2) If $\Omega \in M(\beta)$ then $U \notin \Omega$ and $U \subset V$ implies that $V \notin \Omega$. (3) If $\Omega \in M(\beta)$ then $U \in \Omega$ and $V \in \Omega$ implies that $U \cup V \in \Omega$. (4) $\bigcap_{\Omega \in M(\beta)} \Omega = \{\phi\}$, the set whose only element is the null set.

THEOREM. *If X is a regular T_1 space with open subbase α_0 and there is a family β of finite character satisfying conditions* 1) *through* 4), *then X is compact if and only if no covering of X by sets of α_0 is a member of β.*

LEMMA. *If X is a topological space with open subbase α_0 and there is a family β of finite character satisfying conditions* 1) *and* 2), *then no open covering is a member of β if and only if no covering by sets of α_0 is a member of β.*

By taking β to be the family of open collections which have no finite subcollection covering X we have the following as a corollary to the Lemma.

(Alexander) A topological space X with open subbase α_0 is compact if and only if every covering by sets of α_0 has a finite subcovering.

S. R. KNOX, *Acting Secretary*

THE MARCH MEETING OF THE MICHIGAN SECTION

The annual meeting of the Michigan Section of the Mathematical Association of America was held on March 28, 1959 at Michigan State University, East Lansing, Michigan in conjunction with the meeting of the Michigan Academy of Science, Arts, and Letters. Professor G. Y. Rainich of the University of Michigan and Notre Dame University presided at both the morning and afternoon meetings and at the luncheon-business meeting. A total of 60 persons attended the meeting, including 44 members of the Association.

A nominating committee consisting of Professors C. H. Butler, Chairman, and W. K. Folley proposed Professor W. D. Baten of Michigan State University for Chairman, Professor E. E. Moise of the University of Michigan as Vice-Chairman and Professor L. E. Mehlenbacher of the University of Detroit as Secretary-Treasurer. The slate was elected unanimously.

Professor J. S. Frame of Michigan State University gave the Presidential address on *A bridge to relativity theory* before the Michigan Academy of Science, Arts, and Letters.

The committee on the Mathematics Prize Competition consisting of Professors F. L. Celauro, Chairman, Central Michigan University; A. J. Lohwater, University of Michigan; A. W. Jacobson, Wayne State University; W. E. Deskins, Michigan State University; and J. B. Eckstein, University of Detroit, gave a report of the second annual prize competition held in over 400 Michigan High Schools to over 8000 participating students.

The following papers were presented:

1. *Generalized derivatives*, by Professor Jacob Korevaar, University of Wisconsin. (By invitation).

Two wild animals in the mathematical zoo are the stickle-back (a continuous nowhere differentiable function), and the perfect sneak (a continuous strictly increasing function with zero derivative almost everywhere). In the physicist's jungle the δ animal, although untamed there, performs useful services. By now δ and the related improper animals have been completely do-

mesticated in mathematics. They have been caught in the theory of distributions. This theory is explained and its applications to these functions demonstrated.

2. *Comparison of estimates of the total number of cows in a certain area of Michigan by the ratio and the non-ratio methods*, by Professor W. D. Baten, Michigan State University.

The ratio method is as follows. Let x_i and y_i respectively represent the number of acres and cows in the ith farm. Let the ratio, R_n, be defined as $R_n = \sum y / \sum x$ where the numerator is the sum of the cows in a random sample and the denominator is the number of acres in the farms in the sample.

An estimate of the total number of cows in the area is $y_R = R_n X$ where X is the total number of acres in the area. This estimate is compared with another estimate that does not involve the relationship between x and y. The standard deviations of these estimates are used in the comparison.

3. *Three-dimensional poker hand frequencies by combinatorial analysis and by I.B.M. 704*, by Professor W. W. Funkenbusch, Michigan College of Mining and Technology.

The three dimensional deck consists of 64 cards, uniquely determined by 4 suits, 4 denominations, and 4 colors. In three dimensional poker, two cards are of the *same kind* if they have two common coordinates. Any one particular card can be used to help form only one unit. The numbers of Four of a Kind, Full House, Three of a Kind, Two Pair, One Pair, and No Pair hands can be determined analytically by decomposition into 21 *linkage types*. An I.B.M. 704 program for frequency determination, due to D. L. Shell of General Electric, is given.

4. *The development of the mathematical theory of linear perspective*, by Professor P. S. Jones, University of Michigan.

The high-lights of the development were presented from its beginning with Leone Battista Alberti in 1435 through the use of double orthogonal projection by Piero della Francesca and Albrecht Durer to the generalizations and extensions of Brook Taylor and J. H. Lambert. Connections with coordinates, descriptive and affine, as well as projective geometry were pointed out as well as suggestions for possible uses of these materials in teaching—especially in teacher preparation.

5. *A numerical methods program at General Motors Institute*, by Mr. M. L. DeMoss, General Motors Institute.

This paper outlines four intensive training courses offered at General Motors Institute for corporation people actively engaged in the solution of engineering problems using digital computers. The courses include iterative solution of algebraic and transcendental equations, matrices, interpolation and curve fitting, and numerical solution of differential equations. The author indicates that three of the subject courses are to be integrated and offered as an elective to General Motors Institute engineering students. He suggests that such a course could profitably be offered to mathematics majors by any liberal arts college not having a digital computer.

6. *Some applications of Boolean algebra to the nervous system*, by Sister Marian Joan, S.N.J.M., University of Detroit, introduced by the Secretary.

In this paper it was shown that Boolean algebra can be applied to nerve networks in order to analyze patterns of action known to occur, as in the phenomenon described by McCulloch and Pitts (*Bulletin of Mathematical Biophysics*, V. 1943) by which a person feels heat when he touches a cold object momentarily, but feels cold when he holds the object. It was also shown how Boolean algebra can be applied to testing neural models for fulfillment of the biological requirements for such models, by testing Piéron's neurological explanation of a possible mechanism for color vision published in 1952.

7. *Using $y = mx + b$ to score mathematics tests*, by Professor H. B. Anderson, Michigan College of Mining and Technology.

Linear grading is often discarded because the application is cumbersome or the results are unreliable. This paper suggests that the slope, determined by a minimum and a maximum performance, be set on an ordinary slide rule to simplify grade computations. A method of quiz construction based on core material is suggested for use in conjunction with linear grading. Using a departmental final involving about 400 students as a standard of comparison, linear grading gave a correlation coefficient of .85 against .73 for conventional methods as applied to mathematics students at the Michigan College of Mining and Technology.

8. *Capacity in two and three dimensions*, by Professor J. L. Ullman, University of Michigan.

Volume is a function on sets in three space, and area is a function on sets in the plane. They are related by the simple observation that the volume of a right cylinder of unit height is equal to the area of the cross section.

Capacity is a function on sets in three space, and there is another notion, also called capacity, defined for sets in the plane. The simple expedient that relates area and volume fails for the two notions of capacity, and a more elaborate procedure is developed that achieves this end. This procedure enables one to use intuition of electrostatics, which is closely related to the capacity of three dimensional sets, to the capacity of sets in the plane.

F. A. BEELER, *Secretary*

THE MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The thirty-ninth regular meeting of the Southern California Section of the Mathematical Association of America was held at the University of Redlands, Redlands, California, on March 14, 1959. Professor P. J. Kelly, Chairman of the Section, presided. The registered attendance was 132, including 96 members of the Association.

At the business meeting Professor R. A. Dean, Chairman of the Nominating Committee, reported that the following officers were elected by mail ballot for the next academic year: Chairman, Professor D. H. Hyers, University of Southern California; Vice-Chairman, Professor R. C. James, Harvey Mudd College; Secretary-Treasurer, Mr. R. B. Herrera, Los Angeles City College. The Nominating Committee appointed the following members to the Program Committee for the coming year: Professor Tom Apostol (Chairman), California Institute of Technology; Professor C. V. Holmes, San Diego State College; Professor P. J. Kelly, University of California, Santa Barbara; Mr. Fred Marer, Los Angeles City College; Dr. R. W. Rector, Ramo-Wooldridge Corporation. In regard to contests, the following motion was adopted at the business meeting: "The national office is hereby requested to conduct the national contest in the Southern California Section, starting in 1960, in those cities and rural areas in which local contests are not operating. A list of such areas will be furnished to the national office by the Section." (There are at least five major contests in areas of the Section at present.)

The following program was presented:

1. *Man versus machines*, by Dr. E. M. McCormick, Naval Ordnance Laboratory, Corona, California, introduced by the Secretary.

The relationships between mathematicians and automatic digital computers were considered. The emphasis was on the fact that these computers were developed originally as strictly mathematical tools, but they are now being used by a number of other disciplines. This has resulted in applications to translation of languages, retrieval of information, abstracting of articles, and other interesting fields. Anthropomorphic designations for computers and some of their functions were also considered.

2. *The geometry of numbers*, by Professor Kenneth Rogers, University of California, Los Angeles, introduced by the Secretary.

Minkowski's theorem on linear forms asserts that if we have n linear forms of determinant

one, then integer values not all zero can be found for the variables so as to make all the forms less than or equal to *one* in absolute value. It was proved by Hajos, that if the forms cannot all be made less than *one* then they are equivalent to a set whose matrix has *ones* down the diagonal and *zeros* above it. A new method for trying to prove Hajos' theorem was indicated.

3. *Decomposition*—one-hour invited address—by Professor Hans Zassenhaus, McGill University and California Institute of Technology.

The forms under which the Clavius postulate, that the whole is equal to the sum of its parts, appear in various mathematical fields, were traced.

4. *Auditory images of non-accelerated sources of sound*, by Professor Hugh Hamilton, Pomona College.

When an airplane traveling in a straight line at constant speed occupies any position P it reports itself to the ears of an observer on the ground at O (if at all) as being at some earlier position I (or two such positions, I_1 and I_2). Angular relationships at O between P and I (or I_1 and I_2) are developed, among them this: at supersonic speeds (when there is for a while no "I" but ultimately both "I_1" and "I_2") the average of the angular positions of I_1 and I_2 at O is always 90° behind the angular position of P at O. The techniques used are exclusively those of tenth grade geometry.

5. *Professor Begle's School Mathematics Study Group*, by Mr. L. C. Lay, Pasadena City College, and Mr. William Wooton, Pierce Jr. College, Los Angeles.

The scope of the School Mathematics Study Group and its objectives for the improvement of the teaching of mathematics in the secondary schools were presented. The progress of the Monograph Project of this group was reported, and mention was made of the plans for improving the training of mathematics teachers. The experimental textbooks, which they are writing for grades seven through twelve, were discussed.

6. *A stability theorem for non-linear ordinary differential equations*, by Professor H. A. Antosiewicz, University of Southern California.

The title refers to a theorem, essentially due to Malkin, on the uniform-total stability of the solution $y(t)\equiv 0$ of a vector differential equation (1) $y'=f(t, y)$ with $f(t, 0)\equiv 0$. It states that if $f(t, y)$ is Lipschitzian in y then uniform-asymptotic stability of $y(t)\equiv 0$ implies uniform-total stability; and if $f(t, y)$ is linear in y then exponential-asymptotic stability of $y(t)\equiv 0$ is equivalent to uniform-total stability. The importance of this theorem was discussed for problems in which the asymptotic behavior of the solutions of an equation (2) $x'=f(t, x)+g(t, x)$ with $g(t, x)$ small, in some sense, for large t, was studied by comparing the solutions of (2) with those of (1).

R. B. HERRERA, *Secretary*

THE APRIL MEETING OF THE IOWA SECTION

The 46th meeting of the Iowa Section of the Mathematical Association of America was held at Iowa Wesleyan College, Mt. Pleasant, Iowa, April 17, 1959. Professor E. N. Oberg, Chairman of the Section, presided. Total attendance was 50, including 23 members of the Association.

Routine business was considered during the afternoon meeting. The following officers were elected: Chairman, Professor J. J. L. Hinrichsen, Iowa State College; Vice-Chairman, Professor H. C. Trimble, Iowa State Teachers College; Secretary-Treasurer, Professor E. L. Canfield, Drake University.

The following papers completed the program:

1. *Wedges: A discussion of partially ordered linear spaces*, by Professor M. M. Day, University of Illinois. (By invitation).

The useful properties of a partial order relation ($\geq$) among elements of a linear space L can

be characterized by geometric properties of the set P of elements ≥ 0. Examples were given. Fixed vectors and fixed directions of linear operators preserving order were discussed next. Monotone linear functionals, their monotone linear extensions, and the geometric interpretation of the theorems which are known were mentioned.

2. *A necessary condition for the completeness of a family of probability measures*, by Professor R. V. Hogg, State University of Iowa.

A family $\{F(x;\theta);\theta\in\Omega\}$ of distribution functions is complete if $\int_{-\infty}^{\infty}u(x)dF(x;\theta)=0$, for all $\theta\in\Omega$ and any real $u(x)$ that is absolutely integrable with respect to $\{F\}$, implies that $u(x)=0$ almost everywhere in $\{F\}$. Let $\phi(t;\theta)=\int_{-\infty}^{\infty}\exp(itx)dF(x;\theta)$ be the characteristic function of $F(x;\theta)$ A family $\{\phi(t;\theta);\ \theta\in\Omega\}$ is c-complete if $\int_{-\infty}^{\infty}[\alpha(t)+i\beta(t)],\ \phi(t;\theta)dt=0$ for all $\theta\in\Omega$ and any real $\alpha(t)$ and $\beta(t)$ that are absolutely integrable, implies that $\alpha(t)=0$ and $\beta(t)=0$ almost everywhere Lebesgue. A necessary condition that the family $\{F(x;\theta);\ \theta\in\Omega\}$ be complete is that the family $\{\phi(t;\theta);\theta\in\Omega\}$ be c-complete.

3. *Method of solving a class of mixed boundary value problems*, by Professor Don Kirkham, Iowa State College.

It is shown, by solving two practical potential problem examples, how mixed boundary value functions in a class of problems may be developed, over a boundary, into an infinite series of harmonic functions, leading to the solution. The development is accomplished by introduction of an auxiliary infinite series valid over the portion of the boundary where the normal derivative is given. One of the examples is a two-dimensional one for which a solution can also be obtained by conformal transformations. The solution by the two methods agree. The other example involves Bessel functions. The method of this paper does not require the breaking up of the region in question into auxiliary spaces.

4. *Waring's problem, modulo p, and the representation symbol*, by Sister M. Anne Cathleen Real, Marycrest College, presented by Miss Nancy Ketelaar, introduced by the Chairman.

The representation symbol $[a, b, c]$ is the statement that an integer of n-ic type a is congruent to the sum of an integer of n-ic type b and an integer of n-ic type c. The symbol is extended to include any definite number of elements. New properties, together with a list of symbols involving the n-ic types of specific integers, are derived for use in studying Waring's problem, modulo p, for a particular exponent n. Let $T_p(n)$ be the least number such that every integer is congruent to the sum of $T_p(n)$ or fewer n-ic residues. Then for primes of the form $22k+1$, $k>3$, $2\leq T_p(11)\leq 4$.

5. *An Iowa mathematics test*, by Professor O. C. Kreider, Iowa State College.

An Iowa college mathematics test was constructed for Iowa high school seniors, the results to be used by the Iowa private and state colleges for admissions, scholarships, and placement. Three members of the test construction committee were from private colleges and three from state colleges. Fifty per cent of the items test mechanics, the other fifty per cent require thinking. The test items were selected from arithmetics, algebra (I and II), geometries (plane, solid and analytic), trigonometry, and miscellaneous subjects (statistics, sets, and logic).

6. *An explicit example*, by Professor A. T. Craig, State University of Iowa.

An elementary explicit example of a nonnormal bivariate distribution that has normal marginal distributions is given. Let $f(x, y)=f_1(x, y)+kf_2(x, y)$ where $-\infty<x,\ y<\infty,\ 0<|\rho|<1$, $f_1(x, y)=[2\pi(1-\rho^2)^{1/2}]^{-1}\exp[-(x^2+y^2-2\rho xy)/2(1-\rho^2)]$, and $f_2(x, y)=f_1(x, y)[x^2-(1+\rho^2)xy/\rho+y^2-(1-\rho^2)/2]\exp[-(x^2+y^2-2\rho xy)/2(1-\rho^2)]$. Now $|f_2(x, y)/f_1(x, y)|<M$ so that $f_1(x, y)[1+kf_2(x, y)/f_1(x, y)]\geq 0$ if $|k|<1/M$. Moreover, $\int_{-\infty}^{\infty}f_2(x, y)dx=\int_{-\infty}^{\infty}f_2(x, y)dy=0$ so that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}f(x, y)dydx=1$. That is, $f(x,y)$ is a nonnormal joint probability density function but each marginal distribution is normal. If one wishes to take $\rho=0$ in $f_1(x, y)$, he may at the same time replace, in the definition of $f(x, y)$, $f_2(x, y)$ by $f_1(x, y)(xy)\exp[-(x^2+y^2)/2]$.

7. *Some integrals and series involving Legendre associated functions that arise in added mass theory*, by Professors L. Landweber and Matilde Macagno, Iowa Institute of Hydraulic Research, State University of Iowa, presented by Professor Macagno.

The added mass of a sphere performing high frequency horizontal oscillations when half submerged in a liquid has been evaluated. The problem consists of solving Laplace's equation with the velocity potential vanishing on the free surface. The value of the added mass coefficient, *i.e.*, the ratio of the kinetic energy of the fluid to the kinetic energy of the fluid mass displaced by the hemisphere, is obtained by solving finite and infinite series containing integrals of the Legendre associated functions. The coefficient is found to be $C_H = 4/\pi - 1 = 0.2732 \cdots$.

8. *Shears and inequalities*, by Professor Bernard Vinograde, Iowa State College.

The purposes of this note are (1) to establish the non-cycling of a shear-translation procedure for solving linear inequalities, and (2) to compare the logic of a test for inconsistency with that of the elimination method as used by H. W. Kuhn (*Solvability and consistency for linear equations and inequalities*, this MONTHLY, vol. 63, 1956, pp. 217–232).

9. *Unconditional probability*, by J. G. Baron, M.D., Iowa City, Iowa.

The concept of conditional probability is used to calculate an unknown probability from known ones. This is done by a certain restriction of the possible cases. The usual expressions are *if known* or *if given*. A simple example is presented in which such a formulation leads to a contradiction. This contradiction is discussed. It is shown that the expressions mentioned may involve a volitional act of an informer. This makes the use of the concept of mathematical probability impossible. If the expression *known as a result of a random experiment* is substituted, then the paradox is eliminated.

10. *Note on a limiting distribution*, by Professor D. A. Jones, State University of Iowa, (read by title only).

Let X be a real random variable of the continuous type possessing a probability density function (pdf), $f(x)$, which is continuous on the interval (a, b), positive almost everywhere [Lebesgue] on $[a, b]$ and zero outside of $[a, b]$. Let X_1 and X_n denote the smallest and largest items, respectively, of a random sample of size n from the distribution. Let $Q(x)$ be a measurable function essentially bounded [Lebesgue] on (a, b); we may assume $E[(Q(X)]=0$ and $E[\overline{Q(X)^2}]=1$. Then $y=n\int_a^{X_1} f(x)dx$, $v=n\int_{X_n}^b f(x)dx$, and $z=(1/\sqrt{n})\sum_{\alpha=1}^n Q(X_\alpha)$ have a limiting distribution given by the joint pdf: $(1/\sqrt{(2\pi)})\exp(-y-v-z^2/2)$ for $y>0$, $v>0$, all z; and 0 elsewhere.

11. *An application of generalized means*, by Professor S. D. Nolte, Iowa State College, introduced by the Secretary.

The generalized mean $M(x, y)$ was defined to be $\psi^{-1}[p\psi(x)+q\psi(y)]$ where $p, q>0$, $p+q=1$ and $\psi(x)$ is monotone and continuous. This mean was applied to the second difference $\Delta^2(f; x, h) = f(x+h)+f(x-h)-f(x)$ to form a generalized second difference $\Delta_\psi^2(f: x, h) = M_\psi[f(x+h), f(x-h)] - f(x)$.

A study was made of functions whose generalized second differences satisfy certain conditions. Maxima of classes of generalized quasismooth functions were examined.

E. L. CANFIELD, *Secretary*

THE APRIL MEETING OF THE KANSAS SECTION

The forty-fourth annual meeting of the Kansas Section of the Mathematical Association of America was held at Marymount College, Salina, Kansas, on April 11, 1959, in conjunction with the annual meeting of the Kansas Association of Teachers of Mathematics. Professor P. S. Pretz, Chairman, presided at the sessions. Of the 159 persons registered, 78 are members of the Association.

The following officers were elected for one year terms: Chairman, Professor J. D. Haggard, Kansas State College of Pittsburg; Vice-Chairman, Professor W. D. Bemmels, Ottawa University; Secretary-Treasurer, Miss Helen Kriegsman, Kansas State College of Pittsburg.

At the joint session, held in the morning, Professor B. W. Jones, University of Colorado, delivered a paper entitled *A method for solving some quadratic Diophantine equations.*

The following short papers were presented at the afternoon session:

1. *A geometric number system*, by Sister Helen Sullivan, Mount St. Scholastica College.

A geometric number system is developed by employing a procedure parallel to that used in the construction of an algebraic number system. Equivalence classes of integers, rationals and reals are defined in terms of appropriate flat pencils. The operations of addition and multiplication are achieved by setting up flat pencils in an involution. Comments (without proofs) were given on the field properties in each system. Carefully drawn illustrations for adding and multiplying the lines representing equivalence classes of integers, rationals and reals were shown by use of an opaque projector. The isomorphisms present were also discussed.

2. *Series transformations*, by Professor J. R. Hanna, University of Wichita.

A brief review of the series transformation, $M\{F(t)\}=f(s)=\sum_{t=0}^{\infty}K(s,t)F(t)$ was outlined. Special attention was given to a few properties of the series transformation used by Laplace. In this transformation, the kernel $K(s, t)=s^t$ was used. By employing the Cauchy product, a convolution theorem was established.

3. *Curves in Euclidean space*, by Professor W. L. Stamey, Kansas State University.

Some possible definitions for the concept of a curve are given. The applicability of each possible definition is illustrated by examples of point sets which, according to the given definition, would be curves and examples of point sets which would not be curves. All the examples are subsets of Euclidean two- or three-dimensional space. The trial definitions lead to the definition of a curve as a one-dimensional continuum with dimension defined in the sense of Menger and Urysohn.

4. *A report on the current offerings in mathematics through the calculus at the various colleges in the state of Kansas*, by Professor C. B. Read, and Professor Agnes Nibarger, University of Wichita.

A questionnaire sent Kansas colleges, asking about mathematics courses through calculus indicates relatively rare offering of combination algebra-trigonometry-analytics but considerable offering of analytics-calculus. Schools with such offering form a minority, but *students involved* a majority. Large schools often offer the fused course in addition to separate courses. Some possible reasons for the trend toward combined courses are: a saving in class time spent developing subject matter, a saving in money spent for textbooks, a reduction in teaching load. A discussion was given of actual achievements made by the change and of some problems likely to occur.

5. *The modernization of mathematics*, by Professor O. J. Peterson, Kansas State Teachers College.

The paper presented the nature and purposes of modern secondary mathematics and the work of supporting groups, with special reference to the School Mathematics Study Group, of which Professor E. G. Begle, Yale University, is Executive Director. Questions discussed related to (1) the need for modernized elementary college texts, (2) the need for the establishment of institutes for capable college teachers with conventional mathematics backgrounds, and (3) the articulation of present college freshman courses with preparation in modern high school mathematics.

Attention was called to the urgency of solutions of these and other problems which colleges

will face when considerable numbers of high school graduates enroll with preparation in the new mathematics.

HELEN KRIEGSMAN, *Secretary*

THE APRIL MEETING OF THE METROPOLITAN NEW YORK SECTION

The eighteenth annual meeting of the Metropolitan New York Section of the Mathematical Association of America was held at Polytechnic Institute of Brooklyn on April 18, 1959. Dr. Ernst Webber, President of Polytechnic Institute of Brooklyn gave the address of welcome and discussed *The impact of mathematics on engineering education.* Professor Azelle B. Waltcher, Collegiate Vice-Chairman of the Section, presided at the morning session and Dr. R. N. Walter, High School Vice-Chairman, presided at the afternoon session which was a panel discussion on *The role of geometry in secondary and higher education.* There were 144 persons in attendance, including 79 members of the Association.

Professor James Eastham, Chairman of the Section, presided at the business meeting. Reports were given by Professor Eastham on the Speaker's Bureau; by the Governor, Professor Jewell H. Bushey; the Secretary, Professor June Jensen; and Professor Charles Salkind, for the Committee on Contests and Awards. The following officers were elected: Chairman, Professor Azelle B. Waltcher, Hofstra College; Vice-Chairmen, Professor Jules Russell, Polytechnic Institute of Brooklyn, and Mr. George Grossman, William Howard Taft High School; Secretary, Professor Mary Dolciani, Hunter College; and Treasurer, Mr. Aaron Shapiro, Brooklyn College.

The following papers were presented at the meeting:

1. *Some topics in set theory*, by Professor Smbat Abian, Queens College.

Set theory is found to be essential in formulating the basic concepts of almost every mathematical discipline. The axiom system Z, which is Fraenkel von Neumann's modification of Zermelo's axiom system is described here. The chief motivation in formulating the axioms of system Z is the elimination of set theoretical antinomies. Although Z is sufficient for developing classical set theory, one weakness is that its axioms of subsets and substitution are not proper axioms, but axiom schemas. Also Z imposes some undesirable restrictions on the set concept. The more recent formulations by von Neumann, Bernays and Gödel, of the axiomatization of set theory have been attempts to overcome these shortcomings.

2. *Matrix theorems with application to the rate behavior of metabolic systems*, by Dr. J. Z. Hearon, Office of Mathematical Research, National Institute of Arthritis and Metabolic Diseases, introduced by the Secretary.

For a linear physical system with rate matrix K, the principle of detailed balance insures the existence of a positive definite diagonal matrix A such that KA is symmetric. This implies that K is similar to a symmetric matrix. The consequence of these properties are discussed in terms of symmetry properties of the matrix of principle solutions and certain boundary value problems. For a nonlinear system the corresponding properties of the Jacobian matrix are established and discussed in terms of "relaxation time analysis" of systems close to equilibrium.

3. Panel Discussion by Mr. E. C. Douglas, Taft School, representing the Commission on Mathematics.

Since the Commission firmly believes that the study of geometry should continue to be the basis of a full year's work in secondary school mathematics, it has formulated a program for the tenth grade which is directed to the study of geometry. The following are objectives: (1) the acquisition of information about geometric figures in two as well as three dimensions; (2) the development of a better understanding of the deductive system as a way of thinking and acquiring further skill in applying this method to problems in mathematics; (3) the exposure to opportunities for creative and original thinking. The Commission firmly believes that these objectives can be

met by incorporating with plane geometry some coordinate geometry, and essentials of solid geometry and space perception.

4. Panel Discussion by Professor R. J. Walker, Cornell University, representing the School Mathematics Study Group.

Training in clear thinking, precise language, and logical reasoning should appear somewhere in the high school program. Geometry still seems to be the best vehicle to which to attach such training, having more motivation, in its close contact with simple properties of the physical world, than algebra or any non-mathematical subject. Of the several known rigorous developments of Euclidean geometry, the one proposed by G. D. Birkhoff has the advantages of leading rapidly to the more interesting theorems and of making use of the students' knowledge of algebra. Using a modification of the approach, coordinate geometry can be introduced quickly; but the analytic techniques, being largely mechanical in their application are not over-stressed. The two- and three-dimensional aspects of geometry are closely interwoven.

5. Panel Discussion by Professor Walter Prenowitz, Brooklyn College, speaking on *Geometry in the college curriculum.*

JUNE R. JENSEN, *Secretary*

THE APRIL MEETING OF THE MINNESOTA SECTION

The annual spring meeting of the Minnesota Section of the Mathematical Association of America was held on April 25, 1959, at the University of Minnesota in Minneapolis. The meeting was a joint one with the Minnesota Council of Teachers of Mathematics. Professor J. M. H. Olmsted of the University of Minnesota presided at the morning session. The Rev. W. C. Kalinowski, O.S.B., of St. John's University presided at the afternoon session. There were 114 persons in attendance, including 74 members of the Association.

At the business meeting Professor Leon Green of the University of Minnesota reported for the High School Contest Committee on another successful year in giving the Association's high school mathematics contest.

Professor Warren Loud, chairman of the Section's Nominating Committee, nominated the following slate of section officers who were subsequently elected: Chairman, Professor Gerald Heuer of Concordia College; Secretary, Professor F. L. Wolf of Carleton College; Members of the Executive Committee, Professor John Hafstrom of the University of Minnesota, Duluth Branch, Rev. W. C. Kalinowski and Professor James Serrin of the University of Minnesota.

The following papers were presented:

1. *Evaluation of achievement in mathematics*, by Professor P. C. Rosenbloom, University of Minnesota and the Minnesota State Department of Education.

The evaluation of new curricular material is a vast research problem, requiring the cooperation of mathematicians, psychologists, and educators. Existing achievement tests measure only rote mastery of skills. New tests will have to be devised to measure understanding of concepts, powers of generalization and abstraction, interest, attitudes, and appreciations. The organization of the Minnesota National Laboratory for the Improvement of Secondary School Mathematics as an agency of the Minnesota State Department of Education was described.

2. *Panel discussion on introducing "modern" mathematics into the curriculum*, by Professors B. R. Gelbaum, University of Minnesota, Seymour Schuster and R. W. Sloan, Carleton College.

3. *Motivation of secondary school mathematics courses through the modern electronic computer*, by Dr. R. E. Smith, Control Data Corporation.

The essentials of computer programming can be taught to secondary school students. Used

as a special topic it can do the following: 1) open to students the door to the world of practical computations based upon that which the mathematics teachers are now teaching; 2) demonstrate to the student that the "long" and "tiresome" hand method of solving a problem is often the shortest and "nicest" approach for the computer; 3) introduce to the student the need of organization, planning, and logical thinking required to analyze each problem; 4) bring to the student an awareness of the impossible both in capacity and timing; 5) make the student acquainted with the fundamentals of a science which is growing, important in his own state, and may some day claim him as an employee.

4. *Round-off in square root algorithms*, by Dr. D. M. Brown, Remington Rand Division, Sperry Rand Corporation, St. Paul, Minnesota.

Newton's method for solving for the zeros of $y=x^2-B$ provides the algorithm $x_{i+1}=(B/x_i+x_i)/2$ which, for positive x_0, converges to $\sqrt{B}$. Since finding the square root of an integer M, correct to d decimal places can be reduced to finding the root of the integer $(10)^{2d}M$ to the nearest integer; we restrict B to integral values. If we use $t_i=[x_i]$, we find that t_i converges to $\sqrt{B}$, unless $B=C^2-1$, in which case t_i oscillates between C and $C-1$. If we use $x_{i+1}=[(B-1)/x_i+x_i+1]/2$, it can be shown that $[x_i]$ always converges to the nearest integer to B. Hence this last formula provides as good an approximation to $\sqrt{B}$ as one can find.

5. *The Banach-Tarski paradox*, by Professor L. H. Loomis, Harvard University.

A derivation of the Banach-Tarski theorem was given.

6. *A functional inequality* by Dr. George Brauer, University of Minnesota, and Mr. Donald Kurth, Minneapolis-Honeywell Regulator Company, Hopkins Minnesota.

Let $f(x)$ denote a real-valued function and let f^n denote the nth iterate of f. The continuous functions $f(x)$ such that (1) $f^n(x)\geq x$, $-\infty<x<\infty$, for some positive integer n, are characterized here. If n is odd and $f(x)$ is a solution of (1) then $f(x)\geq x$ for all x. If n is even and $f(x)$ is a solution of (1), then either $f(x)\geq x$ for all x or $f^2(x)$ is identically equal to x.

7. *The numerical range of a matrix operator*, by Professor W. S. Loud, University of Minnesota.

The numerical range of a linear operator in a two-dimensional complex vector space is shown to be an ellipse in the convex plane with the two eigenvalues of the operator as foci. If the operator has the matrix $\{a_{ij}\}$, the numerical range is the set of values assumed by $\sum a_{ij}\bar{x}_i x_j$ for all $\{x_1, x_2\}$ with $|x_1|^2+|x_2|^2=1$. When the matrix is reduced to subdiagonal form by a unitary transformation, the quadratic form becomes $\lambda_1|x_1|^2+\lambda_2|x_2|^2+\epsilon x_1\bar{x}_2$, where $\epsilon=0$ if and only if the operator is normal, and $|\epsilon|$ is independent of the diagonalizing unitary transformation. The minor axis of the elipse is $|\epsilon|$.

F. L. WOLF, *Secretary*

THE APRIL MEETING OF THE MISSOURI SECTION

The Missouri Section of the Mathematical Association of America met at Lindenwood College, St. Charles, Missouri, on Saturday, April 25, 1959. Professor H. M. MacNeille, Vice-Chairman of the Section, presided at the morning program session. Professor Francis Regan, Chairman of the Section, presided at the business meeting and the afternoon program session. The total attendance was 56, including 43 members of the Association.

The following officers were elected: Chairman, Professor C. E. Kelly, Central Missouri State College; Vice-Chairman, Professor L. O. Jones, William Jewell College; Secretary-Treasurer, Miss Marian Lesher, Central Missouri State College.

Professor Regan presented the proposed by-laws for the Missouri Section, which were adopted. Mr. Richard Spreckelmeyer, Chairman of the Contest Committee, reported for

that committee. On motion, it was recommended to the new Chairman that Mr. Spreckelmeyer be reappointed to a two year term.

The morning program consisted of a talk by Professor E. E. Moise, University of Michigan, on phases of the work of the School Mathematics Study Group. This was followed by a general discussion.

The afternoon program was held in conjunction with the Missouri Council of Teachers of Mathematics. Professor R. V. Andree, University of Oklahoma, spoke on the subject, *What is this modern mathematics anyway?*

S. LOUISE BEASLEY, *Secretary*

THE APRIL MEETING OF THE NEBRASKA SECTION

The thirty-fifth annual meeting of the Nebraska Section of the Mathematical Association of America was held on April 18, 1959 at the University of Nebraska, Lincoln, Nebraska, in conjunction with the sixty-ninth annual meeting of the Nebraska Academy of Sciences. Professor J. F. Wampler, Chairman of the Section, presided. There were 75 persons present, including 25 members of the Association. The first session, at which two papers were read, was held jointly with the Nebraska Section of the National Council of Teachers of Mathematics.

The following officers were elected for 1959–1960: Chairman, Professor D. W. Miller, University of Nebraska; Vice-Chairman, Professor J. F. Wampler, Nebraska Wesleyan University; Secretary-Treasurer, Professor H. M. Cox, University of Nebraska.

Professor J. M. Earl was continued as Chairman of the Committee on Mathematics Contests. The committee consists of representatives of the Nebraska Section of the Mathematical Association of America, the Nebraska Section of the National Council of Teachers of Mathematics, and of the Nebraska Actuaries Club.

The following papers were presented:

1. *Mathematics curriculum experiments*, by Professor H. L. Prouse, Mankato State College, Mankato, Minnesota.

This year the Mankato State College Campus School has been participating in mathematics curriculum experiments which include: (1) The seventh grade using the School Mathematics Study Group materials; (2) One ninth-grade section using the Illinois First Course materials. Another innovation being tried involves the eighth grade in which a traditional text is used along with supplementary enrichment topics.

Descriptions of the content of the School Mathematics Study Group seventh and eighth grade materials and the Minnesota National Laboratory were given.

2. *The Second Nebraska (Tenth National) Mathematics Contest*, by Professor J. M. Earl, Municipal University of Omaha.

There were 2,443 students from 134 high schools registered for the Contest and 1,862 score sheets were returned for grading. The scores of all of their own contestants were mailed to all competing schools on March 27 together with other materials which included a tally of all team scores and lists of the 20 high team scores and 20 high contestants in both 1958 and 1959. The Contest's new state sponsor, The Nebraska Actuaries Club, has helped greatly with the work and the expense of the Contest, but, again, the main burden has been shouldered by the Bureau of Instructional Research of the University of Nebraska.

3. (By title) *Algebraic factoring: a process of partial addition*, by Mr. G. B. Paulien, Extension Division of the University of Nebraska.

4. *Honors Mathematics Program at the University of Nebraska*, by Professor D. L. Guy, University of Nebraska (read by Professor D. W. Miller).

In the fall of 1958 the University began giving a sequence of three courses for freshman stu-

dents of high calibre. An outline of the selection procedure, content of the course, and a report of results so far is given. In addition, two seminar programs, one for upper classmen and one for beginning graduate students is discussed.

5. *The Riemann zeta-function*, by Mrs. Mildred Gross, University of Nebraska.

The relationship of the Riemann zeta-function to several other arithmetical functions was discussed on an expository basis. The problem of evaluating the function for various values of s was presented, together with a derivation of the value of zeta of 4 using a Fourier series. Also discussed was the analytic continuation of the zeta-function to the entire complex plane and the resulting functional equation which can be used to locate the zeros of the zeta-function. Also, reference was made to Riemann hypothesis.

6. *The concept of function*, by Professor J. F. Wampler, Nebraska Wesleyan University.

This paper consisted of a brief discussion of the historical development of function, remarks concerning functional notation and a comparison of the three main definitions of function. Emphasis was placed upon the role of set theory in the present-day work with functions.

H. M. Cox, *Secretary*

THE APRIL MEETING OF THE OKLAHOMA SECTION

The annual spring meeting of the Oklahoma section of the Mathematical Association of America was held at The University of Tulsa, Tulsa, Oklahoma, on April 10, 1959. Professor D. P. Richardson, Chairman of the Section, presided during the meeting, which was devoted to the reading of mathematical papers of a research or expository nature. There were 70 persons present, including 41 members of the Association.

The following invited addresses of roughly twenty minutes each were presented:

1. *A question on groups and the axioms for the integers*, by Professor J. E. Hoffman, Oklahoma State University.

2. *Runcible groups*, by Mr. A. S. Davis, University of Oklahoma.

A runcible group $G(\cdot)$ consists of a set G and a binary operation $(\cdot)$ satisfying: (1) if $x\cdot y$ is defined and $y\cdot z$ is defined, then either $x\cdot(y\cdot z)$ and $(x\cdot y)\cdot z$ are both undefined, or else both are defined and are equal to the same element in G; (2) G contains an identity element; (3) each element has an inverse in G. Much of group theory, including the Jordan-Hölder theorem, carries over to these "groups without enclosure." If $H(\cdot)$ may be obtained from $G(\cdot)$ by defining products in $G(\cdot)$, then $G(\cdot)$ is said to be groupable into $H(\cdot)$. The class of all runcible groups groupable into a given group forms a complete lattice.

3. *A technique for approaching fixed point theorems*, by Mr. Tetsundo Sekiguchi, Oklahoma State University, introduced by the Secretary.

4. *On computing the fertility of sets of premises in the propositional calculus*, by Professor W. E. Stuermann, University of Tulsa.

This paper examines the condition imposed by a premise conjunction on any conclusion implied by it. The paper then discloses the pattern exhibited by the frequency distribution of all possible truth functions on the basis of the number of clauses in the developed disjunctive normal form—it is Pascal's triangle. This makes it possible to compute the number and frequency distribution of, and to construct, all valid core *conclusions* for a given set of premises. We have, then, a technique for computing the fertility of any given set of premises or the relative fertility of different sets of premises. Examples are given.

5. *The skew cubic*, by Professor N. A. Court, University of Oklahoma.

The skew cubic C_3 may be considered as the locus of the harmonic pole, for a tetrahedron (T) inscribed in C_3 of a variable plane passing through a definite line s. The lines s', s'', s''' which corre-

spond to s in the three skew harmonic homologies having for axes the three pairs of opposite edges of (T) are secants of C, and the four lines s, s', s'', s''' form a hyperbolic group.

The tangents to C_3 at the vertices of (T) meet the respectively opposite faces of (T) in the four vertices of a tetrahedron (U) which forms with (T) a Moebius pair. Given a pair of Moebius tetrahedrons, a skew cubic, and only one, may be circumscribed about each of them so that the lines joining corresponding vertices of the two tetrahedrons shall touch the two cubics at the respective vertices.

6. *Some remarks on indecomposable continua*, by Mr. P. E. Long, Oklahoma State University.

Mr. Long quoted and discussed the definition of an indecomposable continuum as a continuum which is not the union of two of its proper subcontinua, giving several examples and stating two necessary and sufficient conditions that a continuum be indecomposable. He also discussed some recent results in this field.

7. *A characterization of property* (P), by Dr. J. E. Scroggs, University of Arkansas.

A bounded normal operator A on the Hilbert space H is said to have *property* (P) if every invariant subspace of H under A is also reducing. The sequence of operators A_n is said to converge to the operator A in the weak topology for the space of operators on H if $\lim_{n\to\infty} |(A_n x, y)-(Ax, y)|=0$, for every $x, y\in H$. Wermer has shown that a sufficient (but not necessary) condition that A have property (P) is the weak convergence of a sequence of polynomials in A to A^*. A necessary and sufficient condition that A have property (P) is given.

8. *A Toeplitz endomorphism theorem*, by Mr. E. P. Kelley, Jr., Oklahoma State University.

A proof of the following theorem was given. *If G is a group, H a normal subgroup of G, α an endomorphism on G such that $h\alpha=h$ for $h\in H$, then $H\alpha^{-1}=H\times K$.* An endomorphism having this property is called a Toeplitz endomorphism. If G is the additive group of the residue class ring of bounded sequences of real numbers modulo the ideal of null sequences then the regular or Toeplitz matrices induce an endomorphism on G which is an automorphism on the subgroup of convergent sequences modulo the null sequences. The null sequence ideal is maximal in the subring of convergences and the residue class ring is the real field.

9. *Lipschitzian parameterizations and existence of minima in the calculus of variations*, by Dr. G. M. Ewing, The Army Artillery and Missile School, Fort Sill, Oklahoma.

10. *Abstract summability methods*, by Professor R. B. Deal, Oklahoma State University.

11. *A generalized derivative*, by Mr. T. W. Cairns, Oklahoma State University.

12. *Derivation of equations suitable for a statistical study of the amount of excess propellant in a liquid-bipropellant rocket*, by Mr. R. M. McDonald, University of Tulsa and Douglas Aircraft Company, Tulsa, Oklahoma.

The amount of unused propellant remaining aboard a liquid bipropellant rocket at burnout must be known to load the rocket efficiently. Numerous random influences, however, make it impossible to predict this amount analytically. A statistical evaluation of the rocket's propellant utilization parameters is too difficult to obtain. Therefore, we study artificial stochastic models of the necessary parameters. A mathematical model of the average in-flight propellant mixture ratio is developed. Mention is made of the other parameters which determine propellant utilization.

R. V. ANDREE, *Secretary*

THE APRIL MEETING OF THE SOUTHWESTERN SECTION

The annual meeting of the Southwestern Section of the Mathematical Association of

America was held at Arizona State University, Tempe, Arizona, on April 10–11, 1959. Professor J. H. Butchart, Chairman of the Section, presided at the afternoon session on April 10, and also at the morning session on April 11. There were 53 persons in attendance, including 43 members of the Association.

The following officers were elected: Chairman, Professor R. B. Crouch, New Mexico State University; Vice-Chairman, Professor Harvey Cohn, University of Arizona; Secretary-Treasurer, Professor Deonisie Trifan, University of Arizona.

The following papers were presented:

1. *Some conjectures concerning the decomposition of an algebraic integer into squares*, by Professor Harvey Cohn, University of Arizona.

The author made large scale tests to explore the representability of a totally positive integer, as the sum of a given number of quadratic squares; enabling the author to make very many conjectures including a provable conjecture that the integer $a+2b\sqrt{2}$, $a>2b\sqrt{2}\geqq 0$ can always be represented as the sum of four squares in a manner similar to the known case of $\sqrt{5}$ (Götzky, Ann. of Math. vol. 100). In addition, in no case tested from $\sqrt{2}$ to $\sqrt{41}$ were six squares ever needed. Results will appear in Numerical Mathematics, vol. 1. The GEORGE computer at Argonne Laboratories of the AEC was used. Supported by NSF grant No. G-4222.

2. *Orthoptic of the cardioid*, by Professor J. H. Butchart, Arizona State College, Flagstaff.

It is proved synthetically that the orthoptic of a cardioid consists of two branches, one a circle and the other a nodal limaçon.

3. *A theoretical analysis of errors in hyperbolic positioning systems*, by Mr. G. J. Simmons, Sandia Corporation.

The error involved in the use of asymptotic formulas with hyperbolic positioning systems, valid for very distant sources, was investigated for near sources. This error function was expressed in terms of the differential operator $d(x^{-1})/dx$ which generates a set of polynomial coefficients in the series expansion of the error. The computed error surface $\epsilon(\beta, Z)$ as a function of direction and distance was exhibited. Finally this error was shown to be geometrically the angular difference between the radius vector to a point and the asymptote to the hyperbola through the point.

4. *Modernizing the mathematics curriculum*, by Professor Charles Wexler, Arizona State University.

Desirable improvements in the mathematics curriculum in the grade school and high school are weighed against the capabilities of the teachers. (1) Experiments by Beberman, by Rosenbloom, and by the author indicate that the better 30% or 40% of the grade school students can learn the equivalent of one year of algebra and perhaps one year of geometry under the right kind of teaching. NSF Institutes are greatly needed in this area. (2) In the high schools, teachers are more or less ignoring the prodding of the Commission on Mathematics to include set theory, modern algebra, and statistical inference, but instead, and against the words of caution of the Commission and C.U.P., they are enthusiastically including analytic geometry and calculus in accelerated programs. It is suggested that NSF Institutes fall in with this irresistible trend and include refresher work in calculus plus function theory background. (3) In the colleges, the completion of a strong calculus course should take precedence over a course in set theory and modern algebra, although mathematics majors might take such a course concurrently.

5. *Extensions of homomorphisms*, by Professor E. A. Walker, New Mexico State University.

Let G be a reduced Abelian p-group and B a basic subgroup of G. Let α be a homomorphism of B into G. It is shown that α can be extended in at most one way to an endomorphism of G. In particular, an endomorphism of G that leaves a basic subgroup elementwise fixed is the identity

automorphism of G. These results follow from a more general theorem for arbitrary Abelian groups.

6. *Uniform distribution and almost periodic functions*, by Professor G. M. Petersen, University of New Mexico.

A well-known theorem of Weyl states that a sequence is uniformly distributed if and only if for every $h=1, 2, \cdots$, $\lim_{n\to\infty} 1/(n+1)\sum_{k=0}^{n} e(2\pi h s_k)=0$ where we write $e(x)$ for e^{ix}. If $\{s_k\}$ is a sequence, bounded or unbounded, we shall call $\{s_k\}$ uniformly distributed with respect to the sequence $\{\lambda_p\}$ if $\lim_{n\to\infty} 1/(n+1)\sum_{k=0}^{n} e(\lambda_p s_k)=0$ for each p. We then prove the

THEOREM. *Let $f(x)$ be a u.a.p. function with Fourier exponents $\{\lambda_p\}$, then* $\lim_{n\to\infty} 1/(n+1) \sum_{k=0}^{n} f(s_k)=\lim_{T\to\infty} (1/T)\int_0^T f(x)dx$ if $\{s_k\}$ *is uniformly distributed with respect to* $\{\lambda_p\}$.

7. *On ideals in partially ordered groups*, by Professor J. W. P. Mayer-Kalkschmidt, University of New Mexico, introduced by the Secretary.

Let P be a group in which a partial ordering is defined. The author considers the properties (For the notation compare O. Frink, *Ideals in partially ordered sets*, this MONTHLY, vol. 61, 1954, 223–234): P_1: If $x\leq y$, then $axb\leq ayb$; P_2: $a(A^{*+})b=(aAb)^{*+}$; P_3: If J is an ideal then aJb is an ideal; P_4: If J is a completely irreducible ideal, then aJb is a completely irreducible ideal. He shows that the properties P_1, P_2, P_3, P_4 are equivalent, and discusses some implications of this fact. (This research was sponsored by a National Science Foundation Grant.)

8. *Loci associated with families of osculants*, by Mr. Louis Child, New Mexico State University.

At P_0 on a plane curve Γ, a member g_8 of the 6-parameter family G_8 of 8-pointic quartics, meets each member f_8 of the 1-parameter family F_8 of 8-pointic cubics in 8 points at P_0 and ordinarily at 4 additional points, through which passes a unique 4-pointic cubic δ_4 having a node at P_0. The center of curvature Q of the non-tangent branch of δ_4 at P_0 is thus unique for f_8 relative to g_8 so that F_8 determines a curve ϕ of centers for g_8. Each 1-parameter subfamily G of G_8 then defines a 1-parameter set Φ of curves ϕ whose envelope is unique for P_0.

9. *Use of orthogonal functions for least squares approximation with missing points*, by Professor E. L. Walter, New Mexico State University.

The author approximates a function with domain D by a set of functions orthogonal on O, where O has p points and D consists of $p-m$ of these and n additional points. Approximate coefficients are obtained using the $p-m+n$ values over D, then corrected with the solutions of $m+n$ equations. If $m+n<N$, this method is generally easier than the usual one.

10. *On generalized damped oscillations*, by Professor Oswald Wyler, University of New Mexico.

An autonomous system $\dot{x}=f(x, y)$, $\dot{y}=g(x, y)$, with a finite number of critical points, and with a first integral $E(x, y)=\text{const.}$, represents a system of *generalized undamped oscillations*. A second system $\dot{x}=f_1(x, y)$, $\dot{y}=g_1(x, y)$, with the same critical points, and the property that $E(x, y)$ is decreasing along any orbit of this system, and never constant on a non-critical orbit, represents a system of *generalized damped oscillations*. The present paper discusses the geometric theory of generalized damped oscillations. It is shown, among other things, that every critical point at which $E(x, y)$ does not have a maximum (minimum) is the limit point of at least one noncritical orbit as $t\to+\infty$ $(t\to-\infty)$.

11. *Why and how we should correct the mistakes of Euclid*, by Professor P. H. Daus, University of California, Los Angeles.

12. *The absolute value criterion for curve fitting*, by Mr. R. M. Parker, Air Force Missile Development Center, Holloman Air Force Base, New Mexico.

The problem as to whether there might be advantages in making the sum of the absolute

values of the residuals a minimum in fitting a curve to observed data is investigated. The method is shown to be unsatisfactory because of the difficulty of the procedures involved and because we usually obtain, not a unique solution, but a family of solutions.

13. *Calculation of flux using spherical mirrors*, by Professor J. R. Foote, University of New Mexico.

The basic problem is the optimum manner of replacing paraboloidal surface elements in a large solar furnace by spherical mirror approximations. Since at each point of a paraboloid there are two principal radii of curvature, two spherical approximations are considered. Formulas for both cases are derived by which the flux concentration, resulting from collections of such approximating mirrors, at the paraboloid focus can be calculated. Numerical results for one array indicate that over ninety-seven percent of the performance of a single-surface paraboloid can be obtained, other conditions being equal.

14. *Basic earth satellite orbit*, by Mr. H. W. Burnette, Air Force Missile Development Center, Holloman Air Force Base, New Mexico.

A basic earth satellite orbit which is derived from Kepler's laws is described. Some of the factors involved in the generalization of this orbit are included.

15. *Some new techniques in hydrodynamics*, by Professor M. R. Bottaccini, University of Arizona.

The old problem of the resistance to the motion of a body within an ideal fluid has been solved recently. This solution permits evaluation of force and moment on any Rankine body moving in an arbitrary potential flow. The only difficulty in the application is the need to express the potential in terms of singularities. If, however, the potential can be expanded into a series of Legendre functions, it is possible to transform to a series of singularities by an integral transformation which is a simple extension of Maxwell's theory of poles.

16. *Lagrangian interpolation in information theory*, by Mr. Ali Kyrala, Goodyear Aircraft Corporation, Litchfield Park, Arizona.

The sampling theorems of information theory are shown to be a particular type of Lagrangian interpolation. These theorems are often applied in industry to determine the approximate behavior of functions or their ensemble averages from a *denumerable* set of (sample) values. The method of derivation presented is that of *interpolation by entire functions* which demonstrates clearly the *non-uniqueness* of this type of sampling theorem and indicates what kind of assumptions may be made about the asymptotic behavior of the sampled function in order to secure uniqueness. The connection with spectral behavior is briefly discussed.

17. *Some remarks on irreducible bases for infinite symmetric groups*, by Professor R. B. Crouch and Mr. A. Gray, New Mexico State University.

Let S be a countable set. Let $S(d, d^+)$ be the infinite symmetric group on S. Some normal sets of elements of $S(d, d^+)$ are given. These form bases for $S(d, d^+)$. The problem of reducing to irreducible sets is discussed.

18. *A formularization of some experimental data*, by Dr. C. R. Cassity, New Mexico Institute of Mining and Technology.

From approximately 1000 measurements of an independent variable y as a function of four dependent variables, a formula is derived which satisfactorily represents the data. Standard deviations in the coefficients of the formula and the standard error in the measurements are included.

DEONISIE TRIFAN, *Secretary*

THE MAY MEETING OF THE ALLEGHENY MOUNTAIN SECTION

The annual spring meeting of the Allegheny Mountain Section of the Mathematical Association of America was held on May 2, 1959 at the University of Pittsburgh, Pittsburgh, Pennsylvania. The Section Chairman, Professor I. Dee Peters of West Virginia University, presided at the morning session. Professor J. S. Taylor of the University of Pittsburgh presided at the afternoon session. There were 95 persons registered, including 50 members of the Association.

At the business meeting, the following officers were elected: Chairman, Dr. B. H. Mount, Westinghouse Electric Corporation; Secretary-Treasurer, Professor Evan Johnson, Jr., Pennsylvania State University; Executive Committee, Professor E. F. Myers, University of Pittsburgh and Professor A. B. Cunningham, West Virginia University. Professors E. E. Posey and J. H. Neelley, co-chairmen of the Section Committee on High School Contests, reported that 3811 students, representing 192 high schools, participated in the 1959 contest.

The following short papers were presented:

1. *A new method for the approximate solution of differential equations*, by Dr. Philip Cooperman, University of Pittsburgh, introduced by the Secretary.

The method used here is to bound the solutions of a given differential equation by the solutions of *other* differential equations, the initial values being the same for all of the solutions. It will be shown that this process can be applied to a wide class of linear and nonlinear equations. The approximate solution is then the average of the upper and lower bounds, and the error is less than one-half the differences between the upper and lower bounds. The advantage over numerical methods is that the approximation is given explicitly in terms of elementary functions.

2. *Singularities of three-dimensional harmonic functions*, by Dr. R. P. Gilbert, University of Pittsburgh, introduced by the Secretary.

Bergman has considered the class of harmonic functions, which may be generated by the Whittaker operator from algebraic functions of two complex variables, and has given a simple procedure for obtaining the singularities. In this paper the method is generalized to apply to arbitrary functions of two complex variables. It is found that the possible singularities of the harmonic function lie on the "envelope" of a family of complex surfaces. To locate the actual singularities use is made of an inverse Whittaker operator to see which of the possible singularities correspond to singularities of the function of two complex variables.

3. *Logic for applying topological methods to electric networks*, by Dr. R. W. Long, Westinghouse Electric Corporation, East Pittsburgh, Pennsylvania.

A logic is presented which enables one to form the mesh equilibrium equations for an electrical network from raw system data. The branches of a network graph are divided into two sets designated as link branches and tree branches. Introduced are the concepts of a tree limb and limb nodes. Associating each row of a matrix with a link branch and each column with a tree branch, each term of a column will consist of 0, 1, or -1 depending on whether the particular link branch is contained in the definition of the tree branch associated with the column. Designating this matrix by T, the branch impedance matrix by Z, the link currents vector by I_L, and the impressed voltage vector *by* E, the equilibrium equations are $TZT'I_L = E$.

4. *An integral theorem for harmonic vectors in three variables*, by Professor Josephine M. Mitchell, Pennsylvania State University.

Let $\vec{H}(X) = (H_1(X), H_2(X), H_3(X))$ be a harmonic vector defined in Euclidean 3-space: $X = (x, y, z)$. The integral formula $H_1(X) = (1/2\pi i)\int_{\mathfrak{L}} f(u, \zeta, S)\zeta^{-1}d\zeta$ holds, where f is a rational function of u, ζ and S, the arguments being connected by an algebraic equation, $u = x + \frac{1}{2}iy(\zeta + \zeta^{-1}) + \frac{1}{2}z(\zeta - \zeta^{-1})$ and $\mathfrak{L}$ is a simple closed curve in the ζ-plane. Similar formulas hold for H_2 and H_3. A

relation is obtained between a linear combination of transcendental functions, $\int_{X_{k-1}}^{X_k} \vec{H} \cdot d\vec{X}$, increased by period functions and integrals of certain algebraic functions of one complex variable also increased by period functions. This generalizes a theorem of S. Bergman for rational $f(u, \zeta)$.

5. *Note on a paper of Klamkin*, by Mr. H. W. Gould, West Virginia University.

M. S. Klamkin (this MONTHLY, vol. 64, p. 91) has given a set of formulas which purport to express a generalized geometric series in terms of Stirling numbers of the first kind. It is shown in the present paper that the formulas in question are not valid. It is shown that the problem is equivalent to finding an explicit summation expression giving the Stirling numbers of the second kind in terms of the corresponding numbers of the first kind. One solution to this is exhibited. The converse problem was first solved by Schlömilch (Crelle's Journal, vol. 44, 1852).

6. *High school calculus*, by Professor J. H. Neelley, Carnegie Institute of Technology.

This paper gives the experiences, at Carnegie Institute of Technology for the past two years, with freshmen who have had some calculus before coming to the university. It extends the material given under the title, *What to do about a new kind of freshman.* That paper was presented to the M.A.A. last May and appears in the *Mathematical Education Notes* of this issue, pp. 584–586. The conclusion is that calculus should not be taught in high schools.

7. *A new look at the problem of mathematics teaching*, by Professor Martin Levine, Pennsylvania State University, McKeesport Center.

This presentation dealt with the inadequacy of present methods of counseling for the selection of potential teachers. The major difficulty in teaching mathematics was attributed to the inadequacy of the secondary school faculty in their attitude toward mathematics. Today's methods of counseling students regarding their vocational interest in mathematics teaching was analyzed. Referencing the Strong *Vocational Interest Blank*, the criteria used to determine the successful teacher were questioned and thought to be inadequate. Evaluating a good mathematics teacher's interests represents the area which may yield maximum improvement in mathematics education.

8. *The new mathematics curriculum at Grove City College*, by Professor H. F. Bechtell, Grove City College.

Grove City College recently implemented a mathematics program which it felt would make the most effective use of a minimum number of instructors and still satisfy the needs of the liberal arts and engineering students. A careful analysis had disclosed that the standard courses through the calculus in the first two years and then diversification was the best program. The report is a summary of this analysis and is given in view of the fact that these results contradict some of the recent trends in programs for prospective teachers and students requiring a terminal course.

B. H. MOUNT, *Secretary*

THE MAY MEETING OF THE ILLINOIS SECTION

The thirty-eighth annual meeting of the Illinois Section of the Mathematical Association of America was held at Millikin University, Decatur, Illinois, on May 8 and 9, 1959. Professor Arthur Hallerberg, Chairman of the Section, presided at all sessions. There were 79 persons in attendance, including 66 members of the Association, representing 5 high schools and 21 junior colleges, colleges and universities.

Reports were given by the Committee for the Strengthening of the Teaching of Mathematics and the Committee on Contests and Awards. The Section approved a recommendation of Governor-elect Rothwell Stephens that the Chairman appoint a committee to study the role of the Illinois Section in relation to other educational organizations within the state of Illinois. The following officers were elected to serve for the coming year: Chairman, Professor Donald Myers, Millikin University; Vice-Chairman,

Professor Douglas Daly, Illinois Wesleyan University; Secretary-Treasurer, Professor Wayne McGaughey, Bradley University.

The featured speaker following the banquet was Dean W. L. Everitt of the College of Engineering, University of Illinois. He presented his views on Engineering Education in Russia which he formed following a three weeks study and inspection in the U.S.S.R. during December, 1958.

Following a brief welcome by President Paul McKay of Millikin University, the following program was presented:

1. *Reducing levels of abstraction in mathematics for college freshmen*, by Professor Rose Lariviere, University of Illinois, Navy Pier.

The acute necessity of supplying good background workers for high caliber personnel in mathematics and science makes it imperative that the colleges should not delegate the responsibility of developing the less gifted to others. Since the mathematics courses cannot be impoverished, a concerted effort is required in the areas still open to major improvement—the teaching and the text books. To help effect this pedagogical improvement, teachers in action were urged to make suggestions to authors regarding ways of avoiding abstractions and simplifying techniques. A dozen general methods of coping with this problem were given in outline and illustrated. Further specific instances were described in printed material and distributed to the audience.

2. *The teaching of statistics*, by Professor A. L. O'Toole, Western Illinois University.

There still is little statistics instruction in secondary schools; few teachers feel qualified to teach statistics; teacher preparatory institutions have not prepared teachers of statistics; there is no statistics textbook designed for the majority of secondary-school students. In September, 1959, Western Illinois University will inaugurate a program to prepare teachers of statistics. The new program will provide instruction in (1) suitable subject matter of statistics, (2) the objectives of statistics instruction in secondary-schools, and (3) methods and materials of instruction that seem likely to be effective for teaching statistics in high schools and junior colleges.

3. *Equation of a quadric surface through nine points*, by Professor Ruth Rasmusen, Chicago Teachers College.

The problem of writing the equation of a quadric surface passing through nine given points was solved (a) by applying MURT (matrices under row transformations) technique to the solution of nine equations in nine unknowns, and (b) by a generalization for space of the plane geometry problem of writing the equation of a conic section through five points in a plane.

4. *Use of number theory in the curricula*, by Sister Mary Ferrer, Saint Xavier College.

One of the greatest defects of the traditional teaching of mathematics has been the lack of communication between the three levels of learning. The material at the elementary level should not only be teachable but should provide background for high school mathematics as well as for application to other subject matter. The high school mathematics should be a link between elementary mathematics and collegiate mathematics so that progress is consistent.

Number theory supplies a communication between three levels of learning as well as between traditional mathematics and some new mathematical concepts. The concepts of factor, primes and composite numbers, the division algorithm, Euclid's algorithm and congruence were partially developed to show the communication properties and relations of arithmetic, elementary algebra, set theory, groups, rings and fields.

5. *A geometric representation of the tangent of the sum of two angles*, by Professor Larry Wimp, Southern Illinois University.

A geometric construction was made on the unit circle such that by equating different expressions for the length of a line segment one could immediately obtain the identity for the tangent of the sum of two angles, the sum being less than one right angle.

6. *Boolean matrix algebra and linear graphs*, by Professor Franz Hohn, University of Illinois.

In this paper a type of variable linear graph is defined and a Boolean "connection matrix" is associated with each such graph. The algebra of such matrices is then developed and applied to the study of their associated graphs.

7. *Improved programs of instruction in school mathematics*, by Mr. F. B. Allen, Lyons Township High School and Junior College.

This paper reported on the work of the National Council of Teachers of Mathematics Secondary School Curriculum Committee explaining what has been done and the plans for further work during the summer, 1959.

A. W. McGaughey, *Secretary*

THE MAY MEETING OF THE MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The annual spring meeting of the Maryland-District of Columbia-Virginia Section of the Mathematical Association of America was held at Goucher College, Towson, Maryland, on May 2, 1959. Professor M. W. Oliphant of Georgetown University presided. Ninety-seven members were in attendance.

The following officers were elected for this fiscal year: Chairman, Professor E. E. Floyd, University of Virginia; Vice-chairmen, Professor R. A. Good, University of Maryland, and Dr. Daniel Shanks, David Taylor Model Basin; Secretary, Professor D. B. Lloyd, District of Columbia Teachers College; and Treasurer, Professor T. W. Moore, U. S. Naval Academy.

The section voted a letter of rebuke to be sent to the *School Science and Mathematics* magazine for carrying false advertising about the classical insoluble problems of Greek geometry.

The program included the following presented papers:

1. *On the distribution of prime numbers in arithmetic progressions*, by Dr. Daniel Shanks, David Taylor Model Basin, Washington, D. C.

The question is investigated, mostly empirically, in which sense it is true, that there are more primes $\equiv -1 \pmod 4$ than $\equiv +1 \pmod 4$. If $\Delta(N)$ is this excess (for primes $\leq N$) then $\Delta(N) > 0$; $= 0$; and < 0 for 2,995,242; 1352; and 3406 values of $N \leq 3 \cdot 10^6$ respectively.

If $\tau(N) = \Delta(N) \cdot \sqrt{N}/\pi(N)$ then $\tau(1000\, k)$ for $1 \leq k \leq 2000$ has a mean value of 1.06 and a distribution between $\tau = -1/8$ and $\tau = 17/8$ which is roughly symmetric around $+1$. Author conjectures that $\mathrm{Lim}_{\pi \to \infty}\, (1/\mathrm{N}) \sum_2^N \tau(n) = +1$. He then shows that this weakness of $4n+1$ is contained solely in $8n+1$ (or $12n+1$) and not at all in $8n+5$ (or $12n+5$). In contrast $10n \pm 1$ are both equally weak relative to $10n \pm 3$. This difference between mod 8 and 12 on the one hand and mod 10 on the other is explained.

2. *Abstract theory of retrieval coding*, by Dr. C. J. Maloney, Mathematics Division, Biological Warfare Laboratories, Ft. Detrick, Maryland.

The quickening pace and burgeoning volume of research and development, much of it government supported, has led to a crisis in the efforts of librarians and documentalists to "control the record." Attempts to apply newer methods, including electronic computers, to this task have emphasized the need for a general theory of the information retrieval process. It is shown herein that the organization of the record by aspect, introduced during the forties by Batten in England and Cordinnear in France, called here "corbat" indexing, requires the minimum extent of memory, *i.e.*, involves zero memory redundancy.

3. *On the calculation, to a high degree of accuracy, of the modified Bessel functions*, by Dr. F. D. Murnaghan, David Taylor Model Basin, Washington, D. C.

If t is any real number, $[\cosh t+2\cosh(t/2)]/3$ is an upper bound, and $[1+2\cosh(3^{-1/2}t/2)]/3$ is a lower bound, for $I_0(t)$. The mean of these two bounds is again an upper bound, and $[\cosh(2^{-1/2}t)+\cosh(t\cos(\pi/12))+\cosh(t\sin(\pi/12))]/3$ is a lower bound, for $I_0(t)$ and, similarly, the mean of these two latter bounds is an upper bound, and $[\cosh(t\cos(\pi/24))+\cosh(t\sin(\pi/24))+\cosh(t\cos(\pi/8))+\cosh(t\sin(\pi/8))+\cosh(t\cos(5\pi/24))+\cosh(t\sin(5\pi/24))]/6$ is a lower bound for $I_0(t)$. The mean of these two latter bounds is greater than $I_0(t)$ and, if $|t|\leq 10$, this excess is less than 10^{-27}. Similar bounds exist for $I_1(t)$.

4. *The number of vertices of a convex polytope*, by Dr. W. W. Jacobs, American University, and Mr. E. D. Schell, Research Center, IBM Corporation, Yorktown Heights, New York.

In Euclidean space of m dimensions, a convex polytope which has n faces of dimension $m-1$ can have at most

$$\binom{n-r}{n-m}+\binom{n-s}{n-m}$$

vertices, where r and s are the greatest integers in $(m+1)/2$ and $(m+2)/2$ respectively. Also for any n and m, it is possible to construct a polytope with the specified number of vertices.

5. *The mathematics of contract bridge*, by Professor J. A. Tierney, U. S. Naval Academy.

A summary was given of problems essentially mathematical in character which arise in contract bridge. This was followed by a study of techniques for solving card combinations. The *law of balanced distribution*, a new principle of outstanding value to declarer, was presented, followed by a modification involving Bayes' Theorem. Several examples were considered of popular misconceptions involving applications of probability to methods of play.

6. *Number theory and computers*, by Dr. Morris Newman, Applied Mathematics Laboratory, National Bureau of Standards, Washington, D. C.

The high speed computer is an invaluable tool for research in number theory and may be used both to discover theorems and to prove them, when this is possible by numerical computation. In this talk some of the recent progress made in analytic number theory concerned with properties of the coefficients of certain elliptic modular forms and depending on machine computation was described. For example, it has been shown that $p(n)$, the number of unrestricted partitions of the positive integer n, fills all residue classes modulo 13 infinitely often, and satisfies congruences modulo 13 of Ramanujan type.

7. *The position of mathematics in South American higher education*, by Mr. Carlos Fallon, Nems-Clarke Company, Silver Spring, Maryland.

Contrary to the Anglo-Saxon version of the romantic Latin, college mathematics in the larger South American universities is taught with rigor and almost Teutonic thoroughness. Rather than the handmaid of science, mathematics is considered the master-science of all technology. Diverse disciplines are grouped into a single *faculty of mathematical, physical, and natural sciences*. On the negative side, the *intuitively obvious* is sometimes made less obvious by an overly rigorous approach, and classical mathematics is taught to such depth that it excludes from the curriculum certain useful modern techniques.

D. B. LLOYD, *Secretary*

THE MAY MEETING OF THE OHIO SECTION

The forty-third annual meeting of the Ohio Section of the Mathematical Association of America was held at Miami University, Oxford, Ohio, on Saturday, May 9, 1959. Professor L. E. Bush, Chairman of the Section, presided at the morning and afternoon

sessions. There were 68 persons registered in attendance, including 55 members of the Association.

Officers elected for the coming year are: Chairman, Professor W. R. Van Voorhis, Fenn College; Secretary-Treasurer, Professor Foster Brooks, Kent State University; Third member of the Executive Committee, Professor E. T. Stapleford, Kent State University. The Program Committee is: Professor R. W. Shoemaker, University of Toledo, Chairman; Professor C. W. Topp, Fenn College; Professor W. E. Restemeyer, University of Cincinnati.

The following papers were presented:

1. *Some problems in teacher training and retraining*, by Professor L. E. Bush, Kent State University.

The speaker raised the question whether some kind of graduate program satisfying the following requirements should be offered for in-service teachers. (1) The program will take the certified mathematics teacher at the level of mathematical competence at which it finds him. (2) It will lead to a degree which will satisfy the school boards' requirements for a salary differential. (3) It will be possible to satisfy the requirements for the degree in the same amount of time as is required by present degrees in other departments which are offered to mathematics teachers as a means of qualifying for the salary differential. This proposal was followed by a discussion period.

2. *The role of a statistics laboratory on a college campus*, by Professor D. R. Whitney, The Ohio State University. (By invitation).

A statistics laboratory on a university campus should be a consulting agency designed to help researchers from any field in statistics. Three phases of help are indicated: in the design of experiments, in the statistical analysis of data, and in the translation of statistical statements into the language of some particular area of research. The benefits of such a laboratory are that: (1) the campus level of competence in scientific research is raised; (2) a source of research-wise and pedagogically interesting problems in statistics is maintained, and (3) students working in the laboratory acquire some much needed experience.

3. *The impact of the electronic computer on the mathematics curriculum*, by Professor R. J. Nelson, Case Institute of Technology. (By invitation).

The electronic computer has so far had slight effect on the mathematics curriculum with the exception of considerably increased interest in numerical analysis and the introduction of courses in programming (if the teacher of programming happens to be a mathematician). The main impact in the next few years will be increased interest in the theory of computers, computability, algorithms, and in the use of computers for mathematical work: theorem proving, testing cases of conjectures, etc. In the curriculum the effect will probably be increased attention to computational and algorithmic mathematics, particularly the theory thereof.

4. *Solutions of the general Monge equation with some extensions*, by Dr. L. V. Robinson, Wright-Patterson Air Force Base, Ohio.

It is shown that the Monge general partial differential equation can be obtained by combining four identities. From this analysis four total differential equations involving three differentials arise. The usual tests can then be applied as to existence of solutions of these and of the existence of any intermediate integrals. How these methods can be extended is indicated.

5. *Review of basic concepts in approximation theory*, by Dr. Gertrude Blanch, Wright Air Development Center, Wright-Patterson Air Force Base, Ohio. (By invitation).

Numbers entering into calculations are usually only approximate and for that reason continuity in the real sense does not exist in numerical processes. Not every construction that is theoretically possible lends itself to numerical construction. Estimates of the size of the error relating the numerical to the theoretical results are difficult to attain. The effort to do so by means of a com-

puter program, introduced by Ramon E. Moore in "range" arithmetic, is a very hopeful step in the right direction. The need for seeking processes that are numerically stable must be emphasized. Finally, some implications of function theoretical concepts were touched upon.

6. *Minimum variance of estimates under stratified sampling*, by Professor W. R. Van Voorhis, Fenn College.

When a population is stratified for purposes of sampling, the variance of the estimated sample mean, x, depends not only upon the allocation of the sample of size n to the several k strata, but also upon the location of x_i, the points of stratification. Necessary and sufficient conditions to yield minimum variance have been established by Dalenius but these conditions do not yield the explicit values of x_i that must be known before an optimum stratification can be made. It is shown that no general proof of the existence of uniqueness is possible. For the case of "proportional" allocation, it is shown that there exists at least one optimal solution for k strata. A method of successive approximations beginning with a first feasible solution is discussed, and examples are given for several well-known distributions.

7. *Mutation view of conics (shadow transformations and primal states)*, by Dr. Beckham Martin, Owens-Illinois Glass Company, Toledo, Ohio.

In the presentation the following salient remarks were made: (A) There had to be a clean break with conventional geometry, which has reached a state of stagnation, before one could ever hope to reach new pinnacles of achievement. (B) Mutation Geometry is the science of intangible change (shadow-transformations). The discussion began with the general conic equation: (1) $Ax^2+Bxy+Cy^2=Dx+Ey+F$. A primal state number P was calculated: (2) $P=2/(\sqrt{B^2+(A-C)^2}+A+C)$ by which (1) was transformed shadow-wise to its primal state: (3) $ax^2+bxy+cy^2=dx+ey+f$ from which the properties of the representative conic may be read off at sight. Example: The eccentricity is given by (4) $e^2=2-(a+c)$

8. *The 1959 mathematical program*, by Professor R. L. Wilson, Ohio Wesleyan University.

A contrast is made between the type of mathematics currently being used in the physical and nonphysical sciences and the type of mathematics so applied a decade or more ago. Implications are drawn for the mathematical education of students in the various fields of specialization. Alternative suggestions for meeting this situation are made.

9. *Composite pattern of primitive Pythagorean triangles formulated by arithmetical progression*, by Mr. R. J. Irwin, Eddie Painton Associates, Inc., Cleveland, Ohio.

The method presented is believed to be easier and quicker to compile than the methods more commonly used. The non-Pythagorean Triangles are eliminated very readily as they follow a rhythmic appearance in these tables. Periodic checks throughout the tables automatically correct preceding calculations. These tables discovered two (probably typographical) errors in existing published tables. The interesting relation of Pythagorean Triangles to prime numbers is also shown.

FOSTER BROOKS, *Secretary*

THE MAY MEETING OF THE ROCKY MOUNTAIN SECTION

The forty-second annual meeting of the Rocky Mountain Section of the Mathematical Association of America was held at Utah State University, Logan, Utah, on Friday afternoon and evening and Saturday forenoon, May 8 and 9, 1959. The meeting was divided into several sessions with Professors N. C. Hunsaker, Joe Elich, J. H. Barrett, and Harvey Fletcher presiding. There were 94 persons registered for the meeting, including 60 members of the Association.

Officers elected at the meeting for 1959–1960 were: Chairman, Colonel J. W. Ault,

United States Air Force Academy; Vice-Chairman, Professor L. W. Rutland, University of Colorado; and Secretary-Treasurer, Professor F. M. Carpenter, Colorado School of Mines.

The following papers were presented:

1. *A method of approximating the roots of an equation by quadratic formulae*, by Professor Stephen Kulik, Utah State University.

Two zeros of an analytic function $f(z)$ are approximated with a prescribed accuracy by one application of a quadratic formula. The coefficients of the quadratic equation depend on $D_n(z)$ which is calculated recursively,

$$D_n = f'D_{n-1} - f''fD_{n-2}/2! + \cdots + f^{(n-1)}(-f)^{n-2}D_1/(n-1)! + f^{(n)}(-f)^{n-1}D_0/(n-1)!,$$
$$D_0 = 1, D_1 = f', D_2 = f' - f''f; \quad \text{where} \quad D_n = D_n(z), f^{(n)} = f^{(n)}(z), \ n = 0, 1, \cdots.$$

The roots of the quadratic equation converge to the two zeros of $f(z)$ which are nearer to the number z than the remaining zeros.

2. *A note on a simple matrix isomorphism*, by Professor D. W. Robinson, Brigham Young University.

Let C be the field of complex numbers. Let $\phi:\alpha\phi = A$ be the well-known ring isomorphism of C onto a real subsystem of 2-by-2 matrices over C. Let $f(x)$ be a polynomial over C. It is shown that $\phi(f(\alpha)) = f(\phi(\alpha))$ if and only if $f(\bar{\alpha}) = \overline{f(\alpha)}$, where $\bar{\alpha}$ is the complex conjugate of α. This result is then generalized by considering (1) a ring isomorphism of the n-by-n matrices over C onto a real subsystem of the $2n$-by-$2n$ matrices over C, and (2) functions of matrices.

3. *Definition of "plus" and "times" for the natural numbers*, by Mrs. Jean J. Pederson, Olympus Senior High School of Salt Lake City and University of Utah.

A function may be defined as a set of ordered pairs such that no two distinct pairs of the set have the same first element. Introducing the natural numbers according to the technique of Peano, explicit definitions, as sets of ordered pairs, may then be exhibited for the addition and multiplication functions as applied to the natural numbers.

4. *An extreme value problem for honor students*, by Captain R. C. Rounding and Captain R. L. Eisenman, United States Air Force Academy.

This paper explores the relationship between a problem in which the volume of a cylindrical solid is given and the relative dimensions are to be found to minimize surface area, and the corresponding problem of a rectangular region wherein the area is constant and the perimeter is to be minimized. It is presented as a problem for Honor Students as an example of a technique in mathematical research.

5. *The multiplicity and positiveness of the characteristic numbers of second order Sturm-Liouville systems involving generalized boundary conditions*, by Professor L. C. Barrett, South Dakota School of Mines and Technology.

This paper is concerned with Sturm-Liouville systems that possess boundary conditions involving left and right hand limits of the dependent variable and its first derivative at one or more interior points of a given fundamental interval. Two theorems are cited which provide introductory information concerning the nature of the characteristic functions of such systems. A third theorem describes the orthogonality of the characteristic functions. It is then shown that the characteristic numbers are simple roots of the characteristic equation. Sufficient conditions that these characteristic values be non-negative are also given.

6. *Distribution of zeros of solutions of complex differential equations*, by Mr. N. H. Mines, University of Utah.

A zero-free region of a nontrivial solution of the complex differential equation $[K(z)W']'$

$+G(z)W=0$ for $K(z)\equiv 1$ was obtained by E. Hille (*Transactions American Mathematical Society*, vol. 23, 1922). The same method can be used to obtain a zero free region of a nontrivial solution of the general equation requiring only analyticity of the coefficients and $K(z)\neq 0$.

7. *A vector solution of simultaneous linear equations*, by Professor C. A. Grimm, South Dakota School of Mines and Technology.

Three points in the plane, $Ax+By+Cz=D$, $D\neq 0$, are sufficient to determine, to a scalar multiple, A, B, C. The vector $[A, B, C]$ is orthogonal to the plane, and is a scalar multiple of the cross product of two vectors in the plane determined by the three points in the plane. By generalizing this idea a method is developed by which any set of n nonhomogeneous linear equations in n variables may be solved by evaluating one n by n determinant.

8. *Homogeneous production function*, by Professor E. A. Davis, University of Utah.

9. *Lattice points*, by Professor T. M. Apostol, California Institute of Technology (Lecture sponsored by Mathematical Association of America).

10. *The Caltech experiment in calculus*, by Professor T. M. Apostol, California Institute of Technology. (Invited Address).

11. *An undergraduate course on the topology of a line*, by Professor C. E. Burgess, University of Utah. (Invited Address).

The author described an introductory undergraduate topology course which is based upon axioms that describe a linearly ordered, separable, connected space with no first point and no last point. Such a course has been offered at the University of Utah each year for the last several years. A similar address was given before the Wisconsin Section at Whitewater, Wisconsin, May 11, 1957 (this MONTHLY, vol. 64, 1957, p. 627).

12. *Particular solutions for nonhomogeneous, linear, ordinary difference equations*, by Professors Forrest Dristy and L. C. Barrett, South Dakota School of Mines and Technology, presented by Professor Dristy.

In this paper an identity is derived which relates adjoint difference expressions in much the same way that Lagrange's identity of differential equation theory relates adjoint differential expressions. It is then shown how this identity may be used to determine a particular solution of a nonhomogeneous linear ordinary difference equation once the complementary function is known. Thus, the method provides an alternative to the familiar method of variation of parameters.

13. *Application of Fourier series to difference equations*, by Lieutenant J. N. Christiansen, United States Air Force Academy.

A method for obtaining general solutions to linear difference—differential equations is presented. The method is applied to a simple example and the solution is plotted for a special case. The method presented is valuable in that it requires no knowledge of mathematics beyond that usually gained from a course in advanced calculus. It is easy to apply and reduce the solution to quadratures in a very few steps. Also, the solution contains the initial conditions explicitly. The method is analogous to the use of the Fourier transform for finding solutions to partial differential equations.

14. *Geometrical motivations for determinant type proofs of mean value theorems*, by Mr. R. A. Jacobson and Professor L. C. Barrett, South Dakota School of Mines and Technology, presented by Mr. Jacobson.

The usual proofs of the mean value theorems involve the process of applying Rolle's Theorem to functions or determinants happily designed to yield the desired conclusions. The determinants thus employed are usually introduced without comment. In this paper it is shown that these determinants may be motivated by an analysis originating in a geometrical setting.

15. *An approach to the foundations of intuitionism,* by Mr. David Drake, University of Colorado.

The concept of a formal proof was adapted to intuitionist philosophy by replacing axioms and primitive rules of inference with other assertability criteria, such as metamathematical observation. The primitive symbols of logic were defined by means of words having reference to finite and perceptually concrete situations. An axiom of formalized intuitionist logic was then derived—for the given interpretation of symbols—by the modified proof method.

16. *The program of advanced placement in mathematics at Colorado State University,* by Professor F. M. Stein, Colorado State University.

Many students enter Colorado State University with more than the minimum background to enter the regular sequence of mathematics courses. This is first an outline of the method used to select those in this group who could start the sequence at an advanced level, and second a report on the success of the program thus far.

17. *A method of solving Diophantine quadratic equations,* by Professor B. W. Jones, University of Colorado.

Here a method, originating in some ideas of Edgar Emerson, is given for finding all the integral solutions of certain equations of the form $ax^2-by^2=c$ where a, b, and c are positive integers, given a finite number of such solutions. Geometrically this is equivalent to use of a zigzag pattern of lines. This applies also to some quadratic indefinite forms with cross products. In some respects this method is an improvement over the traditional use of the Pell equation.

F. M. CARPENTER, *Secretary*

THE MAY MEETING OF THE WISCONSIN SECTION

The twenty-seventh annual meeting of the Wisconsin Section of the Mathematical Association of America was held on May 2, 1959, at Wisconsin State College, Platteville, Wisconsin, Professor J. V. Finch, Chairman, presiding. There were 69 present, including 33 members of the Association and 30 members of the Wisconsin Mathematics Council, 15 of whom are also members of the Association.

At the business meeting, the following officers were elected for the coming year: Chairman, Professor C. B. Hanneken, Marquette University; Vice-Chairman, Professor Henry Van Engen, University of Wisconsin; Secretary-Treasurer, Sister Mary Felice, S.S.N.D., Mount Mary College.

The following report of the Section's fourth annual mathematics contest for high school students was given by the Contest Committee Chairman, Professor Earl Swokowski, Marquette University:

For the second consecutive year a preliminary contest examination consisting of multiple-choice questions was given for the purpose of enabling the teachers to better select the participants in the final contest. This examination was given to 12,600 students in 283 high schools on February 26, 1959. The top contestant in each school was awarded a certificate.

The final contest was held on April 11, in 27 centers distributed throughout the State, with 900 participating from 187 schools. Cash prizes of \$50, \$25, \$10, and \$1 were awarded to each student in the top 15 percent of the contestants, divided into four groups respectively. In addition initialed M.A.A. award pins were given to the twenty in the top two groups and a certificate to the rest of these groups.

After some discussion during which various suggestions were offered for subsequent contests, the Section Chairman was instructed to continue to carry on the contest as in the past two years.

After an address of welcome by Professor Bjarne Ullsvik, President of Wisconsin

State College, Platteville, the following program was given:

1. *Introduction of algebra through inequalities*, by Mr. Dawson Trine, Wisconsin State College, Platteville, introduced by the Chairman.

This paper was a report from a member of the first year-long Academic Institute of the National Science Foundation held at the University of Wisconsin. It explained a method used the following year to introduce a first course in algebra. The method was basically that of graphing an equation and its two corresponding inequalities, using a number line for each statement. Example problems were given that could be used to introduce various sections of algebra.

2. *Stokes' theorem*, by Professor L. M. Milne-Thomson, Mathematics Research Center, U. S. Army, Madison, Wisconsin, introduced by the Chairman.

It was shown that any function in the xy-plane can be expressed as a function of the complex variable and its conjugate. This leads to a plane formulation of Stokes' theorem (first published by the speaker in 1938), which has numerous applications including the calculation of area, centroid and moments of inertia. The mapping of the cardioid on the unit circle was found and used to illustrate the applications.

3. *Mathematics of ballistic missile guidance*, by Mr. A. J. Pejsa, A. C. Spark Plug Division, General Motors Corporation, Milwaukee, Wisconsin.

The trajectory of a body in the earth's gravitational field is treated by first deriving equations for the case of a nonrotating spherical earth and then superimposing onto them the effects of earth rotation, oblateness, and other lesser nomalies. Error sensitivities indicating the effects of perturbing the various parameters are considered.

A guidance scheme is suggested which would be accurate enough to guide an IREM with tolerable accuracy, using a comparatively simple airborne guidance computer.

4. *The 37th annual meeting of N.C.T.M.*, by Mr. Vincent Brunner, Nicolet High School, Milwaukee, introduced by the Chairman.

The various sections of this meeting greatly emphasized the changes taking place in high school mathematics. These changes involve new methods, new curricula, and new materials. In order to meet the responsibilities imposed on the teachers by these changes, the teachers were urged to take advantage of National Science Foundation Fellowships, in-service training programs, and programs sponsored by industry in an effort to improve background and increase proficiency. Sources of information which are available can be found in the reports of the National Council of Teachers of Mathematics, the Commission of the College Entrance Examination Board, University of Illinois Mathematics Project, University of Maryland Mathematics Project, and other studies.

5. *Mathematics in Sweden*, by Dr. L. F. Wahlstrom, Wisconsin State College, Eau Claire.

The first-year mathematics student at the university in Sweden may take as much as 15 hours of instruction in mathematics alone, and he may spend as much as 45 hours per week on the study of the subject outside of class. Mathematics may be the only subject he is studying during his first year. Although lecture classes may run as large as 200 students or more, small supervised study groups help the student to master the material. Written examinations last four to five hours and are given periodically. The student takes the examination when he is ready for it. If he passes the written examination, he then takes an oral one.

6. *Large section classes in mathematics*, by the Reverend L. J. Heider, S.J., Marquette University.

SISTER MARY FELICE, *Secretary*

PITY THE POSTMEN!

"Mathematical Minds" means "Migrant Members"

"Figure"-tively speaking—1,866 changes of address were completed for members of the Mathematical Association of America during the period beginning June 1958 and ending June 1959.

That is why we are issuing this plea:

> Please notify us as promptly as possible when you have moved, as address changes take from four to six weeks to effect.
>
> Also, please send us any change of title, business or institutional affiliation and academic degree. In this way we can maintain accurate and up-to-date records.

MATHEMATICAL ASSOCIATION OF AMERICA
University of Buffalo
Buffalo 14, New York

CALENDAR OF FUTURE MEETINGS

Fortieth Summer Meeting, University of Utah, Salt Lake City, Utah, August 31–September 3, 1959.

Forty-third Annual Meeting, Conrad Hilton Hotel, Chicago, Illinois, January 28–30, 1960.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Grove City College, Grove City, Pennsylvania, April 30, 1960.

ILLINOIS, Illinois Wesleyan University, Bloomington, May 13–14, 1960.

INDIANA

IOWA, State University of Iowa, Iowa City, October 16, 1959.

KANSAS, Kansas State College of Pittsburg, April 30, 1960.

KENTUCKY, University of Kentucky, Lexington, April, 1960.

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, American University, Washington, D. C., December 5, 1959.

METROPOLITAN NEW YORK

MICHIGAN, University of Michigan, Ann Arbor, March 26, 1960.

MINNESOTA

MISSOURI, Central Missouri State College, Warrensburg, April 23, 1960.

NEBRASKA, University of Nebraska, Lincoln, April 23, 1960.

NEW JERSEY, Princeton University, November 7, 1959.

NORTHEASTERN, Boston College, Chestnut Hill, Massachusetts, November 28, 1959.

NORTHERN CALIFORNIA, University of California, Berkeley, January 16, 1960.

OHIO, Kent State University, May 7, 1960.

OKLAHOMA, Oklahoma City University, October 23, 1959.

PACIFIC NORTHWEST, State University of Montana, Missoula, June 17, 1960.

PHILADELPHIA, University of Delaware, Newark, November 28, 1959.

ROCKY MOUNTAIN, United States Air Force Academy, Colorado Springs, Spring, 1960.

SOUTHEASTERN, University of South Carolina, Columbia, April 1–2, 1960.

SOUTHERN CALIFORNIA, Los Angeles State College, March 12, 1960.

SOUTHWESTERN, Air Force Missile Development Center, Holloman Air Force Base, New Mexico, April, 1960.

TEXAS, San Antonio College, April, 1960.

UPPER NEW YORK STATE, University of Rochester, May 7, 1960.

WISCONSIN, Mount Mary College, Milwaukee, May 7, 1960.

GEORGE BANTA COMPANY, INC., MENASHA, WISCONSIN, U.S.A.

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THE ROLE OF NUMERICAL ANALYSIS IN AN UNDERGRADUATE PROGRAM*

GEORGE E. FORSYTHE, Stanford University

1. Where do mathematics students go? Before considering the aim of undergraduate mathematics education we should recall approximately who the students are who take college mathematics courses, and where they are going. I assume that everywhere, as at Stanford, most mathematics students major in other fields. Usually students of science and engineering form most of our clientele. The increasing role of mathematics in economics, psychology, etc., is bringing a number of serious students of social science. And we have an increasing number of mathematics majors—students who are naturally dearest to our hearts.

It should be obvious that the engineers and other nonmajors are primarily interested in the *application* of mathematics, and not in its *structure*. The best of the nonmajors will apply mathematical analysis in their professions. A large number of the less gifted, particularly among the engineers, will face a considerable number of computations in their future work, and this may represent the main future use of their mathematical background.

And our majors, where are they bound? One wishes in vain that all the best of them would either teach high school or take graduate work. Of those who attain the Ph.D., about half go into industry and apply mathematics in various areas. Almost all of them will be connected to some degree with automatic computation. But the majority of our undergraduate mathematics majors are lured at once into the market place, where they are greatly in demand as servants of the fast-multiplying family of fast-multiplying computers.

Why are they in such demand? There seem to be over 3000 automatic digital computers now installed in the United States, with more on the way. As a rough estimate, each automatic computer needs to have 10 attendants who serve it as mathematicians—programmers, coders, analysts, supervisors, etc. The resulting requirement for 30,000 computer mathematicians should be compared with the combined membership of the American Mathematical Society, Mathematical Association of America, Society for Industrial and Applied Mathematics, Association for Computing Machinery, Institute of Mathematical Statistics, and American Statistical Association—under 20,000 persons. While some makeshift arrangements are possible, the disparity in numbers is creating the unprecedented demand (and salary) for the new A.B. in mathematics.

It must be noted as a digression that as a practitioner of mathematics, the new A.B. is not in a strong position, despite industry's demand for him. He knows little of what he needs to do a good job—superficially because his teachers have not prepared him to be a practitioner of mathematics, but fundamentally

* Invited address at the Thirty-ninth Summer Meeting of the Mathematical Association of America, August 27, 1958. The preparation of the manuscript was sponsored in part by the Office of Naval Research under project NR 044 211.

because of the brevity of his mathematical education. As a mathematically half-educated man he needs to learn both caution and confidence. But, if he is confident, wise in human relations, aggressive, and lucky, he will become a supervisor of coders—a position in which with his half education he sometimes does serious damage to his company and the reputation of scientific computing. As far as technical advancement is concerned, coding appears to be a terminal occupation, except to those with considerable graduate work in mathematics.

For discussions of mathematicians in the market place, see Rees [12] and Fry [5].

In summary, a very large number of undergraduate students of mathematics are going to apply mathematics in their future work, and much of this application will involve automatic computation. How should this influence our program?

2. Aims of undergraduate mathematics education. In the author's opinion, we should have the following goals for the mathematical education of our undergraduates, whether majors or not.

a. The student should above all learn as much as possible of the structure, significant ideas, and points of view which constitute mathematics. This is the *sine qua non.*

b. The student should learn to read independently the mathematical literature at his level of comprehension, and to use mathematical language to write facts, proofs, and ideas precisely and unambiguously, in a style acceptable to not-too-strict editors of mathematical journals. With the growth of applied mathematics, a good many papers are being written by the A.B. in mathematics; it is unfair to the authors, to editors, and to readers that so many of their authors write as mathematical illiterates.

c. The student should know the tools of the mathematical profession (books and machines), and how and where to find them and to keep up to date on them. Within his general limitations, he should be competent in the library. He should know roughly what bibliographies exist, what tables of useful functions there are, what machines can and cannot do, and so on. Since he is fully competent to learn it, he should be able to code one automatic computing machine quite well. Finally, he should have a general idea of what practitioners of pure and applied mathematics there are, and their abilities. For example, we must teach the future engineer to consult with statisticians, numerical analysts, and other experts at an appropriately early time, instead of trying to go it alone until an emergency arises.

d. The student should have cultivated and practiced the solution of mathematical problems new to him. This is where research ability is built. It hardly matters whether the problems have been solved before by some one else.

e. The student should have gone fairly deeply into some other field of knowledge—preferably some field where mathematics is used, like a physical science or the mathematical aspects of economics. He should have acquired as

much experience as possible in communicating with practitioners of this other field, so as to learn to translate mathematical terms into their language, and vice versa. Here he should have struggled with the mathematical idealization of problems from the real world.

f. The student should have learned to enjoy mathematical study.

It seems to me that these constitute valid goals also for mathematics in grade school, secondary school, and graduate school, if the standards of writing and item *c* are modified appropriately. In graduate school one might wish to add:

g. The student should receive some supervised experience in serving as a mathematical consultant.

3. The abstract versus the concrete. In this post-sputnik era one hears and reads a great deal of criticism of mathematical education at all levels, but particularly in high schools. Many, perhaps most, mathematical critics deplore the lack of rigor and abstraction, and the lack of understanding of the mathematical structure of algebra, geometry, and other subjects. I believe that criticism to be valid. But I have a different complaint—that much teaching is not intuitive enough nor well enough illustrated. I wish mathematics classes would get down to earth more often and make actual application of newly learned theorems to concrete problems of a realistic nature.

In these ideas I agree with Felix Klein [9], and quote from his book, pages 15 and 16: "I have already emphasized the fact that, in the schools, applications accompany arithmetic from the beginning, that the pupil learns not only to understand the rules, but to do something with them. And it should always be so in the teaching of mathematics! Of course, the logical connections, one might say *the rigid skeleton in the mathematical organism*, must remain, in order to give it its peculiar trustworthiness. But the living thing in mathematics, its most important stimulus, its effectiveness in all directions, depends entirely ["entirely" is too strong—G.E.F.] upon the applications, *i.e.*, upon the mutual relations between those purely logical things and all other domains. To banish applications from mathematics would be comparable to seeking the essence of the living animal in the skeleton alone, without considering muscles, nerves and tissues, instincts, in short, the very life of the animal." Klein goes on to say that, while in research differentiation between pure and applied mathematicians may be essential, such differentiation is not reasonable in teaching.

It is not contradictory to criticize in the same breath mathematics teaching for not being abstract enough and for not being concrete enough. I think both these criticisms are frequently valid. Taken together, they imply a *hazy* mathematics course which has neither the beautiful structure of a logical entity nor fascinating application to the physical world. To the extent that mathematics courses are hazy and shapeless, the author claims there are two directions in which reform is needed—to strengthen the skeleton, and to add a variety of applications.

For example, a student with a poor knowledge of high school algebra is weak

on structure—for example, he is unaware that the whole subject flows from a few postulates and principles, or that proofs are just as relevant to algebra as to geometry. But he is also weak in the application of algebra. He is notably unable to convert word problems into algebraic form—always a hard part of algebra. He is also weak in arithmetic, and lacks good intuition about what happens when one gives variables numerical values. (Can $-x$ really be positive?)

The use of high school algebra is purely for an example. Quite parallel instances occur everywhere in school and college.

It must be emphasized that the concrete application of an abstract theory is rarely in practice just a special case of the abstract theory! To illustrate this paradox, consider the multiplication of real numbers x, y. We know that $xy=yx$, and that $x(yz)=(xy)z$. However, in the concrete application to digital computers, one is forced to curtail the decimal (or binary) expansions of numbers at some fixed precision. Thus, x, y, z are represented by *digital* numbers $\bar{x}$, $\bar{y}$, $\bar{z}$. Moreover, exact multiplication of digital numbers does not lead to a digital number, and so one must be content with a *pseudomultiplication* $\times$, where the pseudoproduct is digital. Difficulties arise from the fact that often $\bar{x}\times\bar{y}\neq\bar{y}\times\bar{x}$ and $\bar{x}\times(\bar{y}\times\bar{z})\neq(\bar{x}\times\bar{y})\times\bar{z}$. This is perhaps a typical example of essentially new problems created by concrete applications of a mathematical theory. The problems may in turn lead to a new mathematical theory to solve them.

Since my personal interest is in the concrete side of mathematics, I shall concentrate on it in the following, leaving structure to others more interested and better qualified. For other opinions on the teaching of concrete mathematics, see Tukey [14] and Weyl [15]. Tukey refers to *monastic* and *secular* mathematics.

4. What is numerical analysis? Numerical analysis (apparently a postwar term) is a branch of applied mathematics, namely the art and science of using digital computation to solve scientific problems. Traditionally numerical analysis included such matters as numerical quadrature and the solution of polynomial equations and ordinary differential equations. Since automatic digital computing machines are newly on the scene, the part of numerical analysis dealing with their use in scientific computation is the live area for research. And, in practice, the large machines are dominating scientific computation in this country. Hence the problems of using such machines constitute most of what I consider to be numerical analysis today. Since the machines can deal with vast quantities of data, the solution of *large* problems is frequently asked. For example, one thinks today of solving linear algebraic systems with 100 to 1000 unknowns, where, with a desk calculator, one with 10 unknowns requires most of a day. But, even more than size, the automatic nature of these machines has changed the character of problems. We now consider that numerical analysis covers any type of problem which an automatic computer should be able to solve—any problem which can be reduced to an algorithm or to a flow chart. At this point we consider such things as formal integration of a rational function

as numerical analysis. In fact, most exercises other than "word problems" and proofs which occur in undergraduate courses involve some algorithm, and may be considered as part of numerical analysis!

Numerical analysis has different branches. Among these are the design of algorithms, the analysis of errors, and techniques for coding (mainly the use of machines to prepare codes for machines).

5. What numerical analysis should the undergraduate study? The amount of analysis and algebra at the command of the undergraduate limits the depth to which he can pursue numerical analysis, and in particular error analysis. This tends to force the undergraduate numerical analysis course into a collection of recipes (simple algorithms). But this is hardly an excuse for a course, because the student should be well enough educated to be able to read and apply algorithms for himself. To define an appropriate syllabus for undergraduate numerical analysis is therefore a bit delicate. I consider the following to be about right:

a. He should learn to code one automatic digital computing machine. Actual experience operating a machine is certainly valuable and stimulating, but it is not essential. One can learn to code any machine without seeing it, and indeed all machines are coded before they are built. Except for the specialist, it will suffice to learn to code in an algebraic language like FORTRAN, which can be transformed by special translator codes into machine-language codes. A standard international algebraic language for computers, called ALGOL, is now being developed.

b. He should get a feeling for what automatic computers can do, and what they cannot do. He should learn that they are no substitute for creative thought, and yet that they can do a good deal of what passes for thought in this world. The student must learn the distinction. He should also learn that collaboration with an automatic computer compels precise formulation of the problem, and 100 percent accuracy in preparation of the code. In this respect the automatic computer really forces that precision of thinking which is alleged to be a product of any study of mathematics.

c. He must learn the tools of the computer—what tables, machines, books, codes, bibliographies, and people exist to help him solve future computing problems.

d. He should learn a few typical algorithms, including, for example, Newton's method for finding zeros, Crout's method for solving linear systems, and Euler's method for ordinary differential equations. If not too many algorithms are undertaken, it should be possible to study each fairly deeply.

e. He should have been exposed to some instances of treachery in computing, enough to teach him great caution. For example, he might find out that, whereas the polynomial

$$F(x) = (x-1)(x-2)\cdots(x-20) = x^{20} - 210x^{19} + \cdots + 20!$$

has zeros 1, 2, $\cdots$, 20, the slightly rounded polynomial $F(x)-2^{-23}x^{19}$ has among its zeros numbers near 20.846 and $13.99 \pm 2.5i$ (Wilkinson [**16**]). This example should keep any one from using polynomial-solving routines blindly! We hope it would teach the student that you still have to think in the era of automatic computers.

f. He should have some exposure to careful error analysis. For example, he might have studied Wilkinson's round-off error analysis of Crout's method [**17**]. A fruitful point of view here is to see how much one has to change the data to make the computed answer correct, rather than to ask how wrong the answer is for the given data.

g. He should have a firsthand acquaintance with arithmetical operations on numbers, gained from desk computation and also, if possible, from work with an automatic computer.

Professor Perlis has remarked that, whereas we *think* we know something when we learn it, and are *convinced* we know it when we can teach it, the fact is that we don't *really* know it until we can code it for an automatic computer!

6. How can we teach this numerical analysis? There seem to be two ways of teaching numerical analysis within the course structure of the American college. One is to teach special courses in it, and the other is to mix it into other relevant mathematics courses. Obviously this year's choice between these methods depends on the present existence of teaching materials and trained staff. But one may also ask which would be the preferable solution, assuming that teaching materials and staff will eventually support the prevailing decision.

Many professional numerical analysts, the author included, agree that numerical analysis should be mixed into most undergraduate mathematics courses. From our point of view, many undergraduate exercises in such subjects as calculus, linear algebra, and differential equations are in fact solvable with automatic computers, and hence belong to numerical analysis. We think their real importance is the illumination of mathematical theory, and hence that such exercises should stay where they are now: in the theoretical courses. However, to pick one example from linear algebra, we feel that the conventional system of 3 linear equations in 3 unknowns with small integral coefficients does not furnish a sufficiently bright illumination of linear algebraic theory. And it certainly fails dismally as a fair illustration of linear algebra in industry today. We therefore feel that more realistic examples should be considered in the linear algebra course. One could easily consider 10 or 20 equations in 10 or 20 unknowns, and expect students to produce a complete flow chart, or a code in some algebraic language like ALGOL.

Turning to ordinary differential equations, we feel that no one understands the real nature of a differential equation $dy/dx=f(x, y)$ until he has experimented with computational or graphical procedures for its solution. Such a simple process as Euler's method, while not accurate, is both useful and worthy of study. What normally passes for numerical methods in differential equations

courses bears little resemblance to how such equations are solved in practice. Why not mix realistic practice right with the theory, where they can influence and strengthen each other?

There is one exception, a sub'ect sufficiently demanding and sufficiently remote from other undergraduate mathematics to warrant a special course. This is coding. As we have indicated, we think every undergraduate mathematics student should know how to code some machine fairly well. (I would also include all undergraduate students, for I feel that the computer revolution will have such a great impact on all our lives that every college graduate should understand it intimately. Possibly it will eventually be taught in the ninth grade for the same reason.) Since coding presupposes no mathematics beyond arithmetic, it can be taught to freshmen. I recommend a two-hour-per-week semester course in coding, to be taken as early as possible. There are texts already available for such a course, including one by Andree [1]. The coding course should also include the material in c of Section 5—an exposure to bibliographies, tables, codes, and other library material. For, if numerical analysis is integrated with other mathematics, the student may nowhere else be taught by specialists in numerical analysis.

Until a student actually has a digital computer to work with, the greatest benefit of a coding course is the *point of view.* The student will learn what constitutes creative thought, and what is merely routine. The *flow chart,* almost essential to coding, is one of the greater tools in scientific and technological thinking. It probably originated in the block diagram of the electrical engineer. It would be a useful device in mathematical teaching and thinking. For example, for learning the basic concept of functions of one variable and their composition, the following block diagram seems far superior to the less graphic functional notation $y=f(g(x))$:

$$x \to \boxed{g} \to g(x) \to \boxed{f} \to y.$$

With coding behind him, the student can analyze many algorithms from the point of view of preparing an actual or potential code to solve them. And, to the extent that he has a machine available, he can solve realistic problems in any area.

I should like to point out that numerical analysis was mixed into a theoretical course in mathematics by Dickson in his book [3]. Such a course in the theory of equations was taught for years in American colleges, and is only now beginning to disappear in favor of more linear algebra. Dickson and other authors following his lead included Horner's and Newton's root-solving procedures, evaluation of determinants, solution of linear algebraic systems, and so forth. With the evolution of computing equipment and numerical analysis, much of Dickson's explicit and implicit advice on numerical analysis became obsolete and misleading, but for many of us it was the only numerical analysis we ever saw in college.

Let us consider the common practice of teaching one or more separate courses

in numerical analysis for undergraduates. These courses have not required coding as a prerequisite, although this is changing now. Judging from the textbooks, some of these courses tend to be filled more with algorithms than with analysis, and thus may be shallow for competent students. Milne's book [11] is excellent, but most of it could be read by the student whenever he needed a certain technique. The basic mathematical concepts of the book, like orthogonal polynomial systems and the representation of linear functionals, should really be covered in other courses.

The other extreme in a course is represented by Hildebrand's beautiful book [7]. This includes real analysis of round-off errors, truncation errors, and so on, and is a great pleasure to a numerical analyst. But the typical good undergraduate is simply not enough of an analyst to do justice to the course, if I can judge from Stanford and U.C.L.A. While the instructor can teach the analysis as he goes along, it is questionable whether a special numerical analysis course is the correct place for this.

In at least one university (U.C.L.A.) there are four undergraduate numerical analysis courses covering the fields of interpolation, linear algebra, ordinary differential equations, and approximation, respectively. These are excellent courses, and there seems to be no real reason for not merging most of them with basic undergraduate courses in corresponding subjects—say advanced calculus, linear algebra, ordinary differential equations, and Fourier series. Both sets of courses would profit from the merger. If coding were common knowledge of all mathematics students, there would seem to be no necessary obstacle to the merger.

As nearly as one can tell from Gnedenko's article [6], the Russian practice seems to be to include numerical analysis problems in a mathematical laboratory. In his article Gnedenko lists several required problems in numerical analysis. These problems are large ones, and require plenty of both analysis and computation. How the laboratory is coordinated with the mathematical lectures, we have no idea.

In summary, I favor a special coding course for all students, to be taken early. I should like to see material prepared so that numerical analysis could otherwise be sifted in with the basic mathematical theory, as illustrations of the theory and as motivation for more theory. In the next section are examples of where this mixing might occur.

7. Examples of numerical analysis mixed into other courses. In this section I shall outline places where numerical analysis comes into standard courses in college mathematics, and in particular where problems should be treated with flow charts or codes. Naturally these vary from small problems to quite large ones.

Analytical geometry. Almost any exercise, like finding the bisector of the angle between two given lines $ax+by+c=0$ and $a_1x+b_1y+c_1=0$, where the constants are arbitrary.

Finding areas and volumes included by lines or planes.

Testing for concurrence or collinearity.

Determining the type of a conic corresponding to a second-degree equation in x and y.

Calculus. Formal integration, *e.g.*, making a flow chart for integrating a general rational function of x of degree no higher than 6, considering all cases.

Studying the asymptotic convergence of Newton's process for getting real zeros of a function. What about multiple roots? Bounding the error in a presumed root.

Use of differentials to get the linearized expression for the error in evaluating a polynomial. Use of polynomials of degrees like 20 or 100 for examples.

Study of sequences of difference quotients whose limit is dy/dx. Accelerating the convergence of the sequences. Central differences versus one-sided differences.

Numerical quadrature on an automatic computer. Study of sequences of quadratures by the trapezoidal rule, as $h \to 0$.

The error in the trapezoidal rule, when $f''(x)$ is integrable, or not integrable.

Upper and lower bounds for integrals. Study of Riemann sums as a sequence to be accelerated to its limit, the Riemann integral.

Effect of rounding numbers on the sharp discontinuity in the formula for $\int x^n dx$ when $n = -1$. See Forsythe [4].

Algebra. Number of multiplications required to evaluate a polynomial of degree n. See Todd [13].

Rapidly computable definition of determinant of A. (The definition in terms of $n!$ terms is impossibly slow for large n.)

Evaluation of determinants by row elimination. Importance of choosing large pivotal elements. Methods of doing so conveniently with matrices of order 1000 or more, stored on magnetic tape.

Coding the calculation of the row rank of a set of row vectors.

Computation of $\sqrt{a+ib}$ on a computer, with minimum error.

Actually solving linear algebraic systems. Round-off errors.

Actually computing eigenvalues of matrices. The great distinction between symmetric and nonsymmetric matrices in this problem.

Determination of whether a quadratic form is definite.

Solution of linear inequalities. Duality theorems, and their use in computation.

"Finite calculus" of sums and differences.

Convergence of linear iterative processes.

Bounding the round-off errors in pseudoarithmetic operations; see Section 3 above.

Logic. Preparing a code to test the truth or falsity of a class of propositions. All sorts of questions relating computability to the theory of recursive functions.

Advanced calculus. The nature of the remainder in Taylor's theorem.

Interpolation theory and remainders. Convergence of interpolation polynomials.

Direct methods in calculation of minimizing functions in calculus of variations.

Numerical summation of series with a computer. Code for computing $\sum_{n=0}^{\infty} f(n)x^n$, where $f(n)$ is a polynomial of low order, and $|x| < 1$.

Ordinary differential equations. Coding and running Euler's method for solving $dy/dx = f(x, y)$. Bounding its error, and comparing with experience on test problems.

Better methods for numerical integration of same differential equation.

Two-point boundary-value problems.

Computing the solution of an initial-value problem for $y'' + c_1y' + c_2y = 0$, where the characteristic equation nearly has a double root, and where the coefficients are 10-decimal numbers; see [4].

Probability and Statistics. Monte Carlo method for integration. Stratified sampling methods for greater accuracy.

Generating normal random processes on a digital computer, with a prescribed power spectrum density.

Estimating the power spectral density from an observed instance of a random process.

Use of automatic computer to find mean and variance of given data which are very numerous.

Number theory. Generation and testing of pseudorandom numbers for use in Monte Carlo experiments.

Tests for primality of large numbers.

All kinds of combinatorial problems.

Geometry. Iteration of rational transformations—theory and practice.

Study of gradient methods on a computer.

Fourier series. Numerical summation of a given series; acceleration of convergence. Orthogonal polynomials over finite point sets, and their use in curve fitting.

8. Implementing the recommendations. Since it does not seem practical to teach an undergraduate course without following a textbook fairly closely, courses are likely to change only when textbooks change. There is no textbook for a modern integrated version of a pure mathematical field and its numerical application. Moreover, of all the numerical analysis texts published in the past 10 years, only one seems really concerned with the problems of automatic digital computation. While desk calculators are far from dead, existing textbooks make it possible for any one to learn desk-calculator technique very well indeed. Future textbooks in numerical analysis must emphasize the use of automatic machinery. The one book to deal with automatic computation is the recent

Modern Computing Methods [2], regrettably anonymous but actually written by L. Fox, E. T. Goodwin, F. W. J. Olver, and J. H. Wilkinson. This is very clear, too short to be comprehensive, but full of annotated references to major sources.

I therefore seem to be making a recommendation which is almost two generations ahead of the current textbooks. The first generation might follow *Modern Computing Methods* and treat various aspects of computation by automatic computers. The second generation might integrate this material into basic existing mathematics courses. One can hope that some books will leap the whole step at once.

There is one method of accelerating the appearance of textbooks in which mathematical theory and numerical applications are suitably intertwined—namely to write applied supplements to existing theoretical texts. For example, one might prepare a companion text to a textbook in the theory of linear algebra, in which numerical methods are discussed from the point of view of automatic computation, with suitable problems. Such a companion text might presume that the student knows how to code, but that he is learning the theory of linear algebra for the first time from the basic textbook. The companion text would refer to the basic text for theory, and would indicate the appropriate stages at which the companion book should be used.

Such companion texts would be useful for at least the following subjects: calculus, differential equations, statistics, linear algebra, advanced calculus. It would be well to include suitable analytical applications, as well as numerical ones.

My basic recommendation for integrated courses in mathematical theory and application may have to await the appearance of suitable integrated texts or companion texts. What can be done meanwhile, aside from urging one's friends to write expositions?

For one thing, the coding course can be initiated at once anywhere. One needs only a textbook, say Andree [1] or McCracken [10], plus coding manuals which the International Business Machines Corporation or other computer manufacturers would probably be glad to furnish through their local representatives. McCracken's text deals with a mythical machine.

No matter what courses are given, I believe all students of mathematics should have easy access to semiautomatic or fully automatic desk calculating machines, and some problems to solve with their aid. Even such an apparently simple problem as using an 8-bank calculator to find close upper and lower bounds for

$$1.56898x^3 - 0.45678x^2 + 9.99938x - 0.42057 \qquad \text{for } x = \pi/2$$

will give students a surprising amount of insight into some problems of numerical calculation. Some experience with desk calculators is almost an indispensable prerequisite to use of an automatic computer.

If there is an automatic computer, students should be encouraged to use it

to find upper and lower bounds for various quantities which they deal with in their several theoretical courses. Such exercises should develop a good deal of power in numerical analysis.

Finally, until integrated texts are available, it will probably be profitable to offer a specialized course in numerical analysis to undergraduates, using as a text such existing books as *Modern Computing Methods* [2], Hildebrand [7], Milne [11], or—better yet—selections from each. There are other good textbooks also. Such a course should include as exercises, problems which I envision as later appearing in several theoretical courses.

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A CONNECTED TOPOLOGY FOR THE INTEGERS

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A topology D for the positive integers is obtained when those arithmetic progressions $\{an+b\}$ with $(a, b)=1$ are taken as a basis for the open sets. They form a basis because the intersection of two such progressions is of the same type, or empty, as is easily verified. Note that every nonempty open set, being a union of arithmetic progressions, must be infinite.

This topology furnishes an interesting proof of

THEOREM 1. *The number of primes is infinite.*

Proof. If p is prime, the progression $\{np\}$ is closed, since its complement is $\{np+1\}\cup\{np+2\}\cup \cdots \cup\{np+(p-1)\}$, a union of open sets. Consider the union $X=\cup_p \{np\}$ extended over all primes. If this is a finite union of closed sets, then X is closed. But the complement of X is $\{1\}$, which is neither empty nor infinite. Since the complement of X is not open, X cannot be closed, the union is not a finite one, and the number of primes is infinite.

(A similar proof, in a stronger and very disconnected topology, was given by Furstenberg [2].)

THEOREM 2. *The topology D is Hausdorff.*

Proof. Given distinct positive integers s and t, choose a prime p (by Theorem 1) which exceeds max (s, t). Then $\{pn+s\}$ and $\{pn+t\}$ are disjoint open sets which separate s and t.

THEOREM 3. *The topology D is connected.*

Proof. Suppose the integers could be represented as the union of two disjoint nonempty open sets O_1 and O_2. Let $\{a_1n+b_1\}$ be a basis set in O_1, and let $\{a_2n+b_2\}$ be a basis set in O_2. Let α be a multiple of a_1. If α were in O_2, we would have $\alpha=An_0+B$, where $\{An+B\}\subset O_2$. Since $(A, B)=1$, we would have $(\alpha, A)=1$, and hence $(a_1, A)=1$. But then $\{a_1n+b_1\}$ and $\{An+B\}$ would intersect infinitely often, contradicting disjointness of O_1 and O_2. Thus all multiples of a_1 must belong to O_1. Similarly the multiples of a_2 must belong to O_2. But then the *common* multiples of a_1 and a_2 must belong to both O_1 and O_2, which contradicts disjointness.

The author has recently learned that a proof of the connectedness of the topology D, without reference to number theory, was presented by Morton Brown at the April 1953 meeting of the American Mathematical Society in New York [1].

THEOREM 4. *The topology D is not regular.*

Proof. Suppose that open coverings are given for the closed set $\{2n\}$ and for the point $\{1\}$ outside it. Any open covering of $\{1\}$ not intersecting $\{2n\}$

must include a progression $\{en+1\}$, where e is even. That is, $e\in\{2n\}$. Let $\{an+b\}$ be the member of the open covering of $\{2n\}$ which contains e, so that $e=an_0+b$. Since $(a, b)=1$, we have $(a, e)=1$, whereby $\{an+b\}$ intersects $\{en+1\}$ infinitely often. Thus the closed set $\{2n\}$ and the point $\{1\}$ cannot have disjoint open neighborhoods.

THEOREM 5. *The topology D is not compact.*

Proof. The union $\cup_p \{np-1\}$ extended over all primes is an infinite open covering for the positive integers. Since the omission of any progression $\{nq-1\}$ leaves the number $q-1$ uncovered, the Heine-Borel property fails.

Actually, the topology D is not even locally compact, because every locally compact Hausdorff space is regular. For a proof of this, as well as for the more basic definitions of point-set topology, the reader is referred to [**5**].

Dirichlet's theorem, which asserts that every progression $\{an+b\}$ with $(a, b)=1$ contains infinitely many primes, has an elegant formulation in terms of the topology D.

THEOREM 6. *Dirichlet's theorem is equivalent to the assertion that the primes are a dense subset of the integers in the topology D.*

Proof. Assume first the validity of Dirichlet's theorem. Then every nonempty open set contains primes, so that the primes are a dense subset of the integers. Conversely, assume that the primes are a dense subset. Then every nonempty open set, and in particular all the progressions $\{an+b\}$ with $(a, b)=1$, must contain primes. It is well known [**4**] that if every such progression contains at least one prime, then every such progression contains infinitely many primes. (In topological terminology: "If the closure of the primes is the integers, then the derived set of the primes is the integers.")

It appears unlikely that a complete topological proof of Dirichlet's theorem can be given along these lines without the introduction of powerful new ideas and methods.

Another familiar fact capable of topological formulation is

THEOREM 7. *In the topology D, the interior of the set of primes is empty.*

Proof. If there were an open set consisting entirely of primes, there would be a progression $\{an+b\}$ with $1\leqq b\leqq a$ consisting entirely of primes. But with $n_0=a+b+1$, $an_0+b=(a+b)(a+1)$, which is composite.

It is interesting to consider also the topology D' for the positive integers, which has as a basis those progressions $\{an+b\}$ with $(a, b)=1$ for all $n>N$. (Here N is allowed to assume all values.) This topology is clearly *stronger* than D, although Theorems 1 to 7 are still valid in D'. Moreover, certain theorems related to Eratosthenes' sieve are available in D'. In particular,

THEOREM 8. *The set of positive integers m such that $6m-1$ and $6m+1$ are a pair of "prime twins" is closed in D'.*

Proof. It is known [3] that the numbers m in question are precisely those positive integers *not* expressible in the form $6ab \pm a \pm b$ for any $a \geqq 1$ and $b \geqq 1$. Thus the *complement* of our set is $\bigcup_{b \geqq 1} (6b \pm 1)a \pm b\}$, where each progression is restricted to $a \geqq 1$, and is open because $(6b \pm 1, b) = 1$. The union is open in D', because it is a union of open sets. Thus the integers m for which $6m-1$ and $6m+1$ are both prime form a closed set.

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AN APPLICATION OF TURNS AND SLIDES TO SPHERICAL GEOMETRY

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1. Introduction. The ordinary or Euler spherical triangle, the sides and angles of which are assumed to lie between 0 and π, is determined uniquely by three points on a sphere, provided no two of them are diametrically opposite. Since each side of an Euler triangle has two poles, the choice of a polar triangle is ambiguous; moreover, more than one Euler triangle may exist with parts satisfying the law of cosines on a sphere. Moebius [3], consequently, generalized the concept of a spherical triangle in order to remove these ambiguities by orienting the sides and the angles and allowing them to assume all real values between 0 and 2π. Moebius's device, though adequate for a satisfactory interpretation of the formulas involving the sines and cosines of the sides and angles, did not go far enough. Such formulas Felix Klein [2] called formulas of the first kind to distinguish them from others, essentially different, connecting the sines and cosines of the half-sides and half-angles, which he called formulas of the second kind. Among the latter there are, for example, the elegant formulas first published by Delambre (1807) and later independently by Mollweide (1808) and by Gauss (1809). What is remarkable about the Delambre formulas is that they carry a double sign and that there exist two classes of spherical triangle, one for which the upper sign is valid and another for which the lower sign is. It was Study [4] who grasped the full significance of these formulas; he showed that to establish their full import it was necessary to extend the range of values of the sides and angles of a spherical triangle to include all real values. We intend in this paper to prove Delambre's formulas, and analogous ones for spherical

polygons, by means of simple applications of some elementary transformations of oriented lineal elements on a sphere, which, besides being intrinsically interesting, provide a procedure for obtaining more general results and, in addition, serve to make more natural Study's concept of a spherical triangle. The transformations we have in mind are special cases of one called the *spherical whirl* [1].

2. Turns and slides. Let the unit sphere S have its center at O, the origin of a right-handed orthogonal Cartesian frame $\mathfrak{F}_0$. Let an oriented lineal element* $\mathcal{E}$ be tangent to S at the point P; then we call the great circle passing through P tangent to $\mathcal{E}$ and oriented in the same sense as $\mathcal{E}$, the great cycle of $\mathcal{E}$. We assume all lineal elements are oriented. Let the lineal element $\mathcal{E}_0$ have its point at (1, 0, 0) and let it be directed so that its great cycle passes through (0, 1, 0) and has for its positive orientation the counterclockwise sense when viewed from the point (0, 0, 1). We shall call $\mathcal{E}_0$ the primitive lineal element on S and $\mathfrak{F}_0$, its associated frame.

Let e_0, e_1, e_2, e_3 be the Hamilton quaternion units such that $e_1^2=e_2^2=e_3^2=e_1e_2e_3=-1$, $e_0e_i=e_ie_0=e_i$, and let

$$x = x_0e_0 + x_1e_1 + x_2e_2 + x_3e_3, \qquad \bar{x} = x_0e_0 - x_1e_1 - x_2e_2 - x_3e_3.$$

Then a rotation of S around an arbitrary diameter is given by the Hamilton-Cayley formula

$$N(x)u^* = \bar{x}ux, \qquad N(x) = x\bar{x},$$

where u and u^* are unit vectors (pure quaternions) with their initial points at O. The components x_i of x are the homogeneous Euler parameters of the rotation. Let $\mathcal{E}$ be an arbitrary lineal element on S', a unit sphere concentric with S, and let x be the quaternion of the rotation of S such that $\mathcal{E}_0 \to \mathcal{E}$; then we let, as we may, the components of x serve as the homogeneous coordinates of $\mathcal{E}$. For convenience, where no confusion will result, we shall let x designate both the rotation $\mathcal{E}_0 \to \mathcal{E}$ and $\mathcal{E}$. Evidently, if x designates $\mathcal{E}$, so does kx, where k is a nonzero scalar. If $N(x)=1$, x is said to be normalized. To any quaternion x, $N(x)\neq 0$, correspond two normalized ones, differing only in sign, namely $\pm x/[N(x)]^{1/2}$. We assume from here on that all quaternions are normalized.

If the rotation x revolves S around the unit vector v through the angle 2θ, we can set $x=-\cos\theta+v\sin\theta$, so that its associated normalized quaternion

$$-x = -\cos(\pi-\theta) + \bar{v}\sin(\pi-\theta), \qquad \bar{v} = -v,$$

designates a rotation of angle $2\pi-2\theta$ around $\bar{v}$. The rotation x also rotates $\mathfrak{F}_0$ into another Cartesian frame $\mathfrak{F}$ situated relative to $\mathcal{E}$ as $\mathfrak{F}_0$ is situated relative to $\mathcal{E}_0$; we shall call $\mathfrak{F}$ the frame associated with $\mathcal{E}$. The point (0, 0, 1) of $\mathfrak{F}$ will be called the pole of the great cycle of $\mathcal{E}$.

A lineal element transformation $x \to x^*$ given by

* A *lineal element* on a sphere S is the combination of a point P on S and a line t tangent to S at P. The lineal element is said to be *oriented* if t is oriented.

$$x^* = xa, \tag{2.1}$$

where $a = -\cos\alpha + u \sin\alpha$, $u^2 = -1$, represents a lineal element rotation by means of which all lineal elements on S are rotated through an angle 2α around an oriented diameter having the direction of the vector u. Now, let us suppose that every lineal element x is rotated through the same angle 2α, but in such a way that each lineal element revolves about an oriented axis situated relative to its associated frame $\mathfrak{F}$ as the axis u is situated relative to the primitive frame $\mathfrak{F}_0$. Such a transformation we have called a *spherical whirl* [1]. Evidently, if the whirl transports $x \to x^*$, $x^*\bar{x} = (x\bar{x})a$, so that the whirl has the equation

$$x^* = ax. \tag{2.2}$$

The whirls (2.2) constitute a three-parameter group $\mathfrak{W}_b$, skew-isomorphic to the group of lineal element rotations (2.1). We shall be particularly concerned with certain one-parameter subgroups of $\mathfrak{W}_3$, namely, the *turns*, the *slides* and the *dilatations*.

A *turn* T_τ is a whirl such that every lineal element is rotated around its point through an angle τ. The group of turns is given by

$$T_\tau\colon x^* = [-\cos(\tau/2) + e_1 \sin(\tau/2)]x. \tag{2.3}$$

A *slide* S_σ is a whirl such that every lineal element slides tangentially along its great cycle through the arc-length σ. The equation of the group of slides is

$$S_\sigma\colon x^* = [-\cos(\sigma/2) + e_3 \sin(\sigma/2)]x. \tag{2.4}$$

It will be observed that the identity turn and identity slide are $T_{2k\pi}$ and $S_{2k\pi}$, respectively, where k is an integer, and that the identity transformation is

$$I\colon x^* = \rho x, \ \rho = \pm 1, \tag{2.5}$$

ρ taking the upper sign when k is odd and the lower one when k is even.

A *dilatation* D_δ is a whirl such that every lineal element x is transported to a position orthogonal to a great cycle c which is itself orthogonal to the great cycle of x, the point of x traversing an arc of length δ on c. The dilatations form a one-parameter group. The sense of c is chosen so that the equation of this group is given by

$$D_\delta\colon x^* = [-\cos(\delta/2) + e_2 \sin(\delta/2)]x. \tag{2.6}$$

The dilatations, unlike the turns and slides, are contact transformations. Evidently,

$$D_\delta = T_{-\pi/2} S_\delta T_{\pi/2} = S_{\pi/2} T_\delta S_{-\pi/2}$$

and

$$T_\alpha T_{-\alpha} = S_\sigma S_{-\sigma} = D_\delta D_{-\delta} = I.$$

We define a lineal element polarity $\mathcal{P}$ by means of the equation

$$(2.7) \qquad \mathcal{P}: x^* = T_\pi D_{\pi/2} x;$$

clearly, we can express $\mathcal{P}$ also in the equivalent forms:

$$(2.8) \qquad \mathcal{P} = T_\pi D_{\pi/2} = T_{\pi/2} S_{\pi/2} T_{\pi/2} = D_{-\pi/2} T_\pi = S_{\pi/2} T_{\pi/2} S_{\pi/2}.$$

With the aid of (2.3) and (2.6) we obtain for $\mathcal{P}$ the equation

$$\mathcal{P}: x^* = (e_1 + e_3) 2^{-1/2} x;$$

from this, or from (2.8), we get $\mathcal{P}^2 x = -x$; that is, $\mathcal{P}^2 = I$. Also by means of these equations we see that

$$(2.9) \qquad S_\sigma = \mathcal{P} T_\sigma \mathcal{P}, \qquad T_\sigma = \mathcal{P} S_\sigma \mathcal{P}.$$

As the lineal element x slides along its great cycle c_1 a distance σ, its polar, $\mathcal{P}x$, turns around its point P_1, which is the pole of c_1, through an angle σ; and, as x rotates around its point P_2 through an angle σ, its polar, $\mathcal{P}x$, slides on its great cycle, of which P_2 is the pole, a distance σ.

Let x and y be two lineal elements on S. If x and y have a common point the whirl $x \rightarrow y$ is a turn; if x and y have a common great cycle the whirl $x \rightarrow y$ is a slide. If they have neither a common point nor a common great cycle, let P_1 and c_3 be the point and great cycle respectively of x; P_3 and c_1 the point and great cycle of y. Let P_2 be a point of intersection of c_3 and c_1, and let c_2 be a great cycle that passes through P_3 and P_1. Let us now subject x to a sequence of slides on the great cycles c_i and turns at the points P_i, as follows:

(1) a slide S_c on c_3, starting from P_1 and terminating at P_2, the slide including possibly any number of circuits of c_3;

(2) a turn T_β at P_2, starting from the direction of c_3 and terminating in a direction properly tangent to c_1, permitting the mobile lineal element to complete any number of revolutions during the turn;

(3) a slide S_a from P_2 to P_3;

(4) a turn T_γ from c_1 to c_2;

(5) a slide S_b from P_3 to P_1; and

(6) a turn T_α from c_2 to c_3. With T_α, the lineal element is brought back to its initial position. The complete circuit is given by

$$(2.10) \qquad T_\alpha S_b T_\gamma S_a T_\beta S_c x = \rho x, \qquad \rho = \pm 1.$$

3. Spherical polygons. We define a Study spherical triangle by means of a circuit (2.10); the slides determine its sides and the turns its angles. Degenerate triangles are not excluded. The first three steps in the circuit (2.10) produce the whirl $x \rightarrow y$. Multiplying the two members of (2.10) on the right by $\bar{x}$ and on the left by $T_{-\gamma} S_{-b} T_{-\alpha}$, we obtain two equivalent forms for the whirl, namely,

$$(3.1) \qquad S_a T_\beta S_c = \rho T_{-\gamma} S_{-b} T_{-\alpha}.$$

By means of great cycles passing through P_1 and P_3 and orthogonal to c_1 and c_3, respectively, the whirl (3.1) can be resolved also into products such as TDS,

SDT, etc., where T, D and S represent suitable turns, dilatations and slides.

Since any quaternion defines a whirl, it can be expressed in any one of these factored forms. Consequently, if we replace x in the rotation formula $u^* = \bar{x}ux$ by one of the sets of three factors into which it can be resolved, we obtain for each set a parametric representation of the rotation, the parameters representing the magnitudes of the slides, turns and dilatations that occur in the product. In particular, we get Euler's parametric form when we let $x = S_a T_\beta S_c$.

Let x be a rotation of angle θ around an axis carrying the unit vector $v = e_1 \cos \lambda + e_2 \cos \mu + e_3 \cos \nu$; then we can let

$$x \equiv -\cos(\theta/2) + v \sin(\theta/2) = T_\alpha S_\beta S_c.$$

Substituting for the turns and slides from (2.3) and (2.4), we obtain, by equating the corresponding components:

$$\begin{aligned}
\cos\frac{\theta}{2} &= \cos\frac{\beta}{2}\cos\frac{a+c}{2},\\
\cos\lambda\sin\frac{\theta}{2} &= \sin\frac{\beta}{2}\cos\frac{a-c}{2},\\
\cos\mu\sin\frac{\theta}{2} &= \sin\frac{\beta}{2}\sin\frac{a-c}{2},\\
\cos\nu\sin\frac{\theta}{2} &= \cos\frac{\beta}{2}\sin\frac{a+c}{2}.
\end{aligned} \tag{3.2}$$

If we multiply both members of (2.10) on the left by $\mathcal{P}$ and insert $\mathcal{P}\mathcal{P} = I$ between the factors in the left-hand member, we obtain:

$$(\mathcal{P}T_\alpha\mathcal{P})(\mathcal{P}S_b\mathcal{P})(\mathcal{P}T_\gamma\mathcal{P})(\mathcal{P}S_a\mathcal{P})(\mathcal{P}T_\beta\mathcal{P})(\mathcal{P}S_c\mathcal{P})\mathcal{P}x = \rho\mathcal{P}x;$$

hence, by virtue of (2.9), we see that as x makes a circuit of the Study triangle (2.10), its polar, $\mathcal{P}x$, makes a circuit of its polar triangle.

Inserting in (3.1) the formulas for the turns and slides given by (2.3) and (2.4), we obtain the formulas of Delambre:

$$\begin{aligned}
\cos\frac{\beta}{2}\cos\frac{a+c}{2} &= -\rho\cos\frac{b}{2}\cos\frac{\alpha+\gamma}{2},\\
\sin\frac{\beta}{2}\cos\frac{a-c}{2} &= \rho\cos\frac{b}{2}\sin\frac{\alpha+\gamma}{2},\\
\sin\frac{\beta}{2}\sin\frac{a-c}{2} &= -\rho\sin\frac{b}{2}\sin\frac{\alpha-\gamma}{2},\\
\cos\frac{\beta}{2}\sin\frac{a+c}{2} &= \rho\sin\frac{b}{2}\cos\frac{\alpha-\gamma}{2},
\end{aligned} \qquad (\rho = \pm 1). \tag{3.3}$$

It will be observed that the left members of the equations (3.3) appear also in equations (3.2); consequently, with every Study triangle there is associated an orthogonal transformation, the axis and angle of which are given by equations (3.2). By cyclic permutation we get two more sets of equations like (3.3).

There are Study triangles for which only those Delambre formulas are valid in which $\rho = +1$, and also others for which the Delambre formulas with $\rho = -1$ are valid. The full significance of the double sign was made clear by Study [**4**, **5**]. In order to formulate Study's principal theorem bearing on this circumstance, let us, with Study, call the triangles belonging to the class in which $\rho = +1$ *proper*, and those belonging to the class in which $\rho = -1$ *improper*. Furthermore, let us consider the six-space S_6 the points of which have for their coordinates the quantities $a, b, c, \alpha, \beta, \gamma$, all of which are unrestricted real numbers. Then Study's theorem can be stated as follows:

The set of points in S_6 which are the images of the aggregate of Study triangles separates into just two continua (compact, connected point sets), one corresponding to the set of proper triangles and the other to the set of improper triangles.

The Euler triangles belong to the set of proper triangles.

In the same manner we can, of course, by means of a circuit of slides and turns, define a spherical polygon of any number of sides; the slides determine the sides in sense and magnitude while the turns, in like manner, determine the angles. The polarity (2.7) assigns to every polygon a unique polar polygon. The sides and angles are unrestricted in magnitude, as in the case of Study triangles. Starting from the equation for a circuit of slides and turns for such a polygon, like equation (2.10) for triangles, we can derive sets of formulas connecting its sides and angles, analogous to those of Delambre. Let it suffice to indicate the results obtainable for a spherical quadrilateral determined by the following sequence of slides and turns:

$$T_\delta S_d T_\gamma S_c T_\beta S_b T_\alpha S_a x = \rho x, \qquad \rho = \pm 1. \tag{3.4}$$

Multiplying both of the members of (3.4) on the right by $\bar{x}$, and on the left by $S_{-c}T_{-\gamma}S_{-d}T_{-\delta}$, we obtain

$$T_\beta S_b T_\alpha S_a = \rho S_{-c} T_{-\gamma} S_{-d} T_{-\delta}. \tag{3.5}$$

Substituting in (3.5) the formulas for the slides and turns given by (2.3) and (2.4), and simplifying the results, we obtain the following four formulas:

$$\left[\cos\frac{\alpha+\beta}{2} + \cos\frac{\alpha-\beta}{2}\right]\cos\frac{a+b}{2} + \left[\cos\frac{\alpha+\beta}{2} - \cos\frac{\alpha-\beta}{2}\right]\cos\frac{a-b}{2}$$
$$= \rho\left\{\left[\cos\frac{\gamma+\delta}{2} + \cos\frac{\gamma-\delta}{2}\right]\cos\frac{c+d}{2} + \left[\cos\frac{\gamma+\delta}{2} - \cos\frac{\gamma-\delta}{2}\right]\cos\frac{c-d}{2}\right\},$$

$$\left[\sin\frac{\alpha+\beta}{2}+\sin\frac{\alpha-\beta}{2}\right]\cos\frac{a-b}{2}+\left[\sin\frac{\alpha+\beta}{2}-\sin\frac{\alpha-\beta}{2}\right]\cos\frac{a+b}{2}$$
$$= -\rho\left\{\left[\sin\frac{\gamma+\delta}{2}+\sin\frac{\gamma-\delta}{2}\right]\cos\frac{c-d}{2}+\left[\sin\frac{\gamma+\delta}{2}-\sin\frac{\gamma-\delta}{2}\right]\cos\frac{c+d}{2}\right\},$$

$$\left[\sin\frac{\alpha+\beta}{2}+\sin\frac{\alpha-\beta}{2}\right]\sin\frac{a-b}{2}+\left[\sin\frac{\alpha+\beta}{2}-\sin\frac{\alpha-\beta}{2}\right]\sin\frac{a+b}{2}$$
$$= -\rho\left\{\left[\sin\frac{\gamma+\delta}{2}+\sin\frac{\gamma-\delta}{2}\right]\sin\frac{c-d}{2}+\left[\sin\frac{\gamma+\delta}{2}-\sin\frac{\gamma-\delta}{2}\right]\sin\frac{c+d}{2}\right\},$$

$$\left[\cos\frac{\alpha+\beta}{2}+\cos\frac{\alpha-\beta}{2}\right]\sin\frac{a+b}{2}+\left[\cos\frac{\alpha+\beta}{2}-\cos\frac{\alpha-\beta}{2}\right]\sin\frac{a-b}{2}$$
$$= -\rho\left\{\left[\cos\frac{\gamma+\delta}{2}+\cos\frac{\gamma-\delta}{2}\right]\sin\frac{c+d}{2}+\left[\cos\frac{\gamma+\delta}{2}-\cos\frac{\gamma-\delta}{2}\right]\sin\frac{c-d}{2}\right\}.$$

Cyclic permutations of the sides and angles provide three more sets of formulas. It will be observed that these formulas are self-polar; moreover, as in the case of Study triangles, there are two kinds of quadrilaterals: those whose sides and angles satisfy the formulas with $\rho=+1$, and those that satisfy the set with $\rho=-1$.

In [1] we defined a *spherical turbine* as a series of oriented lineal elements, the points of which lie on a cycle c, and the great cycles of which make a constant angle α with c. When $\alpha=0$ we have a *tangential turbine*, and, when c is a point cycle, a *point turbine*. The equation in lineal element coordinates of the point turbine at the point e_1 is clearly $\bar{x}e_1x=e_1$. By means of a lineal element rotation (2.1), this is transformed into a point turbine at an arbitrary point having for its cordinates the components of the unit vector r, namely, $\bar{x}e_1x=r$. The dilatation D_δ transforms this point turbine into a tangential one with center at r and radius equal to δ, given by the equation

$$\bar{x}(e_1\cos\delta - e_3\sin\delta)x = r. \tag{3.6}$$

When $\delta=\pi/2$, (3.6) is the equation of a great cycle.

If we let the four components x_i of the quaternion x be the homogeneous coordinates of a point in projective three-space S_3, we obtain a continuous one-

to-one mapping of the lineal elements x on S upon the points x of S_3. By virtue of this mapping, the ∞^6 lineal element *whirl-rotations* $x \to axb$, are mapped on the ∞^6 displacements in elliptic three-space E_3; indeed, to the whirls $x \to ax$ correspond the left translations in E_3, and to the rotations $x \to xb$ correspond the right translations [1]. To the turbines on S correspond one-to-one the lines in E_3. To a slide of the lineal elements on S corresponds a left translation in which the points of E_3 glide on lines belonging to a congruence of Clifford parallels C_1, and to a turn on S corresponds a left translation on the lines of another congruence of Clifford parallels C_2. The sequence of slides and turns in (2.10), defining a triangle on S, corresponds to a sequence of left translations that a point in E_3 executes in describing a complete circuit of a skew hexagon, the sides of which belong alternately to C_1 and C_2. Similarly, to a spherical polygon of n sides defined by slides and turns, there corresponds in E_3 a polygon of $2n$ sides that lie alternately on lines belonging to C_1 and C_2. If, however, we let a point in E_3 make a circuit, by means of left translations, of a polygon, say a hexagon, the sides of which are not restricted to C_1 and C_2, the corresponding lineal element on S executes a sequence of six whirls in each of which an "arc" of a turbine is traversed; the figure on S that corresponds to a general hexagon in E_3 is consequently a closed chain of six "arcs" of turbines. For such a figure we could also obtain formulas analogous to those of Delambre; however, the lengths of the "arcs" would be given in an elliptic metric and the formulas would involve additional parameters, namely, the six radii of the turbines and the angles of inclination of the lineal elements in the turbines.

An interesting special case arises when the six turbines are alternately tangential turbines with radii between 0 and $\pi/2$, and point turbines. Such a figure is a "spherical triangle" $\mathbf{t}$ whose sides are oriented arcs of small cycles. Let us assume that the three small cycles have equal radii, and that the sides and angles of $\mathbf{t}$ are measured in the elliptic metric of their image in E_3. It can be easily verified that the tangential whirl that makes all of the lineal elements of the cycle (3.6) advance along its periphery for the distance $a/2$ is given by

$$(3.7) \qquad W_a\colon x \to qx, \qquad q = -\cos(a/2) + (e_1 \cos\delta - e_3 \sin\delta)\sin(a/2).$$

It should be observed that

$$W_a = D_{-\delta} T_a D_\delta, \qquad W_{-a} = D_{\delta'} S_a D_{-\delta'}, \qquad \delta' = \pi/2 - \delta.$$

We can now define the sides and angles of $\mathbf{t}$ by means of the following circuit:

$$(3.8) \qquad T_\beta W_c T_\alpha W_b T_\gamma W_a x = \rho x, \qquad \rho = \pm 1,$$

where $a/2$, $b/2$, $c/2$, are the lengths of the sides of $\mathbf{t}$, and $\alpha/2$, $\beta/2$, $\gamma/2$, are the magnitudes of its angles. From (3.8) we obtain

$$(3.9) \qquad W_b T_\gamma W_a = \rho T_{-\alpha} W_{-c} T_{-\beta}.$$

Substituting for the turns and tangential whirls in (3.9) the expressions given by (2.3) and (3.7), we obtain the following formulas:

$$\cos\delta\sin\frac{\gamma}{2}\sin\frac{a+b}{2}-\cos\frac{\gamma}{2}\cos\frac{a+b}{2}$$
$$=\rho\left[\cos\delta\sin c\sin\frac{\alpha+\beta}{2}-\cos\frac{c}{2}\cos\frac{\alpha+\beta}{2}\right],$$

$$\cos\delta\cos\frac{\gamma}{2}\sin\frac{a+b}{2}-\cos 2\delta\sin\frac{\gamma}{2}\sin\frac{a}{2}\sin\frac{b}{2}+\sin\frac{\gamma}{2}\cos\frac{a}{2}\cos\frac{b}{2}$$
$$=-\rho\left[\cos\delta\sin\frac{c}{2}\cos\frac{\alpha+\beta}{2}+\cos\frac{c}{2}\sin\frac{\alpha+\beta}{2}\right],$$

$$\sin 2\delta\sin\frac{a}{2}\sin\frac{b}{2}\sin\frac{\gamma}{2}-\sin\delta\sin\frac{a+b}{2}\cos\frac{\gamma}{2}=\rho\sin\delta\cos\frac{\alpha-\beta}{2}\sin\frac{c}{2}\cdot$$

When $\delta=\pi/2$, $\mathfrak{t}$ becomes a Study triangle, and these three formulas reduce to Delambre's. However, since the sides and angles are measured in an elliptic metric in which π is the magnitude of one complete revolution of x around its point, as well as of a complete circuit by x of its great cycle, we accordingly let a, b, c be the lengths of the sides and α, β, γ the magnitudes of the angles, to restore the spherical metric.

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AN INEQUALITY ARISING IN GENETICAL THEORY

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1. The study of genetics provides simple examples of the use of such mathematical tools as probability theory, calculus, and matrices: and it may well be of interest to teachers of mathematics and statistics. Recently one of us (C. A. B. S.) came across an unusual inequality in a genetical problem, and the other author (H. P. M.) succeeded in proving its correctness. The inequality is as follows:

Let v, u be arbitrary nonzero real $k\times 1$ and $k\times k$ matrices respectively, with all their elements nonnegative, and with u symmetrical, i.e., $u^T=u$, where u^T denotes the transpose of u. Then, for any positive integer n,

$$(v^Tu^nv)(v^Tv)^{n-1} \geqq (v^T)uv^n \tag{1}$$

with equality if and only if v is a latent vector of u.

We have received alternative proofs of the first part of this result from M. M. Crum and R. Scheuer, and a weaker result has been published by S. P. M. Mandel and I. M. Hughes [4] since this paper was first written. We explain here why this inequality is of interest, as well as giving a proof.

2. One of the most striking of biological phenomena is the great variability of many species of plants and animals, even when raised in a nearly constant environment. For example, some men are short, others tall, some are clever, others stupid, and some dark, others fair. It is far from being immediately obvious why this should be so. For if natural selection leads to the survival of the fittest, why has humanity not been reduced to a single uniform type of maximum fitness? It would be very interesting to know the reason; but it would also be very presumptuous at present to attempt a detailed explanation for qualities such as height, intelligence, or hair color, since our knowledge of how these are inherited is very incomplete. Instead, we here consider a rather simplified model. In fact, in what follows both the biological facts and the theoretical model have been, strictly speaking, oversimplified, but with the essential features retained. Those who wish to pursue the matter further can read C. C. Li [3] or O. Kempthorne [2], who give references to other work on the subject.

Many properties of living organisms are determined by objects called *genes*. For example, there are three types of genes, usually called A, B and O, that determine the ordinary blood group in man; each individual carries just two such genes, so that there are 6 distinct blood groups, or pairs of genes, namely OO, AO, AA, BO, BB, AB. Their practical importance is that it is dangerous to transfuse blood from an individual carrying an A gene into one without an A gene, *e.g.*, AO or AA blood may not be transfused into a BO or BB patient, and a similar rule holds for the B gene. (It can be deduced that when only transfusion is considered it is unnecessary to distinguish between the types AO and AA, which are lumped together as *group* A; similarly $BO+BB=$ *group* B.) Mendel's laws state that each child gets at random one of the two genes carried by its father, and independently and at random one of the two carried by its mother. Thus, if the father is AB and the mother OO the child is equally likely to get A or B but must get an O; it is therefore equally likely to be AO or BO. A child of an AB father and AO mother is equally likely to be AA, AO, AB, or BO.

In general we consider a set of k types of genes or *alleles*, $G_1, \cdots, G_k$, with the property that any individual carries just two such genes, possibly of the same type. In a large population of N individuals there will be a certain number

N_{rs} of type G_rG_s: the ratio $N_{rs}/N=p_{rs}$ is the frequency of type G_rG_s in the population. The frequencies p_{rs} describe the population as far as the genes G_r are concerned. Thus in London approximately 7 per cent of persons have blood group AA, 35 per cent AO, 0.4 per cent BB, 8 per cent BO, 3 per cent AB, and 47 per cent OO (*blood group O*). The population of N individuals can also be considered as one of $2N$ genes. Thus, since every G_1G_1 individual carries two G_1 genes, and every G_1G_r $(r\neq 1)$ carries one, the number of G_1 genes is

$$N_1 = 2N_{11} + N_{12} + \cdots + N_{1k}.$$

A division by $2N$, the total number of genes, shows that the frequency of G_1 is

$$f_1 = N_1/2N = p_{11} + \tfrac{1}{2}p_{12} + \cdots + \tfrac{1}{2}p_{1k}, \tag{2}$$

and similarly for $f_2, \cdots, f_k$. Thus in London the A gene has a frequency $7+\frac{1}{2}\cdot 35+\frac{1}{2}\cdot 3=26$ per cent. In a large population it is generally found that the frequencies are the same in males and females, and so equality is assumed in the following argument.

Evolutionary genetics is the study of how a population gradually changes from generation to generation. For simplicity we here assume the generations to be separate, as with annual plants, which have one generation each year. $p_{rs}(g), f_r(g)$ represent the frequencies in the gth generation. Now there is a saying that "marriage is a lottery." If this is taken to mean that marriage partnerships are formed at random, it seems to be true as far as blood groups and similar qualities are concerned. For while one may choose a partner for charm, intelligence, beauty or common interests, it isn't likely that his or her blood group will seem particularly adorable. We make the assumption that the genes under consideration do not affect the chances of having children: this means that a baby chosen at random gets at random one of the genes carried by a random father, and similarly and independently a random gene via the mother. Thus the chance of a baby of generation $(g+1)$ being of type G_rG_r is $f_r(g)\cdot f_r(g)$, while the chance of being G_rG_s $(r\neq s)$ is

$$f_r(g)\cdot f_s(g) + f_s(g)\cdot f_r(g) = 2f_r(g)\cdot f_s(g)$$

since either the G_r gene comes from the father and the G_s from the mother or vice versa. But babies with different genes will have different liabilities to disease or accident: we let w_{rs} be the chance of a baby of type G_rG_s surviving to adult life. The matrix w of the w_{rs} accordingly represents the effect of natural selection. The frequencies $p_{rs}(g+1)$ of the various types G_rG_s among adults of generation $(g+1)$ will be in the ratios of the frequencies at birth multiplied by the survival probabilities, *i.e.*, for the types G_1G_1, G_1G_2, $G_1G_3, \cdots$ they will be in the ratios

$$\begin{aligned} &w_{11}[f_1(g)]^2 : 2w_{12}f_1(g)f_2(g) : 2w_{13}f_1(g)f_3(g) : \cdots \\ &\qquad = p_{11}(g+1) : p_{12}(g+1) : p_{13}(g+1) : \cdots . \end{aligned} \tag{3}$$

To simplify the notation we introduce here certain conventions. Unless otherwise stated p_{rs}, f_r stand for the frequencies $p_{rs}(g), f_r(g)$ in the gth generation, and P_{rs}, F_r for the corresponding values in the following, $(g+1)$th, generation. After a summation sign Greek letters will be used to denote the suffixes to be summed over, and these will be allowed to range over all possible values unless the contrary is indicated: for example, $\sum f_\alpha = 1$, and (2) becomes $f_r = \frac{1}{2}p_{rr} + \frac{1}{2}\sum p_{r\alpha}$.

Now it may happen trivially that all the products $w_{rs}f_rf_s$ are zero, so that there are no survivors in generation $g+1$; we suppose this case excluded. Since the frequencies $P_{rs} = p_{rs}(g+1)$ must add up to unity, and their ratios are given by (3), their actual values are

$$P_{rr} = w_{rr}f_r^2/W(\boldsymbol{f}), \qquad P_{rs} = 2w_{rs}f_rf_s/W(\boldsymbol{f}) \qquad (r \neq s),$$

where $W(\boldsymbol{f})$ stands for the quadratic form $\boldsymbol{f}^T\boldsymbol{w}\boldsymbol{f} = \sum w_{\alpha\beta}f_\alpha f_\beta > 0$. From (2) the gene frequencies in generation $g+1$ are

$$F_r = f_r \sum w_{r\alpha}f_\alpha/W(\boldsymbol{f}) = \tfrac{1}{2}f_r[W(\boldsymbol{f})]^{-1}\partial W(\boldsymbol{f})/\partial f_r, \tag{4}$$

which we write for brevity as $\boldsymbol{F} = \phi(\boldsymbol{f})$. This is a recurrence relation connecting the gene frequencies in generations g and $g+1$. It is easily seen that if $w_{rs}f_rf_s > 0$ then $w_{rs}F_rF_s > 0$ so that, by assuming that the products $w_{rs}f_rf_s$ do not all vanish, we have ensured that the population will continue to exist in generation $g+2$, and the frequencies in that generation are obtained from those of generation $g+1$ by equations similar to (4). By induction, the population continues to exist in every subsequent generation $g^* > g$, and $W(\boldsymbol{f}(g^*)) > 0$, and a relation like (4) connects every two successive generations.

A real numerical example is provided by Allison [1]. In central Africa there are genes affecting the hemoglobin (officially called Hb^A, Hb^S, but here we use the letters G_1, G_2, for consistency) such that G_1G_1 type individuals (the normal European type) are susceptible to malaria, while the G_2G_2 tend to die in childhood from a severe sickle-cell type of anemia. Their survival chances are, according to Allison, approximately in the ratio $w_{11}:w_{12}:w_{22} = 0.8:1:0.2$, so that the recurrence relation is, by (4),

$$F_1 = f_1(w_{11}f_1 + w_{12}f_2)/(w_{11}f_1^2 + 2w_{12}f_1f_2 + w_{22}f_2^2), \tag{5}$$

where $f_2 = 1 - f_1$. Thus if we take, say, $f_1(1) = .1$ as initial value, the frequency of G_1 in successive generations will be $f_1(2) = .280$, $f_1(3) = .464$, $f_1(4) = .579$, $\cdots$ tending eventually to .8, whereas if initially $f_1(1) = .9$ the successive values are .889, .878, .868, etc., again tending to the limit .8. From this it seems plausible that whatever frequencies (other than 0 or 1) the genes G_1 and G_2 have initially, they will approximate to $f_1 = .8$, $f_2 = .2$ respectively after a few generations; $f_1 = .8$ is a *stable equilibrium*. Thus all three types G_1G_1, G_1G_2, G_2G_2 will continue to exist indefinitely in the population *as a result of natural selection*. Of course, if initially $f_1 = 0$, it will always remain so, for in our model if a gene is not present in any one generation it cannot appear in any subsequent generation. (In nature

it could be introduced by migration or mutation.) We call an equilibrium *degenerate* if any of the f_r are there zero, otherwise *nondegenerate*.

An example of a stable equilibrium with three genes G_1, G_2, G_3 is shown in Figure 1. Here the selection matrix is taken to be

$$\mathbf{w} = \begin{bmatrix} .4 & .7 & .6 \\ .7 & .1 & .9 \\ .6 & .9 & .4 \end{bmatrix}.$$

The small circles show the values of the three gene frequencies f_1, f_2, f_3 plotted as trilinear coordinates, which is possible since $f_1+f_2+f_3=1$, and the initial values are various points near the perimeter of the triangle. The sequences of points clearly appear to tend to an equilibrium at $E=(\frac{1}{6}, \frac{1}{3}, \frac{1}{2})$.

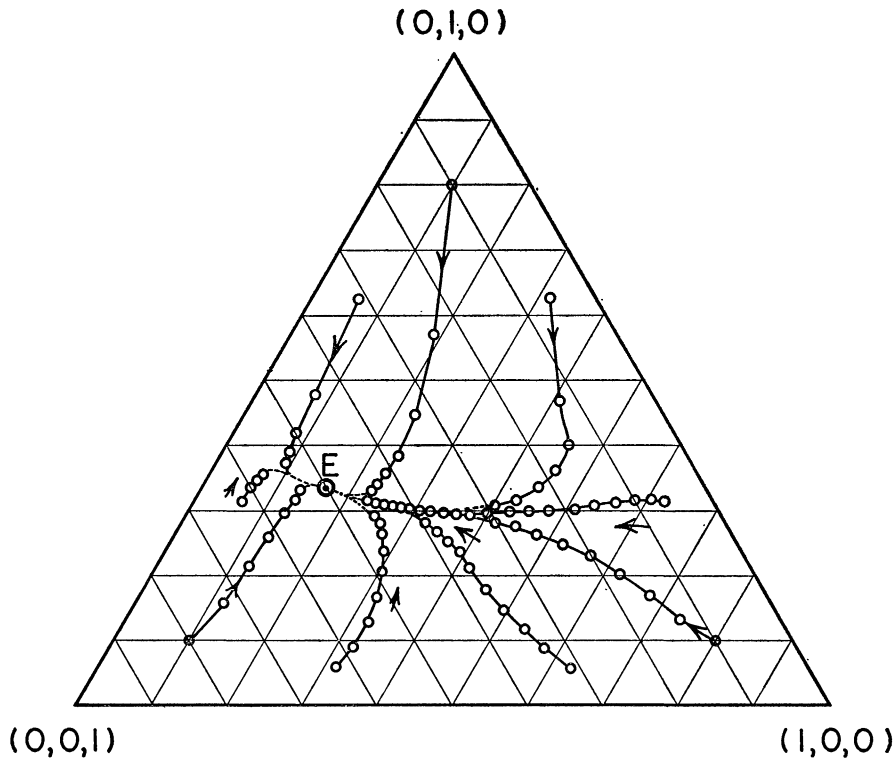

Fig. 1

3. It is therefore an important genetical problem to know when such stable equilibria exist, and whether a population not initially in equilibrium must tend to one under the action of selection.

When only two genes G_1 and G_2 are concerned, a complete analysis can be

made by elementary methods. The cases in which $w_{11}w_{22}=0=w_{12}$ are trivial in the sense that at least one gene is completely lost after the first generation. We therefore exclude these, as well as the cases for which $w_{11}f_1^2=w_{12}f_2f_2^2=w_{22}f_2=0$, already excluded in Section 2. For other cases the analysis is most easily done in terms of the ratio $r=f_1/f_2=f_1/(1-f_1)$, which has the range of values $0\leqq r\leqq\infty$; if $R=F_1/F_2$ the recurrence relation (5) becomes

$$R = r(w_{11}r + w_{12})/(w_{12}r + w_{22}) \tag{6}$$

(and $R=\infty$ when $r=\infty$). An equilibrium point is one at which $R=r$. From (6) we see that there are always two degenerate equilibria $r=0$ and ∞ (at which only one gene is present); there is also a third, nondegenerate equilibrium $r=q$ where

$$q = (w_{12} - w_{22})/(w_{12} - w_{11}) \tag{7}$$

provided that $0<q<\infty$, unless we have $w_{11}=w_{12}=w_{22}$, and q is indeterminate: but then $R=r$ for all r, and any population is in a state of (neutral) equilibrium. Apart from this, the following cases may occur:

(I) $w_{11}<w_{12}>w_{22}$. A nondegenerate equilibrium q exists. Furthermore, it is stable, and whatever initial value other than 0 or ∞ we take for r, the population will eventually tend to the equilibrium q. For from (6) and (7)

$$\frac{R-q}{r-q} = \frac{w_{11}r + w_{22}}{w_{12}r + w_{22}} = 1 - \frac{(w_{12} - w_{11})r}{w_{12}r + w_{22}}\cdot$$

Thus if we choose arbitrarily two positive numbers r', r'' such that $r'<r<r''$, $r'<q<r''$, we have

$$0 < (R-q)/(r-q) < 1 - c, \tag{8}$$

where $c=(w_{12}-w_{11})r'/(w_{12}r''+w_{22})>0$. The distance $|r-q|$ of r from the equilibrium q is diminished by a factor smaller than $1-c$ in one generation. It follows that $r'<R<r''$, the argument repeats in the next and every subsequent generation, and the distance tends to zero with (at least) geometric rapidity.

The other cases which follow can be covered by a similar type of argument, so we omit details:

(II) $w_{11}>w_{12}<w_{22}$. There is an unstable equilibrium at q; from smaller values of r the sequence tends to 0, and from larger ones, to ∞.

(III) $w_{11}\leqq w_{12}\leqq w_{22}$ (but not $w_{11}=w_{12}=w_{22}$). For any initial r, except ∞, successive values tend to 0, the only stable equilibrium.

(IV) $w_{11}\geqq w_{12}\geqq w_{22}$ (but not $w_{11}=w_{12}=w_{22}$). As case (III), with 0 and ∞ interchanged.

4. In general, if $f(1)$ is the initial gene-frequency vector, the vectors in subsequent generations are $f(2)=\phi(f(1))$, $f(3)=\phi(f(2))$, and so on, where the func-

tion ϕ is defined explicitly by (4); but it is difficult to get any explicit expression for $\boldsymbol{f}(i)$ for general i, or study the properties of the sequence directly. However, much information can be got from the following two theorems. (From now on, it is to be always understood that a frequency vector, such as $\boldsymbol{f}$, is to be restricted to lie in the simplex [all $f_i \geqq 0$, $\sum f_\alpha = 1$] and in the region where $W(\boldsymbol{f}) > 0$, unless the contrary is explicitly stated.)

THEOREM 1. *$\boldsymbol{E}$ is an equilibrium point if and only if either (trivially) only one $E_i \neq 0$ or the function $W(\boldsymbol{f})$ has a stationary point at $\boldsymbol{f} = \boldsymbol{E}$ when variation is allowed only in the components f_i for those i for which $E_i \neq 0$; if $E_j = 0$, f_j is held fixed at 0. In particular, a nondegenerate equilibrium $\boldsymbol{E}$ (i.e., one with all $E_i > 0$), is a stationary point for $W(\boldsymbol{f})$ with the f_i varying freely (apart from the necessary restriction $\sum f_\alpha = 1$).*

Proof. Suppose, for example, that $E_1 = E_2 = 0$, $E_i > 0$ $(i > 2)$. The necessary and sufficient conditions for $W(\boldsymbol{f})$ to be stationary at $\boldsymbol{E}$ subject to the conditions $f_1 = f_2 = 0$, $\sum f_\alpha = 1$ are that for some λ

$$\partial W(\boldsymbol{f})/\partial f_3 = \partial W(\boldsymbol{f})/\partial f_4 = \cdots = \partial W(\boldsymbol{f})/\partial f_k = \lambda \tag{9}$$

when $\boldsymbol{f} = \boldsymbol{E}$. By Euler's Theorem $\sum f_\alpha \partial W(\boldsymbol{f})/\partial f_\alpha = 2W(\boldsymbol{f})$, *i.e.*, $\lambda = \frac{1}{2}W(\boldsymbol{E})$. Thus (9) is equivalent to

$$f_i = \tfrac{1}{2} f_i [W(\boldsymbol{f})]^{-1} \partial W(\boldsymbol{f})/\partial f_i \tag{10}$$

for all i, when $\boldsymbol{f} = \boldsymbol{E}$: by (4) this means that $\phi(\boldsymbol{E}) = \boldsymbol{E}$, *i.e.*, $\boldsymbol{E}$ is by definition an equilibrium.

COROLLARY. *An equilibrium point $\boldsymbol{E}$ is either isolated or is on at least one straight segment on which every point $\boldsymbol{f}$ is an equilibrium point with $W(\boldsymbol{f}) = W(\boldsymbol{E})$.*

Let us consider, for example, those points $\boldsymbol{f}$ for which

$$f_1 = f_2 = 0, \qquad \partial W/\partial f_3 = \cdots = \partial W/\partial f_k, \qquad \sum f_\alpha = 1,$$

where, for the moment, we do not require every f_i to be nonnegative. Since these equations are linear in the components f_i they are satisfied either at no point, or at one point, or at every point of some r-dimensional linear subspace with $1 \leqq r \leqq k-3$. As in the proof of the theorem, on such a subspace $W(\boldsymbol{f})$ is stationary at every point for variation of $\boldsymbol{f}$ within the subspace, and thus $W(\boldsymbol{f})$ must clearly be constant on the subspace. On imposing the restrictions $f_i \geqq 0$ and using the theorem we see that the set of equilibrium points $\boldsymbol{E}$ of the special type with $f_1 = f_2 = 0$ is either empty, or contains only one point, or consists of an r-dimensional convex body (with $1 \leqq r \leqq k-3$) on which $W(\boldsymbol{f})$ is constant. Similarly for equilibrium points of the other special types. Since the number of possible types is finite (actually $2^k - 1$), it is clear that an equilibrium point must either be isolated or belong to a convex body (corresponding to one of the special types) in which every point is an equilibrium point and $W(\boldsymbol{f})$ is constant. This suffices to establish the assertion in the corollary.

THEOREM 2. $W(F)=W(\phi(f))\geqq W(f)$, *with equality if and only if* f *is an equilibrium.*

(We recall that values of f at which $W(f)=0$ have been explicitly excluded: they make F indeterminate.)

Proof. (We assume the correctness of (1), which will be proved later.) Write $v_i=\sqrt{f_i}$ and $u_{ij}=v_iw_{ij}v_j$. Then $W(f)=\sum w_{\alpha\beta}f_\alpha f_\beta=v^Tuv$ and, from (4),

$$\begin{aligned} W(F) &= \sum \left[w_{\alpha\beta}f_\alpha \sum w_{\alpha\delta}f_\delta\cdot f_\beta \sum w_{\beta\epsilon}f_\epsilon\right]/[W(f)]^2 \\ &= \sum v_\delta u_{\delta\alpha}u_{\alpha\beta}u_{\beta\epsilon}v_\epsilon/(v^Tuv)^2 = (v^Tu^3v)/(v^Tuv)^2. \end{aligned}$$

Since $v^Tv=\sum v_\alpha^2=\sum f_\alpha=1$, the inequality $W(F)\geqq W(f)$ is equivalent to $(v^Tu^3v)(v^Tv)^2\geqq(v^Tuv)^3$. Furthermore from (9) and (10) the condition for equilibrium is that for each i either $f_i=v_i^2=0$ or $2\sum w_{i\alpha}f_\alpha=\lambda$, that is, for every i,

$$2\sum v_iw_{i\alpha}v_\alpha^2=\lambda v_i, \quad \text{or} \quad uv=\tfrac{1}{2}\lambda v,$$

i.e., that v is a latent vector of u. This reduces Theorem 2 to the case $n=3$ of (1).

5. Now Theorem 2 implies that

$$W(f(1))\leqq W(f(2))\leqq W(f(3))\leqq\cdots. \tag{11}$$

If the function $W(f)$ is represented by its graph, a "surface" in k dimensions, then the sequence $\{f(r)\}$ is represented by a sequence of points climbing up the surface. It is natural to suppose that they will eventually tend to a "peak," which will be a point of stable equilibrium. This picture of the genetic behavior of populations is due to Sewall Wright [5], but his formal proofs do not seem to be complete. In any case, unless we impose further restrictions upon the function $W(f)$ the situation may be complicated by the presence on the surface of saddle points, and also of level ridges or plateaus: the sequence of points may converge to a saddle point, which will be a point of unstable equilibrium, and (although we conjecture the contrary) it might have several limit points instead of converging to a unique limit. However, with the aid of Theorems 1 and 2, we are able to demonstrate the following theorem, and we shall leave till Section 6 the discussion of various genetically appropriate restrictions.

THEOREM 3. (I) *Any limit point of the sequence* $\{f(r)\}$ *is an equilibrium point.* (II) *Let* $W(f)$ *have a strict maximum at* E *(for admissible values of* f*): then* E *is a stable equilibrium point, i.e., if any point* $f(i)$ *is near enough to* E *then* $f(r)\to E$ *as* $r\to\infty$. (III) *If* E *is an isolated equilibrium point, then either* $f(r)\to E$ *as* $r\to\infty$ *or* E *is not among the limit points of* $\{f(r)\}$.

Proof. Since the sequence $\{W(f(r))\}$ is bounded and, by (11), is monotone, it must have a unique limit W_∞, say, as $r\to\infty$, and $W_\infty\geqq W(f(1))>0$. Also, $f(r)$ lies in the bounded closed region

$$R: \sum f_\alpha = 1, \qquad \min f_\alpha \geqq 0, \qquad W(\boldsymbol{f}) \geqq W(\boldsymbol{f}(1)).$$

Hence the sequence $\{\boldsymbol{f}(r)\}$ must have at least one limit point $\boldsymbol{L}$ in R: for any such $\boldsymbol{L}$ we have $W(\boldsymbol{L}) = W_\infty > 0$, since $W(\boldsymbol{f})$ is a continuous function of $\boldsymbol{f}$. Since $W(\boldsymbol{L}) > 0$ we have $\phi(\boldsymbol{f})$ continuous at $\boldsymbol{L}$. Hence $\{\phi(\boldsymbol{f}(r))\}$ must have $\phi(\boldsymbol{L})$ as a limit point. But this sequence is $\{\boldsymbol{f}(r+1)\}$, and so $\phi(\boldsymbol{L})$ is a limit point of $\{\boldsymbol{f}(r)\}$. Hence, as before, $W(\phi(\boldsymbol{L})) = W_\infty$. Thus $W(\phi(\boldsymbol{L})) = W(\boldsymbol{L})$; whence, by Theorem 2, $\boldsymbol{L}$ is an equilibrium point. We have now proved (I), and also that $W(\boldsymbol{L})$ has the same value for every limit point $\boldsymbol{L}$ of $\{\boldsymbol{f}(r)\}$.

For Part II, since $W(\boldsymbol{E})$ must clearly be positive, we can surround $\boldsymbol{E}$ by two regions S_1, S_2 consisting of the points $\boldsymbol{f}$ in R at which the distance d of $\boldsymbol{f}$ from $\boldsymbol{E}$ satisfies the respective inequalities $d \leqq \rho_1$, $d < \rho_2$, where ρ_1 and ρ_2 are such that $\rho_1 > \rho_2 > 0$ but otherwise are at our disposal. Now $W(\boldsymbol{f})$ has a strict maximum at $\boldsymbol{E}$, and so, by Theorem 1, Corollary, $\boldsymbol{E}$ is an isolated equilibrium point. Hence we can choose ρ_1 to be such that

(i) if $\boldsymbol{f}$ is in S_1 but not at $\boldsymbol{E}$, then $W(\boldsymbol{f}) < W(\boldsymbol{E})$;

(ii) S_1 contains no equilibrium point other than $\boldsymbol{E}$.

Further, since $W(\boldsymbol{E}) > 0$ the function $\phi(\boldsymbol{f})$ is continuous at $\boldsymbol{E}$: hence we can now choose ρ_2 to be such that

(iii) if $\boldsymbol{f}$ is in S_2 then $\phi(\boldsymbol{f})$ is in S_1.

Now $S_1 - S_2$ is a bounded closed region and does not contain $\boldsymbol{E}$: hence the greatest value m taken by $W(\boldsymbol{f})$ in $S_1 - S_2$ must be such that $m < W(\boldsymbol{E})$.

Suppose now that $\boldsymbol{f}(i)$ is in S_2 and near enough to $\boldsymbol{E}$ to make $W(\boldsymbol{f}(i)) > m$. Then, by (11), $W(\boldsymbol{f}(r)) > m$ for $r = i+1, i+2, \cdots$, and so $\boldsymbol{f}(r)$ can never lie in $S_1 - S_2$ if $r \geqq i$. However, $\boldsymbol{f}(i)$ is in S_1 and, by (iii), as r runs through $i+1, i+2, \cdots$ the first point $\boldsymbol{f}(r)$ (if any) to fall outside S_1 would be preceded by a point in $S_1 - S_2$. But this cannot occur: hence $\boldsymbol{f}(r)$ cannot escape from S_1. Since S_1 is bounded and closed, every limit point of $\{\boldsymbol{f}(r)\}$ is thus in S_1. By Part I and condition (ii) it follows that $\boldsymbol{E}$ is the sole limit point. Altogether, if $\boldsymbol{f}(i)$ is near enough to $\boldsymbol{E}$, then $\boldsymbol{f}(r) \to \boldsymbol{E}$ as $r \to \infty$, as required.

For Part III let us suppose that $\boldsymbol{E}$ is a limit point of $\{\boldsymbol{f}(r)\}$: we can then show that $\boldsymbol{E}$ is the unique limit of $\boldsymbol{f}(r)$ by modifying the argument used for Part II. We have to drop the requirement (i), and consequently cannot claim that $\boldsymbol{f}(r)$ is never in $S_1 - S_2$ for $r \geqq i$. However, we observe instead that it can be in $S_1 - S_2$ at most finitely often, since $\{\boldsymbol{f}(r)\}$ has no limit point in $S_1 - S_2$. Since $\boldsymbol{E}$ is a limit point of $\{\boldsymbol{f}(r)\}$ there are infinitely many points $\boldsymbol{f}(i)$ in S_2 and (thus also in S_1), and, as before, for every departure of $\boldsymbol{f}(r)$ from S_1 there is a point of the sequence in $S_1 - S_2$. Hence there can only be finitely many departures, and $\boldsymbol{f}(r)$ must be confined to S_1 when r is large enough. It follows as before that $\boldsymbol{f}(r) \to \boldsymbol{E}$ as $r \to \infty$.

6. We remark that better results than those in Theorem 3 can be obtained under various restrictions. Firstly, when there is a nondegenerate equilibrium $\boldsymbol{E}$ at which $W(\boldsymbol{f})$ has a strict maximum it is not difficult to show that this is the

only local maximum point: for any section of the "W-surface" by a plane passing through $\boldsymbol{E}$ will be parabolic with the vertex of the parabola at $\boldsymbol{E}$. Thus $\boldsymbol{E}$ must be the only stable equilibrium. In other cases there can be more than one stable equilibrium, necessarily degenerate: a simple case is

$$W(f_1, f_2, f_3) = [(f_1 - 2f_2)^2 + 3(f_1 + f_2 + f_3)^2]/10 = [3 + (f_1 - 2f_2)^2]/10,$$

with local maxima at (1, 0, 0) and (0, 1, 0).

Secondly, by Part III of Theorem 3, $\{\boldsymbol{f}(r)\}$ *must converge to a unique limit in all cases in which there are only isolated equilibria* (*or, equivalently, in which there is only a finite number of equilibria*). From the corollary to Theorem 1 and from the conditions for an equilibrium point [see (9) and (10) above] it is clear that every equilibrium point will be isolated unless one or more of certain determinants, formed from the numbers $w_{\alpha\beta}$, happen to vanish. In this sense any cases in which $\{\boldsymbol{f}(r)\}$ might not converge are exceptional, and we should be unlucky if we encountered such a case in a real situation.

Further, although in our deterministic model convergence to a "saddle point" equilibrium $\boldsymbol{E}$ is possible, yet such an equilibrium is unstable, and in practice random fluctuations would almost certainly lead the system to some point $\boldsymbol{f}'$, say, with $W(\boldsymbol{f}')$ appreciably greater than $W(\boldsymbol{E})$, and after $\boldsymbol{f}'$, since $W(\boldsymbol{f})$ must increase, $\boldsymbol{f}$ would be kept away from the neighborhood of $\boldsymbol{E}$. In view of all this Sewall Wright's picture of the process [see Section 5] is "practically" correct.

7. It only remains to prove the inequality (1). If we write $M_n = \boldsymbol{v}^T\boldsymbol{u}^n\boldsymbol{v}$, and exclude the trivial case $\boldsymbol{v}=0$, the inequality can be written

$$\psi(v) = M_n/M_0 - (M_1/M_0)^n > 0 \tag{12}$$

under the conditions (α) all $v_i \geqq 0$, (β) all $u_{ij} = u_{ji} \geqq 0$, (γ) $\boldsymbol{v}$ is not a latent vector of $\boldsymbol{u}$. We know that there is a real orthogonal matrix, $\boldsymbol{R}$, say, which reduces $\boldsymbol{u}$ to principal axes, *i.e.*, $\boldsymbol{R}^T\boldsymbol{R} = \boldsymbol{I}$, $\boldsymbol{R}^T\boldsymbol{u}\boldsymbol{R} = \boldsymbol{t}$ where $t_{ij} = 0$ when $i \neq j$. Let $\boldsymbol{a} = \boldsymbol{R}^T\boldsymbol{v}$; then

$$M_n = \boldsymbol{v}^T\boldsymbol{u}^n\boldsymbol{v} = \boldsymbol{a}^T\boldsymbol{t}^n\boldsymbol{a} = \sum t_{pp}^n a_p^2, \tag{13}$$

where by condition (γ) $\boldsymbol{a}$ is not a latent vector of $\boldsymbol{t}$, *i.e.*, there exist i, j, such that

$$a_i \neq 0, \quad a_j \neq 0, \quad t_{ii} \neq t_{jj}. \tag{14}$$

Now M_n is the nth moment of a distribution of masses a_i^2 at respective distances t_{ii} from the origin. When all $t_{ii} \geqq 0$, (12) is a standard result, true even if (α) and/or (β) do not hold. For on allowing n to take all real values, a straightforward differentiation and use of Cauchy's inequality show that $(d/dn)^2 \ln M_n > 0$, *i.e.* the graph of $\ln M_n$ as a function of n is concave upwards. Hence of two adjacent chords of the graph the one on the right must have the greater slope; if $h < m < n$,

$$\frac{\ln M_n - \ln M_m}{n - m} > \frac{\ln M_m - \ln M_h}{m - h},$$

or

(15) $$M_n^{m-h} M_h^{n-m} > M_m^{n-h},$$

and (12) is the particular case $m=1$, $h=0$. (*A fortiori* (15) is true if n and h are even, even if not all $t_{ii} \geqq 0$.) However, if some t_{ii} are negative and n is odd (15) need not necessarily hold, even if (α) and (β) are satisfied. For example, take $\mathbf{v} = \left[\begin{smallmatrix}2\\1\end{smallmatrix}\right]$, $\mathbf{u} = \left[\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right]$; then $M_0 = M_2 = 5$, $M_1 = M_3 = 4$, and $M_3M_1 < M_2^2$ and $M_3^2M_0 < M_2^3$.

We now establish the truth of (12) under conditions (α), (β) and (γ) by *reductio ad absurdum*. Suppose it untrue; choose $\mathbf{u}$, $\mathbf{v}$ of the smallest possible dimensions k to satisfy (α), (β) and (γ) but not (12). Clearly $k > 1$; and we can suppose all $v_i > 0$, for if any $v_i = 0$, then on suppressing the ith row and column of $\mathbf{u}$ we have *a fortiori* a contradiction of (12) in a smaller number of dimensions. Thus $\psi(\mathbf{v})$ depends only on the direction of the vector $\mathbf{v}$, is a continuous function of $\mathbf{v}$, is strictly positive on the boundary of the set of $\mathbf{v}$'s that satisfy (α), where some $v_i = 0$, and is negative or zero at some point $\mathbf{v}^*$ that lies inside this set and also satisfies (γ). Thus among points satisfying (α) there is at least one point $\mathbf{V}$ at which the function $\psi(\mathbf{v})$ takes its minimum value $m \leqq 0$ and is stationary. If $m < 0$ such a point $\mathbf{V}$ must satisfy (γ), since it is easy to see that $\psi(\mathbf{v}) = 0$ if $\mathbf{v}$ is a latent vector of $\mathbf{u}$. If $m = 0$ we may clearly take $\mathbf{V}$ to be $\mathbf{v}^*$, and again $\mathbf{V}$ will satisfy (γ). Write $\psi(\mathbf{v}) = \omega(\mathbf{a})$, where $\mathbf{a} = \mathbf{R}^T\mathbf{v}$, and write $\mathbf{A} = \mathbf{R}^T\mathbf{V}$: then at $\mathbf{a} = \mathbf{A}$ the condition (14) is satisfied and $\omega(\mathbf{a})$ is stationary, so that $\partial\omega(\mathbf{a})/\partial a_i = 0$ at that point. But, by (13), $\partial M_n/\partial a_i = 2a_i t_{ii}^n$, so that for each i either $A_i = 0$ or $M_0 t_{ii}^n - M_n - n(M_1/M_0)^{n-1}(M_0 t_{ii} - M_1) = 0$ at $\mathbf{a} = \mathbf{A}$. In the latter case t_{ii} is a root of the equation

(16) $$t^n - nt(M_1/M_0)^{n-1} + (n-1)(M_1/M_0)^n - \psi(\mathbf{V}) = 0.$$

The left-hand side of (16), considered as a function of t only, has a unique minimum for nonnegative t when $t = M_1/M_0$ and its value is then $-\psi(\mathbf{V}) \geqq 0$: thus the only possible nonnegative root of (16) is $t = M_1/M_0$, and so $t_{ii} \leqq M_1/M_0$ whenever $A_i \neq 0$. But $\mathbf{A}^T\mathbf{t}\mathbf{A}/\mathbf{A}^T\mathbf{A} = M_1/M_0$, and so $\sum A_\rho^2 t_{\rho\rho} = (M_1/M_0)\sum A_\rho^2$, which can accordingly only be satisfied if $t_{ii} = M_1/M_0$ whenever $A_i \neq 0$. This contradicts (14) for $\mathbf{a} = \mathbf{A}$ and so the theorem is established.

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SEPARABILITY CONDITIONS FOR SOME SELF-ADJOINT PARTIAL DIFFERENTIAL EQUATIONS

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1. Introduction. Let L_{xy} be a linear partial differential operator with independent variables x and y. Denote by T a real transformation $x=x(u, v)$, $y=y(u, v)$ with finite, nonvanishing Jacobian. Let the transformed operator be given by $TL_{xy}=L_{uv}$. If L_{uv} can be written in the form

$$F(u, v)(L_u + L_v), \tag{1}$$

where L_u and L_v involve u alone and v alone, respectively, the operator L_{xy} and the equation $L_{xy}[\psi]=0$, are said to be separable in the coordinates u, v. We shall call u, v a separable system of coordinates if L_{xy} is separable in u, v.

Separability conditions and separable systems of coordinates for Laplace's equation, the Helmholtz equation and the Schrödinger equation are well known. (See, for example, [1] to [4].) While these constitute the most important cases from the point of view of applications, it is of interest that results can be obtained for more general types using only elementary methods. In this paper we shall discuss separability of the equations

$$\text{(2a)} \qquad L[\psi] = 0, \qquad\qquad \text{(2b)} \qquad L[\psi] + k\psi = 0,$$

where

$$L[\psi] = (A\psi_x)_x + (B\psi_y)_x + (B\psi_x)_y + (C\psi_y)_y. \tag{3}$$

Here A, B and C are functions of x and y, k is a constant, and subscripts refer to differentiation. We shall determine necessary and sufficient conditions that (2a) and (2b) be separable in coordinates u, v and for the special case $A=C$, $B=0$ we shall determine the separable systems of coordinates.

We assume throughout that all functions are twice differentiable in a region R of the xy-plane and the corresponding region S of the uv-plane where S is defined by the transformation $T\colon x=x(u, v)$, $y=y(u, v)$. We further assume that (2a) and (2b) are elliptic, *i.e.*, $AC-B^2>0$ in R. The equations remain elliptic under the transformation T but not necessarily self-adjoint. However it is easy to show that the transformed equation becomes self-adjoint when multiplied by the Jacobian $J=\partial(x, y)/\partial(u, v)$.

2. Separability of (2a). We define

$$\begin{aligned} N[\psi] &= A\psi_x^2 + 2B\psi_x\psi_y + C\psi_y^2, \\ M[\psi, \phi] &= A\psi_x\phi_x + B(\psi_x\phi_y + \psi_y\phi_x) + C\psi_y\phi_y. \end{aligned} \tag{4}$$

In order that (2a) be separable in u, v the transformation T must be such that

$$\begin{aligned} N[u] = F(x, y)U_1, \qquad M[u, v] &= 0, \qquad N[v] = F(x, y)V_1, \\ L[u] = F(x, y)U_2, \qquad L[v] &= F(x, y)V_2, \end{aligned} \tag{5}$$

where F is an arbitrary function of x, y and where here and in what follows, U_i are functions of u alone, V_i are functions of v alone, $i=1, 2, 3, 4$. We assume that U_i, V_i are different from zero for all i.

The transformation T when applied to (2a) gives, after multiplying by J,

$$(A'\psi_u)_u + (B'\psi_v)_u + (B'\psi_u)_v + (C'\psi_v)_v = 0, \tag{6}$$

where $A'=JN[u]$, $B'=JM[u, v]$, $C'=JN[v]$. Hence conditions for separability are also

$$JN[u] = U_3V_3, \qquad M[u, v] = 0, \qquad JN[v] = U_4V_4. \tag{7}$$

This latter set of conditions is equivalent to (5) in the sense that (7) implies (5). This equivalence can be established without difficulty by making use of the self-adjoint property of (6), namely

$$\frac{\partial}{\partial u}(JN[u]) = JL[u], \qquad \frac{\partial}{\partial v}(JN[v]) = JL[v].$$

Let $\bar{u}$, $\bar{v}$ be defined as follows: $\bar{u}=\int U_1^{-1/2}du$, $\bar{v}=\int V_1^{-1/2}dv$. We now prove the following theorem:

THEOREM 1. *The necessary and sufficient conditions that* (2a) *be separable in* u, v *are* 1) $\bar{u}$ *and* $\bar{v}$ *satisfy the Beltrami equations*

$$A\bar{u}_x + B\bar{u}_y = W\bar{v}_y, \qquad B\bar{u}_x + C\bar{u}_y = -W\bar{v}_x \tag{8}$$

with $W=(AC-B^2)^{1/2}$, *and* 2) *functions* $U(u)$, $V(v)$ *exist such that*

$$W = U(u)V(v). \tag{9}$$

We first prove the conditions are necessary. From (5) we have $N[\bar{u}]=N[\bar{v}]=F$, $M[\bar{u}, \bar{v}]=0$, and from these the Beltrami equations follow at once. Then from the Beltrami equations we get

$$N[\bar{u}] = W(\bar{u}_x\bar{v}_y - \bar{u}_y\bar{v}_x) = HW(U_1V_1)^{-1/2}$$

with $H=1/J$. Combining this result with equations (7) gives $JN[\bar{u}]=U_3V_3U_1^{-1} = W(U_1V_1)^{-1/2}$ and similarly $JN[\bar{v}]=U_4V_4V_1^{-1}=W(U_1V_1)^{-1/2}$. Hence, (9) follows immediately.

To show that the conditions are sufficient, we note first that the equations $N[\bar{u}]=N[\bar{v}]=F$, $M[\bar{u}, \bar{v}]=0$, follow from the Beltrami equations. Moreover, as above, we have $N[\bar{u}]=HW(U_1V_1)^{-1/2}=HU(u)V(v)(U_1V_1)^{-1/2}$ and a similar equation for $N[\bar{v}]$. These equations are obviously equivalent to (7) which completes the proof of the theorem.

3. Separability of (2b). If we transform (2b) by means of T, multiply by J and make use of (6) and (7), we get

$$\frac{1}{U_4}(U_3\psi_u)_u + \frac{1}{V_3}(V_4\psi_v)_v + \frac{k\psi}{N[u]} = 0. \tag{10}$$

Thus we have the theorem:

THEOREM 2. *The necessary and sufficient conditions that* (2b) *be separable in* u, v *are that* $\bar{u}$, $\bar{v}$ *satisfy* (8), U, V *exist which satisfy* (9) *and* $N[\bar{u}]$ *is such that*

$$1/N[\bar{u}] = f(u) + g(u). \tag{11}$$

We omit the proof which follows readily from Theorem 1 and proceed to discuss the special case $A = C$, $B = 0$. This is a special case that occurs frequently in physical applications, *e.g.*, in the study of vibrations of nonhomogeneous membranes or flow of heat in substances whose thermal conductivity is a function of position.

4. Differential equations for f, g, $F = \log U$, $G = \log V$. With $A = C$, $B = 0$ the separability conditions of Theorems 1 and 2 become

$$u_x = v_y, \quad u_y = -v_x, \quad A(x, y) = UV, \quad J = UV(f(u) + g(v)), \tag{12}$$

where we have omitted the bars from u and v. Thus we shall be concerned with the transformations $T: x = x(\bar{u}, \bar{v})$, $y = y(\bar{u}, \bar{v})$, and this involves no loss in generality.

Let $z = x + iy$ be an analytic function of $w = u + iv$, and write $z = \phi(w)$. Two transformations T_1, T_2 are then defined by $z_1 = \phi_1(w)$, $z_2 = \phi_2(w)$. We shall say that two such transformations are of the same type if ϕ_1 and ϕ_2 are related in the following manner:

1) $z_1 = az_2 + b$, $a \neq 0$,
2) $w_1 = \alpha w_2 + \beta$, $\alpha \neq 0$, α real or purely imaginary.

We have $J = x_u^2 + x_v^2 = |\phi'(w)|^2$ and

$$2 \log |\phi'(w)| = F(u) + G(v) + \log (f(u) + g(v))$$

with $F(u) = \log U(u)$, $G(v) = \log V(v)$. Since $\log |\phi'(w)|$ is a harmonic function we find that

$$(F'' + G'')(f + g)^2 + (f + g)(f'' + g'') - (f'^2 + g'^2) = 0, \tag{13}$$

where here and in what follows, primes indicate differentiation. We omit the details, which amount to straightforward differentiation and integration, and find from (13) that

$$F'' = m^2 f^2 - \frac{nf}{2} + n_1, \qquad G'' = -m^2 g^2 - \frac{ng}{2} - n_1, \tag{14}$$

$$\begin{aligned} f'^2 + m^2 f^4 - nf^3 + p^2 f^2 - qf + r &= 0, \\ g'^2 - m^2 g^4 - ng^3 - p^2 g^2 - qg - r &= 0, \end{aligned} \tag{15}$$

where m, n, n_1, p, q, r are real constants. These equations determine, to within arbitrary constants, the F, G, f and g. We note that putting $f = -g$, $du = idv$, changes the first of (15) into the second and vice versa. Hence we need consider only one of (15) and we choose the one involving g.

It is convenient to define $h(g)=m^2g^4+ng^3+p^2g^2+qg+r$ and write the second of (15) as

$$g'^2 - h(g) = 0. \tag{16}$$

5. Classification of solutions of (14) **and** (15). We shall discuss only those solutions which are real in both f and g. Constants of integration will be designated by a_i, b_i, c_i, u_i, v_i, f_i, g_i with $i=1, 2, \cdots$, and are to be real. The function $n_1(u^2-v^2)/2+a_1u+b_1v+c_1$ will be denoted by $R(u, v)$. We assume $n_1\neq 0$; the extension to $n_1=0$ is obvious. We shall not discuss all possible cases that arise from the solutions of (14) and (15) but only those which seem of most interest and simply state the results for the others.

Case I. $m=n=0$, $\mu^2=-n_1/4$. Define complex constants γ and δ by $\gamma=(a_1-ib_1)/2$, $\delta=$any complex constant whose real part is $c_1/2$. From (14) we find

$$F(u) = -2\mu^2u^2 + a_1u + a_2, \quad G(v) = 2\mu^2v^2 + b_1v + b_2,\ a_2 + b_2 = c_1.$$

(a) $p=q=r=0$. Here $2 \log |\phi'(w)| = R(u, v)+\log (f_1+g_1)$ and

$$\phi(w) = (f_1 + g_1)\int^w \exp(-\mu^2\zeta^2 + \gamma\zeta + \delta)d\zeta.$$

Hence we say that all transformations here are of the type

$$\phi(w) = \int^w \exp(-\zeta^2)d\zeta.$$

(b) $p=0$, $q\neq 0$. In this case we find

$$\phi(w) = C\int^w (\zeta + \zeta_1)\exp(-\mu^2\zeta^2 + \gamma\zeta + \delta)d\zeta,$$

where C and ζ_1 are complex constants. The transformations are of the type

$$\phi(w) = \int^w (\zeta + \zeta_2)\exp(-\zeta^2)s\zeta.$$

(c) $p\neq 0$, $h=p^2(g-g_1)^2$. Transformations are of type (a).

(d) $p\neq 0$, $h=p^2(g-g_1)(g-g_2)$, g_1 and g_2 real and distinct. All transformations here can be shown to be of the type

$$\phi(w) = \int_0^w \sin(\sigma_1\zeta + \zeta_1)\exp(-\zeta^2)d\zeta,$$

where σ_1 and ζ_1 are complex constants.

It is of interest to notice that the cases where F and G are both constant,

which implies $n_1=0$, are the cases in which the separable coordinate systems are the four cylindrical coordinate systems found by Eisenhart [2]. These are the only separable systems for $A(x, y)=\text{const.}$

Case II. $m=0$, $n\neq 0$.

(a) $h=n(g-g_1)^3$. Transformations are of type I(b).

(b) $h=n(g-g_1)^2(g-g_2)$. Transformations are of type I(d).

(c) $h=n(g-g_1)(g-g_2)(g-g_3)$, g_1, g_2, g_3 real and $g_1>g_2>g_3$. The solution here can be found in terms of elliptic functions.* We have

$$g = \{g_1 - g_2 \operatorname{sn}^2 [M(v + v_1), k_1]\} \operatorname{cn}^{-2} [M(v + v_1), k_1],$$

where $k_1^2=(g_2-g_3)/(g_1-g_3)$ and $M=\frac{1}{2}n(g_1-g_3)$, and

$$f = (g_1 - g_2) \operatorname{sn}^2 [M(u + u_1), k_2] - g_1$$

with $k_2^2=1-k_1^2$. In the remainder the modulus k_1 is to be associated with v and k_2 with u. Explicit reference to these moduli will be omitted. Also for brevity we write v for $v+v_1$ and u for $u+u_1$. Thus we have

$$f + g = (g_1 - g_2) \mid \operatorname{sn} Mw \mid^2 (1 - \operatorname{sn}^2 Mv \ \operatorname{dn}^2 Mu)/\operatorname{cn}^2 Mv,$$

where $w=u+iv$ and the modulus of sn Mw is k_2.

The functions F and G can be expressed in terms of the theta functions, thus:

$$F = \mu^2 u^2 + a_1 u + a_2 + 2 \log \theta(Mu, k_2),$$
$$G = - \mu^2 v^2 + b_1 v + b_2 + 2 \log \theta(iMv, k_2),$$

where $\mu^2=n_1/2+(n/4)[g_3+(g_1-g_3)E/K]$, and K and E are complete elliptic integrals.

Finally we find that

$$\begin{aligned} 2 \log \mid \phi'(w) \mid &= \mu^2(u^2 - v^2) + a_1 u + b_1 v + c_1 \\ &\quad + \log [(g_1 - g_2) \cdot \mid \operatorname{sn} (Mw) \cdot \Theta(Mw) \mid^2 \Theta(0)^2], \end{aligned}$$

and hence the transformations for this case are of the type

$$\phi(w) = \int^w \Theta(\sigma_1 \zeta + \zeta_1) \operatorname{sn} (\sigma_1 \zeta + \zeta_1) \exp (-\zeta^2) d\zeta.$$

(d) $h=n(g-g_1)[(g-a)^2+b^2]$, a and b real constants. This case can also be expressed in terms of the elliptic functions. We omit the details and give only the result that the transformations are of the type

$$\phi(w) = \int^w \{\Theta(\sigma_1 \zeta + \zeta_1)[1 - \operatorname{cn} (\sigma_1 \zeta + \zeta_1)]\}^{1/2} \exp (-\zeta^2) d\zeta.$$

* The necessary details concerning elliptic functions can be found in [5].

Case III. $m\neq 0$.

(a) $h=m^2(g-g_1)^4$. Transformations are of type I(a).

(b) $h=m^2(g-g_1)^3(g-g_2)$. Transformations are of type I(b).

(c) $h=m^2(g-g_1)^2(g-g_2)^2$. Transformations are of type I(a).

(d) $h=m^2(g-g_1)^2[(g-a)^2+b^2]$. Transformations are of type I(a).

(e) $h=m^2(g-g_1)(g-g_2)(g-g_3)^2$. Transformations are of type I(d).

(f) $h=m^2(g-g_1)(g-g_2)(g-g_3)(g-g_4)$. This case can be shown to yield transformations that are of type II(c).

(g) $h=m^2(g-g_1)(g-g_2)[(g-a)^2+b^2]$. This case can be shown to yield transformations that are of type II(d).

The results of this section can be summarized as follows: The transformations which separate (2b) with $A=C$, $B=0$ and which are defined by the real and imaginary parts of $z=\phi(w)$ with $z=x+iy$ and $w=u+iv$, are of the type

$$\phi(w) = \int^{w} \chi(\zeta) \exp(-\zeta^2) d\zeta, \tag{17}$$

where $\chi(\zeta)$ assumes one of the values, 1) 1, 2) $\zeta+\zeta_1$, 3) $\sin(\sigma_1\zeta+\zeta_1)$, 4) $\Theta(\sigma_1\zeta+\zeta_1)$ $\operatorname{sn}(\sigma_1\zeta+\zeta_1)$, 5) $\{\Theta(\sigma_1\zeta+\zeta_1)[1+\operatorname{cn}(\sigma_1\zeta+\zeta_1)]\}^{1/2}$.

The coefficient $A=A(x, y)=U(u(x, y), v(x, y))$ are those for which separable transformations exist. Although all transformations which are separable are given by (17) it is obvious that an explicit expression for A cannot be written in most cases.

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AN OPTIMAL SEARCH PROCEDURE*

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The problem which we are concerned with is the following. We are given a set of objects, one of which has been previously selected without our knowledge, as the one which we must determine. In addition, we have assigned to each object a probability of having been selected, the sum of all such probabilities being unity. We must determine the selected object by a series of yes-no questions, arbitrarily dividing the set of objects into two groups and asking which it is contained in. We continue this yes-no process until the selected object is known. For example, we might use the yes-no process to determine which letter of the alphabet was randomly selected from a book, given the probabilities (the frequencies) of each letter. We would now like to know what the best strategy of division is, where by a strategy we mean a plan which indicates exactly what the next division will be, given that the selected object is contained in any particular group of the original objects. For example, given four objects with assigned probabilities (.4), (.3), (.2), (.1), our strategy might be, first divide in the following way: (.4, .1), (.3, .2). Then, if the object is learned to be in the left-hand group, further divide, (.4), (.1), and if it is in the right-hand group, divide, (.3), (.2). The expected value, that is the probable number of divisions for this strategy, would be two.

The questions which will now be answered are: Given any set of objects, what strategy should be used to obtain the smallest expected value (the optimal strategy)? What is the smallest expected value?

It is important to observe that although this seems like a method of repeated division, it can be viewed rather as one of combination. We could start with the entire set as in the above strategy, and describe where the first division, and subsequent divisions dependent upon the outcome of the preceding ones, are going to be made. Instead, we can proceed in precisely the reverse manner, beginning with the several isolated objects and indicating a series of repeated combinations, the first few of which correspond to the last few divisions, and the final one to the first division. Since in our strategy divisions are always into two groups, each of the combinations will similarly involve two groups.

In the above example then, the method of combination corresponding to the strategy used would be one of the following two. First, combine the two objects with the probabilities (.4) and (.1); next combine those with probabilities (.3) and (.2). Finally combine the two groups corresponding to (.4, .1) and (.3, .2). The second method of combination would simply contain a reversal of the first two steps of the preceding one. It is seen from this example that a particular strategy need not have a unique method of combination. However, it should be fairly evident that any method of combination will always have only one division strategy corresponding to it. Therefore, in order to determine

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the optimal strategy of division of a set of objects, it is only necessary to determine a corresponding method of combination.

A concept of importance is that of the n-set. If we are given a set of n objects, then we call the array of the corresponding probabilities, arranged from left to right in descending order of magnitude, an n-set. This left to right arrangement has no significance except that it will be convenient later on. When we combine two of the objects of the n-set, and assign the sum of the two probabilities as the probability of the new "object", we have essentially reduced the n-set to an $(n-1)$-set. Any method of combination is then a way of making a combination of two objects $n-1$ times until the 1-set is reached.

Another concept which we will use is that of the partial sum. At any stage of combination, the sum of probabilities of all the pairs of combined objects we call the partial sum. The probability of one of the original objects can be included more than once in a partial sum, since after it has been combined to form a new object, this latter object may again be combined.

We can now state the main theorem, although it will not be proved until later, following a series of lemmas.

THEOREM. *The optimal rule of combination is: At each stage combine the objects with the two smallest probabilities.*

LEMMA 1. *The method of combination of any n-set is, after the first step, precisely the same as that of some $(n-1)$-set.*

Proof. In the method of combination of the n-set, there must be a first combination. Therefore, the $(n-1)$-set equivalent to the n-set, except for the replacement of these two objects by the combined one, has, from then on, the same method of combination as the n-set. We have, in fact, referred to the n-set as being reduced to the $(n-1)$-set by the combination of the two objects. It follows that the expected value of the n-set should be that of the $(n-1)$-set plus the probability that we will have to ask another question if the object is contained in the two combined ones. Since the latter is exactly the sum of the two original probabilities, we can write the equations

$$E_r = E_{r-1} + (p_m + p_n), \qquad E_1 = 0,$$

where E_r is the expected value of a given r-set, and p_m and p_n are the probabilities of the combined objects. If we proceed by induction, the result is that E_n is the sum of $n-1$ combinations of probabilities and the problem is really how to combine probabilities two at a time so that the total of these probabilities is a minimum. Each original probability is going to be combined one or more times.

LEMMA 2. *In an optimal method of combination, a larger probability cannot be combined more times than a smaller one.*

Proof. If we had a method whereby a larger probability were combined more

often than a smaller one, we could interchange the two probabilities in our operations and get a smaller expected value.

LEMMA 3. *We are given the optimal value of an n-set, and any two probabilities p_r and p_s where $p_r > p_s$. If we increase p_r by any amount k, and decrease p_s by the same amount, the optimal value of the transformed set cannot be greater than that of the original.*

Proof. By Lemma 2, p_r cannot be combined a greater number of times than p_s and so if we use the same method of combination for the transformed set we can add plus k no more than minus k. It is impossible to get a greater expected value in this way. In addition, this may not even be the optimal method for the transformed set.

We are now in a position to give an inductive proof of the main theorem.

Proof of the main theorem. The rule obviously holds for all the 1-sets, of which there is only one. We shall now assume that it holds for every r-set, where r assumes the values $1, \cdots, n-1$. If it can be shown that the rule is true for any n-set, then the theorem is proved, by induction.

We are given an arbitrary n-set, arranged in descending order of size, $p_n, \cdots, p_1$. There are three possible first combinations:

(1) A combination of objects r and s, that is, an addition of p_r and p_s, where neither r nor s is equal to 1 or 2.

(2) A combination of r and s where one of r and s is equal to either 1 or 2.

(3) A combination of 1 and 2.

It must be shown that the third method is at least as good as the other two.

(1) Combining r and s gives a partial sum of (p_r+p_s) and the $(n-1)$-set $p_n, \cdots, (p_r+p_s), \cdots, p_1$. The next combination must be 1 and 2 according to the inductive assumption, giving a partial sum of $(p_r+p_s+p_1+p_2)$ and the $(n-2)$-set $p_n, \cdots, (p_r+p_s), \cdots, (p_1+p_2), \cdots, p_3$.

If, instead we first combine 1 and 2, then we can next combine r and s, which may not even be optimal. We arrive at the same partial sum and the same $(n-2)$-set. This method is at least as good as combining r and s in the n-set.

(2a) If 1 and r $(r>3)$ are combined, our partial sum is (p_1+p_r) and we have the $(n-1)$-set $p_n, \cdots, (p_1+p_r), \cdots, p_2$. The next combination must be (p_2+p_3); this gives a partial sum of $(p_1+p_2+p_3+p_r)$ and an $(n-2)$-set $p_n, \cdots, (p_1+p_r), \cdots, (p_2+p_3), \cdots, p_i$, where $p_i=p_4$ if $r\neq 4$.

If, instead, 1 and 2 are combined, we have a partial sum of (p_1+p_2) and an $(n-1)$-set $p_n, \cdots, (p_1+p_2), \cdots, p_3$. The next combination can be 3 and r, which may not even be optimal. This gives the same total partial sum as before and an $(n-2)$-set, $p_n, \cdots, (p_3+p_r), \cdots, (p_1+p_2), \cdots, p_i$. By Lemma 3, the

new $(n-2)$-set has an expected value no greater than the original, and since the partial sums are equal, this method must be at least as good as the former combination.

(2b) If 1 and r are combined, where r is 3, then one of two situations can arise. If $(p_1+p_3)>p_4$ the proof is analogous to (2a). If however, $(p_1+p_3)<p_4$, then, after we have combined 1 and 3 we would combine 2 and the combination of 1 and 3. In the second method, after we had combined 1 and 2, we would combine 3 with the combination of 1 and 2. The resulting $(n-2)$-sets would be the same, and the partial sum of the 1 and 2 combination would be lower than that of the 1 and 3 combination. Therefore the latter would be at least as good as the former method.

In the proof of (2) it is possible to interchange the values for p_1 and p_2 so as to prove the theorem for the combination of 2 and r. All possible first combinations in the n-set have been treated and hence the theorem is proved.

A rather interesting situation is that of nonuniqueness. If, at any stage of the method, there exist two or more probabilities which are equal, we can arrange them in any order we like, as long as they are adjacent. Further, if these probabilities represent not individual objects but groups, then the order will affect the final combination procedure. The result will be two or more arrangements which are equally acceptable.

As final remarks, let me indicate the outcome of the example presented at the outset, that of the determination of a letter of the alphabet. We are given a set of probabilities corresponding to each letter and the blank space. They range in size from .001 for the letter Q to .200 for the blank. Using the rule of combination, the expected number of divisions or questions is 4.154.

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SOLUTION OF A SET OF GAMES

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1. Introduction. This paper deals with a family of zero-sum two-person games in the Von Neumann sense [1]. The rules of the games are simple, but their implications are sufficiently obscure that the methods of game-theoretic analysis are necessary to determine the solutions. In this respect, the games fall into a category which, at present, has too few representatives known; namely, those games which (unlike tic-tac-toe) are not so simple that a theoretical analysis is superfluous, and (unlike chess) not so complicated that the analysis is inadequate to provide the solution.

2. Description of the games. The games considered are known by the designation of the "n-coin game," where n is a positive integer. The particular case of the three-coin game will be used to illustrate the rules of play; the general case is completely analogous.

In the three-coin game, each player A and B is supplied initially with three identical coins or other counters.

First Move. Each player chooses, unknown to the other, an integer between 0 to 3 (inclusive) and places that number of his coins to his right hand, which is thereafter kept closed. The number of coins selected by A and B, respectively, in this move will be denoted A_1 and B_1.

Second Move. Player A attempts to guess the total number of coins now held in both player's right hands. He announces his guess, A_2, an integer between 0 and 6.

Last Move. Player B, having heard A's second move, now attempts himself to guess the total. His guess, B_2, must differ from A_2.

The hands are now opened and the correct total ascertained. If either player has guessed correctly, he wins one arbitrary unit from the other. Otherwise, the play is a draw. In practice, when a draw occurs, it is customary to repeat the game with A and B exchanging roles, until one or the other wins. However, this convention need not affect our theoretical studies, which are concerned simply with optimal strategies for A and B in a single play of the game.

3. Heuristic discussion. Each player has some benefits under the rules, and it is not immediately obvious which has the greater. Player A has the choice of seven numbers (0, 1, 2, 3, 4, 5, or 6) for his second move, while B has only six. Moreover, if A guesses the sum correctly, then B's last move is completely futile, since he is defeated already. However, B has the advantage of hearing A_2 before naming B_2, and A_2 may give B some clue as to A_1, A's choice at the first move. For example, if $A_2=6$, then B can be reasonably sure that A's right hand holds three coins. This information will be useless in the case when B is also holding 3, but in any other case, B can be assured of a win.

Note that A can cancel this advantage to B by making a guess of $A_2=3$ at his second move. This reveals nothing to B about A's first move. On the other hand, in order to utilize this method of annulling his opponent's advantage, A must sacrifice his own advantage of complete freedom of choice at Move 2.

The relative importance of each of these features of the game will be revealed in the theoretical analysis below.

4. A simple special case. Before undertaking the analysis of the general n-coin game, we shall consider some special cases. Our first example is the one-coin game. Observe first, that among the possible strategies for either player, there are some which offer no possibility of winning. For example, for A to play $A_1=1$, $A_2=0$ is manifestly absurd. It is possible to exclude such strategies by employment of the dominance principle [1], but it is hardly necessary to use

so sophisticated a technique to effect so elementary a result. In the remainder of this paper, only strategies which give the player a possibility of success ("feasible" strategies), will be considered. This does not destroy the generality of the results, since, as remarked above, all others may be excluded by dominance. Henceforth, we shall use the term strategy to denote feasible strategy, unless otherwise specified. With this convention, A has exactly four strategies:

$$A\text{–I}\ :\ A_1 = 0,\ A_2 = 0, \qquad A\text{–III}:\ A_1 = 1,\ A_2 = 1,$$

$$A\text{–II}\ :\ A_1 = 0,\ A_2 = 1, \qquad A\text{–IV}:\ A_1 = 1,\ A_2 = 2;$$

and B has only two strategies

$$B\text{–I}:\quad B_1 = 0,\ B_2 = \begin{cases} x \\ 0 \\ 1 \end{cases} \text{as}\quad A_2 = \begin{cases} 0 \\ 1, \\ 2 \end{cases}$$

$$B\text{–II}:\ B_1 = 1,\ B_2 = \begin{cases} 1 \\ 2 \\ x \end{cases} \text{as}\quad A_2 = \begin{cases} 0 \\ 1. \\ 2 \end{cases}$$

In case $B_1=0$ and $A_2=0$, B presumably has a lost game, and his second move is immaterial; a similar comment holds for the case $B_1=1$, $A_2=2$. These cases are denoted above by the (purely arbitrary) notation $B_1=x$. As a matter of practical expediency, B may be best advised to take $B_2=1$ in these two cases since this will give him an opportunity to win in the unlikely event that A is playing one of the nonfeasible strategies.

The payoff matrix for the game is given in Table 1.

TABLE 1. PAYOFF FOR THE ONE-COIN GAME (PAYOFF TO A)

	B-I	B-II
A-I	1	−1
A-II	−1	1
A-III	1	−1
A-IV	−1	1

From this table, it can be seen that basic optimal strategy mixes [3] for A include (and are limited to)

I and II in equal proportions,
I and IV in equal proportions,
II and III in equal proportions,
III and IV in equal proportions;

and B has only one optimal strategy, which is to mix B-I and B-II in equal

proportions. The value of the game is zero; no play will end in a draw, and neither player can protect himself against loss if he attempts to take advantage of an opponent's error. In several of these respects, we shall see that the one-coin game is atypical.

5. The two-coin game. When $n=2$, a game is obtained which is typical in all important respects of all higher values of n, and still has few enough possibilities that detailed examination is practical.

It will be convenient to extend our notation as follows: Let $\overline{B}_1$ be player A's guess of B's first move, B_1, *i.e.*, $\overline{B}_1=A_2-A_1$. Then A's strategies in the two-coin game are nine in number and may be tabulated as in Table 2.

TABLE 2. STRATEGIES FOR PLAYER A IN THE TWO-COIN GAME

Strategy number	$\overline{B}_1$	A_1	A_2
A-I	0	0	0
A-II	0	1	1
A-III	0	2	2
A-IV	1	0	1
A-V	1	1	2
A-VI	1	2	3
A-VII	2	0	2
A-VIII	2	1	3
A-IX	2	2	4

The strategies for Player B may be described by the following convention. Any strategy for B determines a column in the payoff matrix, provided the A-strategies are listed down the side as in Table 2. Conversely, any column of nine entries of $+1$, -1, or 0 determines a B-strategy provided certain conditions are observed. The restrictions are stated below for the two-coin game. The form these take in the general case (with parameter n) is given in brackets.

Condition 1. Let the A-strategies be classed into the following three $[n+1]$ mutually exclusive sets:

All strategies for which either $\overline{B}_1=0$, or $\overline{B}_1=1$, $\cdots$, or $\overline{B}_1=n$. Then there must be $+1$ entries in the column defining the B-strategy against all A-strategies in exactly one of these sets, and no other $+1$ entries may appear. Since any B-strategy whatsoever must have a value of B_1, one of the three $[n+1]$ estimates $\overline{B}_1$ must be correct, *i.e.*, lead to a payoff to A of $+1$. We shall denote a B-strategy in which $B_1=i$ as a strategy of Type i.

Condition 2. The remaining 6 $[2(n+1)]$ entries in the column may be filled with 0 and -1 entries, but one and only one -1 entry* may be inserted against

* If nonfeasible B-strategies are considered, this condition should be modified to read "... not more than one -1 entry ...."

any single value of A_2. Thus, strategy A-VI and A-VIII both display $A_2=3$. In the column defining the B-strategy (say of Type 0), player B must select his call B_2 corresponding to $A_2=3$. He may select it so that he will win against A-VI, *i.e.*, $B_2=2$, or against A-VIII, *i.e.*, $B_2=1$, but he cannot choose a strategy which will win against both.

Brief consideration will establish that any column conforming to conditions (1) and (2) above will define a B-strategy, and conversely. It is not difficult to determine all B-strategies in the two-coin game, of which there are exactly ten. These strategies and the payoff matrix for the game appear in Table 3. (In this table $+1$ and -1 are abbreviated to $+$ and $-$ respectively, for brevity.) The reader is urged to construct Table 3 for himself. By so doing, he will greatly facilitate his understanding of the proofs given in subsequent sections.

TABLE 3. PAYOFF MATRIX, TWO-COIN GAME (PAYOFF TO A)

A-Strategies			B-Strategies									
Strategy number	$\bar{B}_1$	A_2	B-I	B-II	B-III	B-IV	B-V	B-VI	B-VII	B-VIII	B-IX	B-X
			Type 0				Type 1		Type 2			
A-I	0	0	+	+	+	+	−	−	−	−	−	−
A-II	0	1	+	+	+	+	−	−	0	−	−	0
A-III	0	2	+	+	+	+	−	0	−	−	0	0
A-IV	1	1	−	−	−	−	+	+	−	0	0	−
A-V	1	2	−	−	0	0	+	+	0	0	−	−
A-VI	1	3	−	0	0	−	+	+	−	−	−	−
A-VII	2	2	0	0	−	−	0	−	+	+	+	+
A-VIII	2	3	0	−	−	0	−	−	+	+	+	+
A-IX	2	4	−	−	−	−	−	−	+	+	+	+

A somewhat tedious but straightforward examination of this payoff matrix suffices to establish the following results.

PROPOSITION 1. *The game is fair.* [2]

PROPOSITION 2. *Player B has several optimal mixed strategies. Every one of the ten B-strategies is active in at least one of the optimal strategy mixes available to B.*

PROPOSITION 3. *Player A has only one optimal mixed strategy; namely, A-III, A-V, and A-VII in equal proportions.*

6. The general theorems. Statements analogous to Propositions 1, 2, and 3 hold in the general case. We shall present proofs of these general theorems. Because of the awkward nature of the operations, it will prove convenient to present the proofs in the language of a specific game, and we shall use $n=3$, the three-coin game, for this purpose. However, it will be observed that the methods we employ may immediately be applied to any value of n.

THEOREM 1. *If $n > 1$, the n-coin game is fair.*

Proof for the three-coin game. Using notation like that introduced in the preceding section, we first show a mixed strategy for B in which the sum of the entries in any row is nonpositive. This implies that if the strategies so selected are played in equal proportions against any A-strategy, pure or mixed, the expected payoff to A will not exceed zero [2].

Select any nonnegative integers, $i \neq j$ $(i, j \leqq 3)$. We now choose B-strategies of Type i and j. In the column defining the Type i strategy, we insert -1 against every A-strategy which has $\overline{B}_1 = j$; similarly, the Type j strategy has -1's inserted in those positions where $\overline{B}_1 = i$. The other -1 entries are immaterial as long as Condition 2 is observed. It is obvious that such strategies satisfy the Conditions 1 and 2 of the preceding section and assure that the row sums are nonpositive. B can now play these two pure strategies with equal weight in a mixed strategy. This completes the proof that B can force an expected payoff of 0 or less.

Player A can force an expected payoff of 0 or more by playing an equal weight mix of the four stragegies:

$$\overline{B}_1 = 0,\ A_2 = 3; \qquad \overline{B}_1 = 1,\ A_2 = 3; \qquad \overline{B}_1 = 2,\ A_2 = 3; \qquad \overline{B}_1 = 3,\ A_2 = 3.$$

By Condition 1, every B-strategy must lose against exactly one of these four. Also, according to Condition 2, each B-strategy will win against one and only one of them. Therefore, any column sum is zero.* This completes the proof. The restriction $n > 1$ is, of course, unnecessary since we have already proved the one-coin game to be fair. However, the method of proof used here requires $n > 1$.

THEOREM 2. *In the n-coin game, any pure B-strategy is active in some mixed optimal strategy for B.*

Proof for the three-coin game. Let S be a given pure B-strategy. We shall produce a four-strategy mix of B-strategies including S which has all row sums nonpositive.

Strategy S is of some type, say i; rename $S = S_i$, and begin construction of the other three strategies in the mix by choosing them to be one of each of the three types among Types 0, 1, 2, and 3 obtained by omitting Type i. Name each strategy by a subscript defining its type, *i.e.*, S_0, S_1, S_2, and S_3, of which one is S and is completely defined, and the others still have -1 entries to be inserted in the columns of the payoff matrix defining them.

We shall insert these -1 entries such that each row has at least one such -1 entry. Since each row (defined by $\overline{B}_1$ and A_2) has exactly one $+1$ entry (by construction) the four strategies S_0, S_1, S_2, and S_3 may be played with equal weight, giving nonpositive payoff to A. This will prove the theorem. We will assign these -1 entries systematically. For each value of A_2, there are one or more A-

* If nonfeasible B-strategies are considered, any column sum is nonnegative.

strategies, *i.e.*, one or more rows in the matrix. Table 4 shows the number of occurrences of each A_2 value.

TABLE 4. OCCURRENCES OF VALUES OF A_2 IN FEASIBLE A-STRATEGIES IN THE THREE-COIN GAME

Values of A_2	0	1	2	3	4	5	6
Number of occurrences	1	2	3	4	3	2	1

We begin by assigning the four -1's which fall on rows where $A_2=3$. One of these already is entered, in the S_i column (since it is a feasible strategy). This lies on a row for which $\overline{B}_1=j$ (say), $j\neq i$. In column S_j insert a -1 against an A-strategy for which $A_2=3$, and $B_1\neq i, j$, say at $\overline{B}_1=k$. This is always possible since for each $\overline{B}_1$, an A-strategy exists with $A_2=3$. Next, in column S_k, insert -1 against an A-strategy for which $A_2=3$, $\overline{B}_1\neq i, j, k$, say at $\overline{B}_1=m$. Lastly, in column S_m, insert a -1 against strategy $A_2=3$, $\overline{B}_1=i$. It is obvious that this construction is always possible and complies with Conditions 1 and 2.

In similar manner, three -1 entries are made against A-strategies with $A_2=4$. One of these is already determined in column S_i. Suppose it occurs against the A-strategy with $\overline{B}_1=j'$. In column $S_{j'}$, insert a -1 against an A-strategy with $A_2=4$, $\overline{B}_1\neq i$, say at $B_1=k'$. This is always possible. In column $S_{k'}$, insert a -1 against an A-strategy with $A_2=4$. This may or may not occur where $\overline{B}_1=i$, but it, too, is always possible, since three distinct values of $\overline{B}_1$ ($\overline{B}_1=1$, 2, and 3) permit A-strategies with $A_2=4$.

Similarly, the three -1 entries for $A_2=2$, the two entries for $A_2=1$, and the two entries for $A_2=5$ can be assigned. The one entry for $A_2=0$ is inserted in any one (or more) of columns S_1, S_2, and S_3; the entry for $A_2=6$ is inserted in any one (or more) of columns S_0, S_1, and S_2. This completes the construction. It can readily be seen that the required conditions are satisfied and the theorem proved.

THEOREM 3. *If $n>1$, in the n-coin game, no optimal strategy for Player A has any active strategies except*:

$$\overline{B}_1 = 0,\ A_2 = n;\ \overline{B}_1 = 1,\ A_2 = n;\ \cdots;\ \overline{B}_1 = n,\ A_2 = n.$$

Proof for the three-coin game. These strategies have been shown to be active in one optimal strategy mix. We now prove that no other pure strategy can be active in an optimal mixed strategy for A in the three-coin game.

Let T be an A-strategy with $A_2\neq 3$. First consider the case $A_2<2$. We choose the following two-strategy mix for B: (1) S_2, a Type 2 strategy with -1 entries against T and every A-strategy for which $\overline{B}_1=3$, (2) S_3, a Type 3 strategy with -1 entries against every A-strategy for which $\overline{B}_1=2$. Other -1 entries are immaterial as long as condition 2 is satisfied. If these strategies are played with equal weights, a nonpositive expected payoff to A is assured for every pure A-strategy, and a negative payoff of $-1/2$ is obtained when strategy T is played in particular. Similarly, if $A_2>4$, a mix of a Type 0 and Type 1 strategy

suffices to provide a negative expected payoff.

It remains to consider the case $A_2=2$ or 4 (in general, $A_2=n\pm 1$). Suppose $A_2=2$. For this case, we construct a four strategy optimal mix for B in a manner similar to that employed in the proof of Theorem 2. We begin by defining four strategies of Types 0, 1, 2, and 3, denoted S_0, S_1, S_2, and S_3, respectively. S_3 will be called the key strategy. Suppose strategy T is defined by $\overline{B}_1=i$, $A_2=2$. Choose an integer $j\leqq 3, j\neq i$. In column S_j, insert a -1 against strategy T. This is possible since $\overline{B}_1(T)=i$, and $j\neq i$. Now in column S_i, find an A-strategy with $A_2=2$, $\overline{B}_1\neq i$, j. Such a strategy exists, say $\overline{B}_1=k$, since three occurrences of $A_2=2$ are shown in Table 4. Furthermore, $k\neq 3$, since the A-strategy $\overline{B}_1=3$, $A_2=2$ is not feasible. In column S_k, assign a -1 against the A-strategy with $A_2=2$, $\overline{B}_1=j$. Also, in column S_i, assign a -1 against the A-strategy with $A_2=2$, $\overline{B}_1=k$.

Now enter a -1 in column S_3 (the key strategy) against T. This is the fourth -1 entry against A-strategies having $A_2=2$, and therefore, the second such entry against T. The remaining entries in column S_3 are entered in any manner to create a feasible strategy and satisfy Condition 2. The remaining entries in columns S_0, S_1, and S_2 are filled in the manner described in the proof of Theorem 2 (except for the -1's on lines where $A_2=2$ which have already been defined). The resulting strategy mix has one $+1$ and one or more -1 on every line, and in particular on line T, it has two -1's. Hence, when these strategies are played in equal proportion against any A pure strategy, the payoff to A is nonpositive, and against T in particular, it is negative and equal to $-1/4$.

This proves the theorem for the case $A_2=2$. For $A_2=4$, the proof is similar, except that S_0 is the key strategy. This completes the proof of the theorem.

Remark. It is clear from Theorem 3 and the proof of Theorem 1 that A must play these strategies with equal weights.

7. Concluding remarks. Despite the content of Theorem 1, it seems apparent that the advantage in the game lies with Player B wherever $n>1$. He has a wide selection of optimal strategy mixes, and in view of Theorems 2 and 3, he can select them in such a way that he can penalize A for any departure from the strategies in A's single optimal mix, while simultaneously protecting himself against any loss. Player A, on the other hand, is limited to his single mixed strategy, and may not deviate from it without risking loss. Unfortunately for A, while his strategy assures a nonnegative expected payoff, it also assures a nonpositive payoff, even if B blunders. While a quantitative measure of the value of such a property of a game is not generally agreed on [3], it is difficult to imagine any interpretation which would not make this circumstance a plus value for B.

It has been pointed out to the author that in view of this, a wiley A-player may choose a nonfeasible strategy such as $A_1=3$, $A_2=2$ in order to force a drawn play. In accordance with the convention mentioned in Section 2, this player would then attain on the next play the B-role, which is preferable. The inter-

pretation of this strategem will be left to the reader.

It will be noted that in the three-coin game, one half of the plays end in draws (provided both players play perfectly). This may be seen by examination of the proof of Theorems 1 and 3. The single feasible strategy mix which A will employ will win one play out of four against any B-strategy, and lose one out of four against any feasible B-strategy, and draw in the remaining two plays. In general, the ratio of draws to total plays is readily seen to be $(n-1)/(n+1)$.

An interesting consequence of the theory is that B may play the game with only one coin for any n. In the proof of Theorem 1, a construction is given which provides an optimal mixed strategy for B comprising Type 0 and 1 strategies only. The content of Theorems 2 and 3 must be modified, of course, if B has only one coin. However, the methods employed in the proof of the theorems may readily be applied to determine the new relationships.

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MATHEMATICAL NOTES

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NOTE ON STIRLING'S FORMULA

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Stirling's asymptotic formula, namely,

$$n! \sim \sqrt{(2\pi n)}\cdot(n/e)^n, \qquad n \to \infty,$$

is usually proved by showing that

$$n! = \sqrt{(2\pi n)}\cdot(n/e)^n\cdot e^{\gamma_n}, \qquad 0 < \gamma_n < 1/(12n), \qquad n = 1, 2, \cdots. \tag{1}$$

Herbert Robbins (this MONTHLY, vol. 62, 1955, pp. 26–29) has shown by an elementary method that the estimate for γ_n in (1) can be replaced by the improved estimate

$$1/(12n+1) < \gamma_n < 1/(12n), \qquad n = 1, 2, \cdots. \tag{2}$$

In this note, we shall prove, by a simple refinement of the argument of Robbins, the stronger and more precise result

(3) $$n! = \sqrt{(2\pi n)}\cdot(n/e)^n\cdot\exp\left(\frac{1}{12n} - \frac{\theta_n}{360n^3}\right), \quad 0 < \theta_n < 1, \quad n = 1, 2, \cdots,$$

as envisaged by Euler's summation formula.

For the proof, let $a_n = n!n^{-1/2}(e/n)^n$, $l_n = \log a_n$, $n = 1, 2, \cdots$. We have

$$\frac{a_n}{a_{n+1}} = \frac{1}{e}\left(1 + \frac{1}{n}\right)^{n+1/2}, \qquad l_n - l_{n+1} = \left(n + \frac{1}{2}\right)\log\left(1 + \frac{1}{n}\right) - 1.$$

Using the familiar series

$$\log\left(1 + \frac{1}{n}\right) = \frac{2}{2n+1}\left(1 + \frac{1}{3(2n+1)^2} + \frac{1}{5(2n+1)^4} + \cdots\right),$$

we have

(*) $$l_n - l_{n+1} = \frac{1}{3(2n+1)^2} + \frac{1}{5(2n+1)^4} + \cdots.$$

First we derive, as usual, from (*):

$$0 < l_n - l_{n+1} < \frac{1}{3(2n+1)^2}\left(1 - \frac{1}{(2n+1)^2}\right)^{-1} = \frac{1}{12n(n+1)}.$$

On rewriting this in the form $0 < l_n - l_{n+1} < (1/12)\{(1/n) - (1/(n+1))\}$, we get on the one hand

(4) $$l_n > l_{n+1}, \qquad n = 1, 2, \cdots$$

and, on the other,

(5) $$l_n - \frac{1}{12n} < l_{n+1} - \frac{1}{12(n+1)}, \qquad n = 1, 2, \cdots.$$

From (4) and (5) we obtain the known result that the sequence of intervals $I_n: l_n - (1/(12n)) < \lambda < l_n$, $n = 1, 2, \cdots$, whose lengths evidently tend to zero, forms a nest assuring existence of the limit

(6) $$\lambda = \lim_{n\to\infty} l_n.$$

This, in turn, assures the existence of the limit

(7) $$\alpha = \lim_{n\to\infty} a_n = e^\lambda > 0.$$

We use the known value

(8) $$\alpha = (2\pi)^{1/2}.$$

Indeed, (8) follows, after (7), on letting $n \to \infty$ in the easily verified relation

$$\frac{a_n^2}{a_{2n}} = \left(\frac{2}{n}\right)^{1/2} \frac{2\cdot 4\cdot \cdots \cdot (2n)}{1\cdot 3\cdot \cdots \cdot (2n-1)}$$

and using Wallis' well-known product for π.

We now proceed to improve (4), so as to lead to (3), by finding a suitable series majorized by (*). The series obtained by replacing the pth term in (*) by $3^{-p}(2n+1)^{-2p}$ for each p was employed by Robbins and led him to (2). It fails, however, to yield the more precise estimate we have in mind. On the other hand, if we retain the second term there and allow the others to be replaced as above, we still get a series majorized by (*) and it is interesting to find that it just yields our result. In fact, we have

$$\begin{aligned} l_n - l_{n+1} &> \frac{1}{3(2n+1)^2}\left(1 - \frac{1}{3(2n+1)^2}\right)^{-1} + \left(\frac{1}{5} - \frac{1}{3^2}\right)\frac{1}{(2n+1)^4} \\ &= \frac{1}{12 n+1)}\left(1 + \frac{1}{6n(n+1)}\right)^{-1} + \frac{1}{180n^2(n+1)^2}\left(1 + \frac{1}{4n(n+1)}\right)^{-2} \\ &> \frac{1}{12n(n+1)}\left(1 - \frac{1}{6n(n+1)}\right) + \frac{1}{180n^2(n+1)^2}\left(1 - \frac{1}{2n(n+1)}\right) \\ &= \frac{1}{12n(n+1)} - \frac{3n(n+1)+1}{360n^3(n+1)^3}. \end{aligned}$$

As before this gives

$$l_n - l_{n+1} > \frac{1}{12}\left(\frac{1}{n} - \frac{1}{n+1}\right) - \frac{1}{360}\left(\frac{1}{n^3} - \frac{1}{(n+1)^3}\right),$$

which shows that

$$l_n - \frac{1}{12n} + \frac{1}{360n^3} > l_{n+1} - \frac{1}{12(n+1)} + \frac{1}{360(n+1)^3}, \qquad n = 1, 2, \cdots. \tag{9}$$

It follows from (5) and (9) that the sequence of intervals

$$J_n\colon\ l_n - \frac{1}{12n} < \lambda < l_n - \frac{1}{12n} + \frac{1}{360n^3}, \qquad n = 1, 2, \cdots,$$

whose lengths obviously tend to zero, forms a nest assuring the existence of the limit (6), since the endpoints of J_n differ from those of I_n by $o(1)$ as $n \to \infty$.

The result just proved yields what we desire. In fact, we may now write

$$l_n = \lambda + \frac{1}{12n} - \frac{\theta_n}{360n^3}, \qquad 0 < \theta_n < 1, \qquad n = 1, 2, \cdots. \tag{10}$$

Recalling the definitions of a_n and l_n, (10) together with (7) and (8) clearly establishes (3) and the proof is complete.

ESTIMATION IN A CERTAIN PROBABILITY PROBLEM

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An example sometimes quoted as a "nonintuitive" probability is the following: Given a collection of K people, find the probability that some two of them were born on the same day of the year (assuming that no one was born on February 29, and that people are born with equal probability on other days). If K is 23, the probability slightly exceeds 1/2, and if K is 50, the odds are about 33 to 1 in favor of such a pair.

This is of course a special case of the following problem. Let an experiment with N equally likely outcomes be performed K times. What is the probability, $P(N, K)$, that at least one of the outcomes occurs twice? $P(N, K)=1-Q(N, K)$ where $Q(N, K)=N(N-1)\cdots(N-K+1)/N^K=N!/[(N-K)!N^K]$ is the probability that all K outcomes are distinct ($K<N$). For given N and K this may be evaluated directly, or approximated with the aid of Stirling's formula.

The problem becomes more difficult, however, if one assigns values to $P(N, K)$ and N, and attempts to solve for K. We are then confronted with an equation

$$1-\frac{N!}{(N-K)!N^K}=t$$

or, if Stirling's formula is used,

$$\left(\frac{N}{N-K}\right)^{N-K+1/2}e^{-K}=1-t,$$

to be solved for K, where $t=P(N, K)$. Such transcendental equations can be solved approximately, of course, when N and t are specified, but to exhibit $K=f(N, t)$ explicitly seems rather difficult. It is the purpose of this note to show that for $0<t<1$, K is given asymptotically by $L(t)\sqrt{N}$, where $L(t)=\sqrt{-2\log(1-t)}$. Furthermore, except for extreme values of t, this approximation is very good even for small N.

We first prove the

LEMMA. *If $Q(N, K)\geqq a$, where a is constant, $0<a<1$, then $K/N\to 0$ as $N\to\infty$.*

*Proof.** Since $1-x<e^{-x}$ for all x, we have

$$a\leqq Q(N,K)=\left(1-\frac{1}{N}\right)\cdots\left(1-\frac{K-1}{N}\right)$$
$$\leqq\exp\left[-\left(\frac{1}{N}+\cdots+\frac{K-1}{N}\right)\right]=\exp\left[-\frac{K(K-1)}{2N}\right],$$

so that $K(K-1)\leqq 2N\log(1/a)$, which implies the conclusion of the lemma.

* The author is indebted to the referee for suggesting this simpler proof of the lemma.

THEOREM. *Let K, N, and t be related by the equation*

$$1 - \frac{N!}{(N-K)!N^K} = t.$$

(*We may assume all three are continuous real variables,* $0<K<N$.) *Then*

$$\frac{K^2}{-2N\log(1-t)} \to 1 \text{ as } N \to \infty.$$

Thus K is given asymptotically by $K=L(t)\sqrt{N}$, where $L(t)=\sqrt{-2\log(1-t)}$.

Proof. By Stirling's inequality we have

$$\left(\frac{N}{N-K}\right)^{N-K+1/2} e^{-K}\left(\frac{12(N-K)-1}{12(N-K)}\right) < \frac{N!}{(N-K)!N^K} < \left(\frac{N}{N-K}\right)^{N-K+1/2} e^{-K}\frac{12N}{(12N-1)}.$$

Since the middle member above is $1-t$, we have, by inverting,

$$(1-K/N)^{N-K+1/2}e^K = 1/[(1-t)(1+\epsilon)],$$

where $\epsilon\to 0$ as $N\to\infty$ (since $N-K\to\infty$ when $N\to\infty$).

Taking logarithms gives

$$\begin{aligned}
-\log[(1-t)(1+\epsilon)] &= K + (N-K+\tfrac{1}{2})\log(1-K/N) \\
&= K - (N-K)\sum_{r=1}^{\infty}\frac{1}{r}\frac{K^r}{N^r} + \frac{1}{2}\log(1-K/N) \\
&= K - K - \sum_{r=2}^{\infty}\frac{1}{r}\frac{K^r}{N^{r-1}} + \sum_{r=1}^{\infty}\frac{1}{r}\frac{K^{r+1}}{N^r} + \frac{1}{2}\log(1-K/N) \\
&= \frac{1}{2}\log(1-K/N) + \sum_{n=0}^{\infty}\left(\frac{1}{n+1}-\frac{1}{n+2}\right)\frac{K^{n+2}}{N^{n+1}} \\
&= \frac{1}{2}\log(1-K/N) + \frac{K^2}{2N}\left[1+\sum_{n=1}^{\infty}\frac{2}{(n+1)(n+2)}\frac{K^n}{N^n}\right].
\end{aligned}$$

The lemma shows that $K/N\to 0$ as $N\to\infty$, so we have $-\log[(1-t)(1+\epsilon)] =\epsilon'+[K^2/(2N)](1+\epsilon'')$, where $\epsilon'=\frac{1}{2}\log(1-K/N)\to 0$ as $N\to\infty$ and

$$0<\epsilon''=\sum_{n=1}^{\infty}\frac{2}{(n+1)(n+2)}\frac{K^n}{N^n} < \sum_{n=1}^{\infty}\frac{K^n}{N^n} = \frac{K/N}{1-K/N}\to 0$$

as $N\to\infty$. Thus

$$\frac{K^2}{2N} = \frac{-\log[(1-t)(1+\epsilon)]-\epsilon''}{1+\epsilon'},$$

so that

$$\frac{K^2}{[L(t)]^2N} \quad \frac{-\log[(1-t)(1+\epsilon)]-\epsilon''}{-(1+\epsilon')\log(1-t)} \to 1.$$

It is of some interest to see how good the approximation is for several values of N. The author has done this with the aid of the Univac 1103A and some of the results are tabulated below. In this table, K is the least integer such that $N!/[(N-K)!N^K] \leqq 1-t$ and K_1 is the integer nearest $L(t)\sqrt{N}$.

	$t=\frac{1}{32}$		$t=\frac{1}{4}$		$t=\frac{1}{2}$		$t=\frac{3}{4}$		$t=31/32$	
N	K	K_1	K	K_1	K	K_1	K	K_1	K	K_1
4	2	1	2	2	3	2	4	3	5	5
4^2	2	1	4	3	5	5	7	7	10	11
4^3	3	2	7	6	10	9	14	13	21	21
4^4	5	4	13	12	20	19	27	27	42	42
4^5	9	8	25	24	38	38	54	53	84	84
4^6	17	16	49	49	76	75	107	107	168	168
4^7	33	32	98	97	151	151	214	213	337	337
4^8	65	65	195	194	302	301	427	426	674	674
4^9	130	129	389	388	604	603	853	853	1348	1348
4^{10}	259	258	778	777	1206	1206	1706	1705	2696	2696
4^{11}	517	516	1554	1553	2412	2411	3411	3410	5392	5392
4^{12}	1033	1032	3108	3107	4823	4823	6821	6820	10784	10784
4^{13}	2065	2064	6215	6214	9646	9645	13641	13641	21567	21568
4^{14}	4130	4129	12429	12428	19291	19291	27282	27281	43135	43135
4^{15}	8258	8257	24856	24855	38582	38581	54563	54562	86270	86271
4^{16}	16515	16514	49712	49711	77164	77163	109125	109125	172541	172541
4^{17}	33029	33028	99423	99422	154326	154325	218250	218249	345082	345082

AN ANALYTICAL EXPRESSION FOR $[X]$

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THEOREM. *Given the functions*

$$F(X)=\frac{\operatorname{Arcsin}|\sin \pi x|}{\pi} \quad \text{and} \quad G(X)=\lim_{N\to\infty}\{1+|\sin\pi(X-F(X))|\}^N,$$

then for all finite X, $[X]$ (*the greatest integer not greater than* X) *may be written as*

$$[X]=X-|F(X)+2^{1-G(X)}-1|.$$

To prove the theorem it is helpful to establish two lemmas.

LEMMA 1. *For* K *an integer and* $0\leqq b\leqq\frac{1}{2}$, $F(K+b)=b$; *for* $\frac{1}{2}<b<1$, $F(K+b)=1-b$.

LEMMA 2. *For* K *an integer and* $0\leqq b\leqq\frac{1}{2}$, $G(K+b)=1$; *for* $\frac{1}{2}<b<1$, $G(K+b)=\infty$.

To prove Lemma 1, compute $F(K+b)$, where $0 \leqq b \leqq \frac{1}{2}$. Then

$$F(K+b) = \frac{\text{Arcsin}\,|\sin(K\pi + b\pi)|}{\pi} = \frac{\text{Arcsin}\,|\sin(b\pi)|}{\pi} = b.$$

Now consider $F(K+b)$, where $\frac{1}{2} < b < 1$. We have

$$F(K+b) = \frac{\text{Arcsin}\,|\sin(K\pi + b\pi)|}{\pi} = \frac{\text{Arcsin}\,|\sin(b\pi)|}{\pi}$$

$$= \frac{\pi - b\pi}{\pi} = 1 - b.$$

The proof of Lemma 2 is also direct. Consider $G(K+b)$, where $0 \leqq b \leqq \frac{1}{2}$. Then $G(K+b) = \lim_{N\to\infty} \{1 + |\sin(K+b-b)\pi|\}^N = \lim_{N\to\infty} \{1+0\}^N = 1$, where Lemma 1 is used to prove that in this case $X - F(X)$ is the integer K. Then consider $G(K+b)$ for $\frac{1}{2} < b < 1$. We have $G(K+B) = \lim_{N\to\infty} \{1 + |\sin(K-1+2b)\pi|\}^N = \lim_{N\to\infty} \{1 + |\sin(2b\pi)|\}^N = \infty$, since $\sin(2b\pi) \neq 0$ for the assumed range of b.

It is now possible, with the aid of the lemmas, to prove the theorem. We assume that $X = K+b$, where $0 \leqq b < 1$, so that $[X] = K$. Two cases are considered.

Case 1. $0 \leqq b \leqq \frac{1}{2}$. Substituting $X = K+b$ in the formula of the theorem, we find $[X] = K + b - |b + 2^{1-1} - 1| = K$, where each step is justified by one of the lemmas.

Case 2. $\frac{1}{2} < b < 1$. Letting $X = K+b$ in the formula of the theorem, we have $[X] = K + b - |1 - b + 2^{1-\infty} - 1| = K + b - |-b| = K$.

These two cases prove the theorem.

Since the cost, in cents, for a letter of N ounces may be written $C = -A[-N]$, where A is the cost per ounce, it is possible to write an analytical formula for this "postal function."

The writer wishes to express his appreciation for some very helpful suggestions on the part of the referee.

ANOTHER PROOF FOR TWO PROPERTIES OF THE GENERATORS OF A RULED SURFACE

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C. E. Weatherburn [2] has proved the following theorem for ruled surfaces, referred to the generators v = constant and their orthogonal trajectories.

When $\mathbf{a}$ is the unit tangent vector of the generators, so that div $\mathbf{a}$ is equal to the geodesic curvature of the orthogonal trajectories u = constant, ψ is the distance function for the family of generators and K the Gaussian curvature, then

$$(1) \qquad \frac{\partial}{\partial u}\sqrt{(-K)} = \frac{\partial}{\partial u}\left(\frac{M}{\psi}\right) = M_\psi \frac{\partial}{\partial u}\left(\frac{1}{\psi^2}\right) = -2\sqrt{(-K)}\,\text{div}\,\mathbf{a}.$$

This theorem is obtained by using the Mainardi-Codazzi equations. Ram Behari [1] has proved it without reference to these equations. An alternative proof, which only depends on the material in [2], Sections 29, 30 is as follows:

Suppose u denotes the actual distance along a generator; then we have $E=1$, $F=0$. Since the normal curvature in the direction of a generator is zero, it follows that $L=0$, so that the first and second curvature are given by:

$$J = N/G, \tag{2}$$

$$K = -M^2/G. \tag{3}$$

The distance function ψ for the family of generators $v=$constant is the reciprocal of the magnitude of ∇v. Consequently,

$$\psi = 1/|\nabla v| = 1/\sqrt{(|\mathbf{r}_2|^2/G^2)} = \sqrt{G}, \tag{4}$$

and

$$\operatorname{div} \mathbf{a} = G_1/(2G) = \psi_1/\psi. \tag{5}$$

Since $\psi\psi_{11}+\psi_1^2=1$, integration with respect to u gives $\psi\psi_1=u-\alpha$, and therefore $\psi^2=(u-\alpha)^2+\beta^2$, where α and β are functions of v only. Now

$$K = -\beta^2/\psi^4. \tag{6}$$

On the basis of (5) and (6), we obtain immediately

$$\frac{\partial}{\partial u}\sqrt{(-K)} = \frac{-2\beta\psi\psi_1}{\psi^4}$$

and therefore that

$$\frac{\partial}{\partial u}\log\sqrt{(-K)} = \frac{-2\psi\psi_1}{\psi^2} = -2 \operatorname{div} \mathbf{a}.$$

That is,

$$\frac{\partial}{\partial u}\sqrt{(-K)} = -2\sqrt{(-K)} \operatorname{div} \mathbf{a}.$$

Ram Behari [1] also proved, now with the aid of the Mainardi-Codazzi equations, the following theorem for a ruled surface using an orthogonal coordinate system with $v=$constant as asymptotic lines, so that $L=0$, $F=0$.

If the surface is ruled and the mean curvature varies inversely as the distance function along the generator, then the total curvature is constant along any orthogonal trajectory of the generators.

Here the derivation is somewhat long and complicated. We point out that the result can be derived directly: Since ([2], Sec. 29)

$$\frac{\partial}{\partial v}\left(\frac{M}{\psi}\right) = \frac{\partial}{\partial u}\left(\frac{N}{\psi}\right), \qquad \frac{\partial}{\partial u}(M\psi) = 0,$$

for $M \neq 0$,

$$\frac{\partial}{\partial v}\sqrt{(-K)} = \frac{\partial}{\partial v}\left(\frac{M}{\psi}\right) = \frac{\partial}{\partial u}\left(\frac{MN}{M\psi}\right) = \frac{1}{M\psi}\frac{\partial}{\partial u}(\sqrt{G}\cdot J\cdot G\sqrt{(-K)})$$

$$= \frac{1}{M\psi}\left[\psi^2\sqrt{(-K)}\frac{\partial}{\partial u}(\psi J)\right] = \frac{\psi}{M}\sqrt{(-K)}\frac{\partial}{\partial u}(\psi J).$$

Consequently we have

$$\frac{\partial}{\partial v}\sqrt{(-K)} = \frac{\partial}{\partial u}(\psi J).$$

from which the theorem follows.

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1. Ram Behari, On the generators of a ruled surface, Tôhoku Math. J., vol. 46, 1940, pp. 41–43.
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CLASSROOM NOTES

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THE HIGHER DERIVATIVE TEST FOR EXTREMA AND POINTS OF INFLECTION

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1. Introduction. Consider a real function f of one real variable x. When the second derivative as well as the first vanishes at a point P: $(x_0, f(x_0))$, the second derivative test for determining whether P is an extremum of f fails to apply. Under these circumstances more advanced books on calculus give a higher derivative test for determining whether P is an extremum [**1–9**]. The proof always seems to use a higher order mean value theorem, that is, Taylor's theorem. Usually the first of the higher derivatives which does not vanish at x_0, say $f^{(s)}$, is required to be continuous in an open interval (a, b) which includes x_0 [**1–7**]. However, the hypotheses of the test can be weakened: The existence and be-

havior of $f^{(s)}$ at any point but x_0 is irrelevant; $f^{(s-1)}$ must exist throughout some open interval (a, b) including x_0, but need not be continuous there. Occasionally the test is proved in this more general form by applying Taylor's theorem to $f^{(s-1)}$, then using some extra argument [**8**, **9**]. The present note proves the more general form of the test without using Taylor's theorem, which the author feels is too sophisticated for the average first year student of calculus.

2. Examples. Before beginning the proof we show that the extension of the higher derivative test is not trivial by finding functions whose sth derivative exists at x_0 but is discontinuous there. To these functions the extended test can still be applied at x_0, although the usual test can not. Consider

$$\phi_r(x) = x^r \cos(1/x),\ x \neq 0;\ \phi_r(0) = 0;$$

where $r=\rho/(2\sigma+1)>0$ and ρ, σ are integers. It is not too hard to show that

$$\phi_r^{(s)}(x) = x^{r-2s}\cos\left(\frac{1}{x}-\frac{\pi s}{2}\right) - A_{rs}x^{r-2s+1}\sin\left(\frac{1}{x}-\frac{\pi s}{2}\right) + O(x^{r-2s+2}) \quad \text{for } x \neq 0,$$

$$\phi_r^{(s)}(0) = 0 \qquad \text{for } r > 2s-1,$$

where $A_{rs}>0$ depends only on r and s, and O denotes the order of the other terms as x goes to 0. It follows that $\phi_r^{(s)}$ is defined at 0 but discontinuous there whenever $2s-1<r\leqq 2s$. When $r<2s$, the oscillations of $\phi_r^{(s)}$ tend to infinity as x tends to 0.

Consider now the functions

$$\begin{aligned} f(x) &= x^{2n} + x^{4n-\delta}\cos(1/x), & x \neq 0;\quad & f(0) = 0;\\ g(x) &= x^{2n+1} + x^{4n+2-\delta}\cos(1/x), & x \neq 0;\quad & g(0) = 0;\end{aligned}$$

where $0\leqq\delta=\alpha/(2\beta+1)<1$ and α, β are integers. Then $f'(0)=f''(0)=\cdots=f^{(2n-1)}(0)=0$ but $f^{(2n)}(0)=(2n)!$, and $g'(0)=g''(0)=\cdots=g^{(2n)}(0)=0$ but $g^{(2n+1)}(0)=(2n+1)!$, while neither $f^{(2n)}$ nor $g^{(2n+1)}$ is continuous at 0. By Theorem 1 below f has a minimum at 0 and g has a point of inflection there. Thus Theorem 1 is a nontrivially stronger form of the higher derivative test for extrema and points of inflection.

Two other examples may be of interest. Let ψ_0 be an everywhere continuous, nowhere differentiable function, such as that of Weierstrass. Let $\psi_n(x)=\int_0^x\psi_{n-1}(z)dz$ for $n=1, 2, \cdots$. Consider $h(x)=x\psi_{s-1}(x)$. Then $h^{(j)}(x)=x\psi_{s-j-1}(x)+j\psi_{s-j}(x)$ for $j=1, \cdots, s-1$, while $h^{(s)}(0)=s\psi_0(0)$ but $h^{(s)}(x)$ does not exist for $x\neq 0$.

As a last example let u be defined by

$$u(x) = \begin{cases} 0 & \text{for } x = 0 \\ x^{1+r} & \text{for } 2^{-(2n+1)} < |x| \leqq 2^{-2n} \\ -x^{1+r} & \text{for } 2^{-2n} < |x| \leqq 2^{-(2n-1)} \end{cases} \quad \text{for integer } n,$$

where $r=\rho/(2\sigma+1)>0$ and ρ, σ are integers. Then $u'(x)$ is defined except when $x=2^{-n}$, and $u'(0)=0$. It follows that neither u nor u' is continuous in any open interval (a, b) which includes 0. The existence of its derivative at a point ensures the continuity of a function at the point, but not its continuity in any open interval including that point.

3. The higher derivative test. The higher derivative tests can be stated as follows:

THEOREM 1. *Let n denote a positive whole number, f a real function of one real variable.*

A. If $f'(x_0)=f''(x_0)=\cdots=f^{(2n-1)}(x_0)=0$, *then*:

(1) $f^{(2n)}(x_0)>0$ *implies that $f(x_0)$ is a minimum value of f and that f is concave upwards at the point $(x_0, f(x_0))$;*

(2) $f^{(2n)}(x_0)<0$ *implies that $f(x_0)$ is a maximum value of f and that f is concave downwards at the point $(x_0, f(x_0))$.*

B. *If $f''(x_0)=f'''(x_0)=\cdots=f^{(2n)}(x_0)=0$, then: $f^{(2n+1)}(x_0)\neq 0$ implies that $f(x_0)$ is not an extreme value of f, even if $f'(x_0)=0$, but that $(x_0, f(x_0))$ is a point of inflection of f.*

(1) *If $f^{(2n+1)}(x_0)>0$, then f is concave downwards to the left, concave upwards to the right of x_0.*

(2) *If $f^{(2n+1)}(x_0)<0$, then f is concave upwards to the left, concave downwards to the right of x_0.*

(3) *If in addition $f'(x_0)=0$, then $f^{(2n+1)}(x_0)>0$ implies that f is increasing, and $f^{(2n+1)}(x_0)<0$ implies that f is decreasing at the point $(x_0, f(x_0))$.*

First we prove Theorem 1-A for $n=1$. We assume as already known the following theorems [10].

THEOREM 2. *If g is a function and $g'(x_0)\neq 0$ exists, then there is an open interval (a, b) including x_0 such that $g(x)$ exists for every x within the interval and $[g(x)-g(x_0)]/(x-x_0)$ maintains the same sign as $g'(x_0)$ for every such x but x_0.*

If $g'(x_0)>0$, then $g(u)<g(x_0)<g(v)$ whenever $a<u<x_0<v<b$.

If $g'(x_0)<0$, then $g(u)>g(x_0)>g(v)$ whenever $a<u<x_0<v<b$.

In applying Theorem 2 we shall consistently take (a, b) to be the most inclusive open interval satisfying the conditions of that theorem.

THEOREM 3. *If $g'(x)$ exists for every x in an open interval (a, b) and maintains the same sign throughout, then:*

(1) *$g'(x)>0$ for every x in (a, b) implies that g is strictly increasing there and that $g(a+)<g(u)<g(v)<g(b-)$ whenever $a<u<v<b$;*

(2) *$g'(x)<0$ for every x in (a, b) implies that g is strictly decreasing there and that $g(a+)>g(u)>g(v)>g(b-)$ whenever $a<u<v<b$.*

Suppose then that $f'(x_0)=0$ and $f''(x_0)>0$. Then, applying Theorem 2 to the function f', $f'(u)<0<f'(v)$ whenever $a<u<x_0<v<b$. It follows from Theorem 3

that $f(a+)>f(u)>f(x_0)$ and $f(x_0)<f(v)<f(b-)$, that is, $f(x)>f(x_0)$, where $x\neq x_0$ is any other point in (a, b), and $f(x_0)$ is a local minimum value of f. For every x in (a, b) but $x_0, f(x)$ lies above the line $y=f(x_0)$, which is the tangent line to the graph of f at the point $(x_0, f(x_0))$. Hence f is concave upwards at x_0. Similarly, if $f'(x_0)=0$ and $f''(x_0)<0$, one can prove the second half of Theorem 1-A for $n=1$. Theorem 1-A with $n=1$ is commonly called the second derivative test for extrema.

We now prove Theorem 1-A for general n. Suppose that $f^{(2n-1)}(x_0)=0$ and $f^{(2n)}(x_0)>0$. Then, applying Theorem 2 to the function $f^{(2n-1)}$, one finds that $f^{(2n-1)}(x)$ exists for every x in (a, b), the open interval, including x_0, where $[f^{(2n-1)}(x)-f^{(2n-1)}(x_0)]/(x-x_0)>0$ for every x but x_0. The second derivative test is applied to the function $f^{(2n-2)}$ and one finds that $f^{(2n-2)}(x)>f^{(2n-2)}(x_0)=0$, where $x\neq x_0$ is any other point in (a, b). The same line of argument is used when $f^{(2n-1)}(x_0)=0$ and $f^{(2n)}(x_0)<0$. The existence of $f^{(2n-1)}(x)$ for every point x in (a, b) ensures the continuity there of f and its derivatives of order less than $2n-1$.

The second derivative test can not be used further, for by hypothesis $f^{(2r)}(x_0)=0$ for $r=1, \cdots, n-1$. To continue the proof we establish the following lemma.

LEMMA 4. *Let h be a function for which $h''(x)$ exists for every x in an open interval (a, b), and let $h'(x_0)=h''(x_0)=0$ for x_0 some point of (a, b).*

(1) *If $h''(x)>0$ for every x in (a, b) except x_0, then, within (a, b), h' is strictly increasing, $h(x_0)$ is the minimum value of h, and h is concave upwards.*

(2) *If $h''(x)<0$ for every x in (a, b) except x_0, then, within (a, b), h' is strictly decreasing, $h(x_0)$ is the maximum value of h, and h is concave downwards.*

Suppose that $h'(x_0)=h''(x_0)=0$ and that $h''(x)>0$ for every other x in (a, b) except x_0. Then, applying Theorem 3 to the function h', h' is strictly increasing in the open intervals (a, x_0) and (x_0, b) and $h'(a+)<h'(u)<h'(x_0-)=0 =h'(x_0+)<h'(v)<h'(b-)$ whenever $a<u<x_0<v<b$. Thus h' is strictly increasing throughout the open interval (a, b). Since $h'(u)<0<h'(v)$, it follows from Theorem 3 that $h(a+)>h(u)>h(x_0)$ and $h(x_0)<h(v)<h(b-)$, that is, $h(x)>h(x_0)$, where $x\neq x_0$ is any other point in (a, b), and $h(x_0)$ is a local minimum value of h. For every x in (a, b) but x_0, $h(x)$ lies above the line $y=h(x_0)$, which is the tangent line to the graph of h at the point $(x_0, h(x_0))$. Hence h is concave upwards at x_0. One can readily prove that h is in fact concave upwards for every x in (a, b). However, we do not need this to establish our main result and omit the proof of this detail. Similarly if $h'(x_0)=h''(x_0)=0$ and if $h''(x)<0$ for every other x in (a, b) except x_0, one can prove the second half of Lemma 4.

We return now to the proof of Theorem 1-A for general n. Suppose we have established that $f^{(2r)}$ is continuous throughout the open interval (a, b) and that $f^{(2r)}(x)>f^{(2r)}(x_0)$ whenever $x\neq x_0$. By hypothesis $f^{(2r-1)}(x_0)=f^{(2r)}(x_0)=0$. Then by Lemma 4, $f^{(2r-1)}$ is strictly increasing throughout (a, b), $f^{(2r-1)}(u)<0<f^{(2r-1)}(v)$ whenever $a<u<x_0<v<b$, $f^{(2r-2)}(x_0)$ is the minimum value of $f^{(2r-2)}$ within

(a, b); and $f^{(2r-2)}$ is concave upwards throughout (a, b). Again, $f^{(2r-2)}$ is continuous throughout (a, b), $f^{(2r-2)}(x) > f^{(2r-2)}(x_0)$ there whenever $x \neq x_0$, and by hypothesis $f^{(2r-3)}(x_0) = f^{(2r-2)}(x_0) = 0$. If this argument can be started for some integer $r = s$, it can be continued for smaller integers until we reach $r = 1$. But we earlier established the proper conditions for this argument for $r = n-1$. At the $(n-1)$st step, with $r = 1$, we find that f' is strictly increasing throughout (a, b), $f'(u) < 0 < f'(v)$ whenever $a < u < x_0 < v < b$, and conclude that $f(x_0)$ is the minimum value of f within (a, b), and f is concave upwards throughout (a, b). Thus the first half of Theorem 1-A is established for general n. The same line of argument is used to prove the second half of Theorem 1-A for general n.

To prove Theorem 1-B apply Theorem 1-A to the function f'. Suppose that the other proper hypotheses hold and that $f^{(2n+1)}(x_0) > 0$. Then $f''(u) < 0 < f''(v)$ whenever $a < u < x_0 < v < b$, as we found in the proof of Theorem 1-A, both when $n = 1$ and in the general case. Thus f has a point of inflection at the point $(x_0, f(x_0))$, and f is concave downwards in the open interval (a, x_0), concave upwards in the open interval (x_0, b). Furthermore $f'(x_0)$ is the minimum value of f' in (a, b), and f' is continuous throughout this interval. If $f'(x_0) \geqq 0$, then $f'(x) > 0$ for every other $x \neq x_0$ in (a, b). Then f' does not change sign in (a, b), so $f(x_0)$ is not an extreme of f, but rather f is strictly increasing throughout (a, b). In particular this is true for $f'(x_0) = 0$, proving the first half of 1-B-3. On the other hand, if $f'(x_0) < 0$, then by Theorem 2 there is an open interval (c, d), possibly a subinterval of (a, b), where $[f(x) - f(x_0)]/(x - x_0) < 0$ for every x in (c, d) but x_0. Then $f(u) > f(x_0) > f(v)$ whenever $c < u < x_0 < v < d$, and again $f(x_0)$ is not an extreme value of f. Thus 1-B-1 is established. Similarly, with the other proper hypotheses and with $f^{(2n+1)}(x_0) < 0$, one can prove 1-B-2 and the second half of 1-B-3.

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ON TWO FUNCTIONAL EQUATIONS*

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It is well known that the only real-valued continuous solutions of the functional equation

$$f(x+y)=f(x)+f(y) \tag{1}$$

are of the form $f(x)=kx$ where k is a real number. In this note we shall solve two functional equations which may be given to a beginning analysis class as applications of (1). Both of these equations arise naturally in describing the one-parameter groups of affine transformations in the euclidean plane.

THEOREM 1. *The only continuous solutions of the equation*

$$f(x+y)=f(x)+f(y)+a(1-A^x)(1-A^y), \tag{2}$$

where a and A are real numbers and $A>0$, *are of the form*

$$f(x)=kx-a(1-A^x) \tag{3}$$

where k is a real number.

Proof. Let us first find differentiable solutions of (2). Holding y constant and differentiating, we get

$$f'(x+y)=f'(x)+a(1-A^y)(-A^x\ln A).$$

Now holding x constant and differentiating, we obtain $f''(x+y)=a(\ln A)^2A^{x+y}$. Let $u=x+y$; then $f''(u)=a(\ln A)^2A^u$. $f'(u)=\int_0^u a(\ln A)^2A^t dt=a(\ln A)\left[A^t\right]_0^u$ $=a(\ln A)(A^u-1)$ and $f(u)=a\int_0^u(\ln A)(A^t-1)dt=a\left[A^t-t(\ln A)\right]_0^u$. Therefore, $f(u)=aA^u-au(\ln A)-a$. Let f_1 be any other continuous solution of (2); then $g=f-f_1$ is also continuous. We notice that

$$g(x+y)=(f-f_1)(x+y)=f(x)-f_1(x)+f(y)-f_1(y).$$

This relation shows that g is of form (1) and, therefore, there exists a real number c such that $g(x)=cx$. Hence, any continuous solution of (2) is of form (3).

THEOREM 2. *The only continuous solutions of* $f(x+y)=A^xf(y)+A^yf(x)$, *where A is a positive real number, are of the form* $f(x)=kxA^x$ *where k is a real number.*

Proof. Let $g(x)=A^{-x}f(x)$ and observe that g satisfies (1). Hence, there exists a real number k such that $g(x)=kx$. Therefore, $f(x)=kxA^x$.

* This paper was presented to the Louisiana-Mississippi Section of the Mathematical Association of America on February 21, 1958.

ON THE SYNTHETIC DIVISION PROCESS

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The authors present here a pair of proofs of a familiar result from college algebra. Stated without proof in many reputable texts, the theorem, with either of the proofs here given, has proved generally palatable to those freshmen conversant with the axiom of mathematical induction. The content of the theorem is, very roughly, that if nonintegral entries arise in the standard synthetic division process, then the rational number being tested is not a root of the equation in question. The result may be stated precisely as follows:

THEOREM. *Let p/q be a nonzero rational root of the polynomial equation $f(x) = a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$, where each a_k is an integer and $(p, q) = 1$. Let $b_0 = a_0$, and, for each positive integer k not exceeding n, let*

$$b_k = a_0(p/q)^k + a_1(p/q)^{k-1} + \cdots + a_k.$$

Then each b_k is an integer.

First proof. Assume that some $b_k = r/s$ is nonintegral, where $(r, s) = 1$. From the fact that

$$b_k = [a_0p^k + a_1p^{k-1}q + \cdots + a_kq^k]/q^k,$$

it follows that s divides q^k, so that $(p, s) = 1$. Then $(pr, s) = 1$, so that $(pr)/(qs) = (p/q)b_k$ is nonintegral. Then $b_{k+1} = (p/q)b_k + a_{k+1}$ is nonintegral.

$(n-k)$-fold repetition of this argument shows that b_n is nonintegral; in particular, b_n is nonzero.

Second proof. In this proof we show that each of the numbers b_k is an integer divisible by q.

That q divides the integer $b_0 = a_0$ is well known. Now let $0 \leqq k < n$, and suppose that b_k is an integer divisible by q. Then $b_{k+1} = (p/q)b_k + a_{k+1}$ is an integer which, in case $k+1 = n$, is zero and hence is divisible by q. To show that q divides b_{k+1} in case $k+1 < n$, set $x = p/q$ in the equation $f(x) = 0$ and multiply by q^{n-k-1} to get

$$a_0p^n/q^{k+1} + a_1p^{n-1}/q^k + \cdots + a_{k+1}p^{n-k-1}$$
$$= -q[a_{k+2}p^{n-k-2} + \cdots + a_{n-1}pq^{n-k-3} + a_nq^{n-k-2}].$$

Since q divides the right-hand side, it divides the left-hand side, which is precisely

$$b_kp^{n-k}/q + a_{k+1}p^{n-k-1} = p^{n-k-1}[b_kp/q + a_{k+1}] = p^{n-k-1}b_{k+1}.$$

Since $(p, q) = 1$, it follows that q divides b_{k+1}.

AN APPLICATION OF STIRLING'S FORMULA

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Stirling's formula states that $n! \sim (2\pi n)^{1/2} n^n e^{-n}$. Varying only in certain refinements, the majority of proofs for Stirling's formula use the relation

$$\int_a^b \log x \, dx = b \log b - a \log a - b + a. \tag{1}$$

For our purpose (given below), it suffices to know that there exists elementary proofs which, *without using* (1), establish Stirling's formula in a weaker form, *i.e.*,

$$n! \sim C n^{n+1/2} e^{-n}, \tag{2}$$

where C is an unspecified positive constant. Feller [1], Aissen [2], (there may be others) have given proofs of (2). The purpose of this note is to show how Stirling's formula, (2), may be used to establish the validity of (1) for real numbers, $b > a > 0$.

We recall that a continuous function is Riemann-integrable over a closed interval. For $a > 0$, $\log x$ is integrable on $a \leqq x \leqq b < \infty$. Thus, if we divide the interval $[a, b]$ into n equal parts such that $\Delta x_k = (b-a)/n$, $x_k = a + k\Delta x_k$, $k = 0, 1, \cdots, n$, we are assured that

$$\int_a^b \log x dx = \lim_{n\to\infty} S_n = \lim_{n\to\infty} \sum_{k=1}^{n} \{\log (a + k\Delta x_k)\} \Delta x_k. \tag{3}$$

Let $a = r/s$, $b - a = p/q$, where p, q, r, and s are fixed positive integers. Let m be an arbitrary positive integer and define $n = mqs(b-a) = mps$. We note that p and s are fixed here, so that if $m \to \infty$, then $n \to \infty$. Now, $\Delta x_k = (b-a)/n = 1/mqs$ and

$$\begin{aligned} S_n &= \sum_{k=1}^{n} (\log (a + k\Delta x_k)\} \Delta x_k \\ &= \frac{1}{mqs} \sum_{k=1}^{mps} \log \left(a + \frac{k}{mqs}\right) = \sum_{k=1}^{mps} \log \left(\frac{mqr + k}{mqs}\right)^{1/mqs}; \end{aligned} \tag{4}$$

$$e^{S_n} = \prod_{k=1}^{mps} \left(\frac{mqr + k}{mqs}\right)^{1/mqs} = \frac{1}{(mqs)^{p/q}} \left[\frac{(mqr + mps)!}{(mqr)!}\right]^{1/mqs}. \tag{5}$$

At this point, one introduces (proves) Stirling's formula, (2). Thus, if we define $Q(n)$ by the relation, $n! = Cn^{n+1/2}e^{-n}Q(n)$, then $\lim_{n\to\infty} Q(n) = 1$. Using (2), we have

$$\frac{(mqr + mps)!}{(mqr)!} = \frac{Q(mqr + mps)(mqr + mps)^{mqr+mps}e^{-mps}}{Q(mqr)(mqr)^{mqr}} \left(1 + \frac{ps}{qr}\right)^{1/2} \tag{6}$$

$$= \frac{Q(mqr + mps)(mqs)^{mps}e^{-mps}\left(\frac{r}{s} + \frac{p}{q}\right)^{mqr+mps}}{Q(mqr)(r/s)^{mqr}}\left(1 + \frac{ps}{qr}\right)^{1/2}.$$

Substitution of (6) into (5) yields

$$\begin{aligned} e^{S_n} &= e^{-(p/q)}\left(\frac{r}{s} + \frac{p}{q}\right)^{(r/s)+(p/q)}\left(\frac{r}{s}\right)^{(-r/s)} F(p, q, r, s, m) \\ &= e^{a-b}b^b a^{-a} F(p, q, r, s, m), \end{aligned} \tag{7}$$

where

$$F(p, q, r, s, m) = \left[\frac{Q(mqr + mps)}{Q(mqr)}\right]^{1/mqs}\left(1 + \frac{ps}{qr}\right)^{1/(2mqs)}. \tag{8}$$

Let $m \to \infty$. Then $F(p, q, r, s, m) \to 1$, $n \to \infty$, and $S_n \to \int_a^b \log x dx$. Thus, from (7),

$$\int_{a=r/s}^{b=a+(p/q)} \log x dx = b \log b - a \log a - b + a. \tag{9}$$

Thus far, we have established the validity of (1) for *rational* numbers, $b > a > 0$. Because of the continuity property of the function, $\log x$, and because one may choose $a = \{r/s\}$ and $b = \{a + (p/q)\}$ as two rational sequences of numbers approaching two distinct, positive, irrational numbers as limits, respectively, the extension of the validity of (1) for *irrational* numbers is immediate. Thus, (1) is valid for all real numbers $b > a > 0$.

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MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN A. BROWN, University of Delaware, AND
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GUIDE ON PURCHASE OF EQUIPMENT

The Council of Chief State School Officers with the assistance of representatives from science, including staff members of the National Bureau of Standards, published in August, 1959 a very comprehensive guide for the purchase of equipment under the provisions of National Defense Education Act of

1958. The guide includes specifications on some 1000 items of equipment recommended for teaching science, mathematics, and modern foreign languages in elementary and secondary schools. There also is included an extensive bibliography for elementary and secondary school libraries. The project was in charge of Dr. Edgar Fuller, executive secretary of the Council. Advice on the equipment and bibliography recommended for use in the teaching of mathematics was given by a group of mathematicians and teachers whose meeting was made possible by the School Mathematics Study Group.

Copies of the publication may be purchased from the state superintendent of public instruction in each of the states, or directly from Ginn and Company.

AAAS STUDY ON THE USE OF SPECIAL TEACHERS OF SCIENCE AND MATHEMATICS IN GRADES 5 AND 6

One of the major activities in the extended Science Teaching Improvement Program of the American Association for the Advancement of Science, under the new three-year grant to AAAS by the Carnegie Corporation of New York, is to be known as the Study on the Use of Special Teachers of Science and Mathematics in Grades 5 and 6. The cooperation of the public school systems in Cedar Rapids, Iowa; Lansing, Michigan; Versailles, Kentucky; and Washington, D. C. have been obtained for this project. In each of these school systems, at least one special teacher of science and at least one special teacher of mathematics will teach only science or only mathematics in grades 5 and 6 to at least four different classes. The teachers have been selected by the school systems as those having special training, competence, and successful teaching experience in the areas to which they are assigned.

While the use of special teachers of art, music, and physical education is fairly widespread in American elementary schools, the question of special teachers in the academic disciplines such as science and mathematics remains highly controversial. Majority opinion of experts in elementary education, both administrators and teachers, seems to favor strongly the retention of the "self-contained" classroom at this level. On the other hand, many scientists favor the use of specially prepared teachers particularly because of the new and anticipated demands on the elementary teacher. It is quite impossible under the present methods of training elementary teachers for the elementary teacher to take what appears to be a necessary minimum of course work in science and mathematics, or in the other academic disciplines as well. The National Science Foundation sponsored curriculum studies, which are giving attention to the science curriculum at the elementary level, will stress how serious is the problem. Both the School Mathematics Study Group and the Biological Sciences Curriculum Study are concerned with grades from kindergarten to grade 8, as well as with the high schools, and there are also other privately sponsored activities which are working on materials at the elementary level.

An attempt will be made to provide a careful evaluation of the Study so

that interested persons will have available objective data to support their cases or to persuade them of weaknesses in their present point of view. It is anticipated that the Study will be conducted during a two-year period beginning in the fall of 1959 and it is to be hoped that the Study might be extended to more schools during the second year. (Abstracted from *Science Education News*, September 1959.)

A NEW KIND OF COURSE FOR THE PREPARATION OF TEACHERS*

E. P. NORTHROP, College of the University of Chicago

Three main streams can be discerned in current efforts to improve secondary school mathematics curricula in this country. First, various groups, working at both national and local levels, are formulating proposals for curricular change. Second, various groups, also working at both national and local levels, are preparing new classroom materials. Third, numerous programs, supported by industry and by private and public foundations, are being offered to improve the mathematical and professional competence of teachers.

These efforts have come into being and have grown too large too fast to be well coordinated. Yet their interrelatedness is readily apparent. If the proposals for curricular change are to be understood and not rejected out of hand, then teachers need to be trained in the relevant mathematical subject matter. If the proposals are to be acted upon and not just read and filed away, then teachers need to be supplied with relevant text materials.

These efforts, however well conceived and executed, will not in themselves suffice. What is lacking? The more thoughtful of the groups proposing curricular change agree that no single curriculum will serve all the students in all the schools. Accordingly, when they produce a particular curriculum by way of illustration of what might be done, they are quick to emphasize that this is only an example, and that the teachers of the country must adapt the suggested changes to their local needs. Again, the more thoughtful of the groups preparing new text materials similarly reject the notion of a universal panacea. Accordingly, when they produce texts by way of illustration of what might be written, they too are quick to emphasize that these are only examples, and that the teachers of the country must adapt the suggested materials to their local needs.

What is lacking now begins to emerge in the picture, as does the place where the lack can be remedied. Overburdened classroom teachers, however well trained or retrained, are about to be faced with a bewildering and often conflicting variety of curricular proposals and text materials. They need help in selecting and organizing materials for use in their own local classrooms. They need, in short, help in learning to think and to write for themselves. The help they need can and should be provided in the third stream of our current efforts, namely, in the programs of re-education being offered to inservice teachers.

* Presented at a panel discussion, "The Training of Secondary School Mathematics Teachers," Annual Meeting of the Mathematical Association of America, Philadelphia, Pa., January 23, 1959.

Let us therefore consider a possible design for a seminar of nine months' duration, intended to provide the individual teacher with a working understanding of curriculum construction, together with working experience in the preparation of materials for use by his own students. It is assumed that the seminar is to be attended by twenty-five or thirty teachers, and is to be conducted concurrently with, or subsequent to, appropriate courses in mathematics proper.

During the first three months of the year, and while the teacher is learning some of the mathematics he will need, the seminar would be devoted to a down-to-earth consideration of factors that have an important bearing on curriculum construction. Examination of a few specific curriculum proposals, including the bases offered in support of them, would lead naturally and quickly to a discussion of such factors as the needs or demands of society, the philosophy of education of the school, the characteristics of its students, the nature of the subject-matter, the formulation of aims or objectives, the selection and organization of learning activities, and testing procedures. Such a discussion, properly conducted, can serve three purposes. First, it can bring order into possibly disordered and unrelated ideas gleaned from earlier courses in education. Second, it can provide the teacher with a framework for the writing efforts to follow. Third, it can provide him with a rationale by which to justify his proposals and sell his product to an audience too often ignored, namely, the folks back home.

The next two months would be devoted to group writing efforts. The class would be divided into groups of four or five teachers each, sorted according to expressed interest in this, that, or the other general topic. Each group, having selected a modest and manageable unit, would settle upon the kind of student for which it is intended and what he is expected to achieve from it, would outline it, and would write a piece of text complete with problem material. It would be made clear at the outset that these initial efforts were not expected to terminate in a polished product—that they were simply practice for a more extensive effort to come. The seminar would continue to meet, though perhaps less frequently, for open discussion of progress, major difficulties encountered here or there, and so on. The final products would be reproduced and distributed to all members of the seminar for analysis and criticism, in the course of which all would learn something about the importance of correctness, clarity, and precision in mathematical exposition. But there is an additional and very real value in this preliminary writing. Here the teacher finds himself, perhaps for the first time, facing up to issues and committing himself to paper, and by doing so in the company of his fellows he gains as much in confidence as he gains in simple experience.

The last four months of the year would be devoted to the major effort and product of the seminar. Each teacher would embark on an individual project in the planning of a year course for use in his own high school by his own students. The project would issue in a substantial report consisting of the following parts:

1. A brief general description of the course.
2. Its justification, addressed to school board and principal, parents and fellow teachers, and including general statements of aims, of shortcomings of the present program, and of strengths of the program being proposed.
3. A full outline of the year's work.
4. A complete treatment of a substantial unit of the course (for example, a month's work, including classroom materials to be used by the student for further study), an examination over the unit, and a teachers' manual for use by other teachers.

During this period the seminar would be disbanded as such, though it could conceivably meet for discussions with visiting representatives of outside groups concerned with curricular proposals or preparation of text materials. Members of the staff would of course make themselves available for individual conferences. It would be desirable, if it could be arranged, to use the fourth and last month of the period for criticism and revision of the reports, especially if they were to be used—as well they might—in partial fulfillment of certain degree requirements.

A seminar of the kind described is perhaps most suitable for use with teachers who have had several years of teaching experience.† It could nevertheless be adapted—no doubt with lowered sights—for use with new teachers in the course of their practice teaching. Indeed, a foreshortened version of the seminar might be imagined for use in a summer program of, say, eight weeks' duration. Here the work would take the form of group projects, with a skilled leader assigned to each of the groups of four or five teachers for two purposes: to direct the writing efforts and, in the course of them, to call attention to the curricular issues involved. In this way the first two parts of the nine-month seminar described above might be successfully telescoped, especially if the several groups concerned were to meet together once a week, under a director of the entire operation, for discussion of current proposals for curricular change.

It is to be anticipated that the individual or group reports issuing from seminars of the kind described in this paper would be of mixed quality. But it can be safely asserted that the author or coauthor of even the weakest report would have grown enormously in critical understanding of his task. It is often said that one does not really know his subject until he tries to teach it. In the same vein, one does not really know his subject or his teaching until he tries to write what he teaches. Even if what he writes is not worth teaching, the experience will have made him a far better and wiser judge of courses and texts that others write for him to use.

† Evidence that it can be done is furnished by the fact that in the 1958–59 Academic Year Institute for High School Mathematics Teachers at the University of Chicago, Mr. Maurice Hartung and the author are in midstream with a second such seminar. The author's indebtedness to Mr. Hartung, and to other colleagues in education and mathematics who assisted in the design and conduct of the seminar, is hereby gratefully acknowledged.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1381. *Proposed by H. E. Hart, City College of New York*

Show that

$$1 + x + x^2/2! + x^3/3! + \cdots + x^{2n}/(2n)! = 0$$

has no real roots.

E 1382. *Proposed by Ian Connell, University of Manitoba*

Let u_n be the nth term of the sequence

$$1, 2, 4, 5, 7, 9, 10, 12, 14, 16, 17, \cdots,$$

where one odd number is followed by two evens, then three odds, etc. Prove that

$$u_n = 2n - [(1 + \sqrt{8n - 7})/2],$$

where square brackets denote the integral part.

E 1383. *Proposed by E. J. Burr, University of New England, N.S.W., Australia*

Let P_n be the problem: "Find a set of $n > 2$ consecutive positive integers, the greatest of which is a divisor of the least common multiple of the remaining $n-1$." Show that precisely one of the problems P_n has a unique solution.

E 1384. *Proposed by J. H. Butchart, Arizona State College*

Construct a circle through two given points, separated by a given circle, which shall cut the given circle at the smallest possible angle.

E 1385. *Proposed by John Burke, Saranac Lake, N. Y.*

A homogeneous die is "loaded" at a single point and is cast n times. The one spot appears n_1 times, the two spot n_2 times, $\cdots$, the six spot n_6 times. Find the (probable) position of the load.

SOLUTIONS

Generalization of a Russian Olympiad Problem

E 1337 [1958, 708; 1959, 424]. *Proposed by the Student-Faculty Colloquium, Carleton College*

What is the largest value of n, in terms of m, for which the following sentence is true? If from among the first m natural numbers any n are selected, among the remaining $m-n$ at least one will be a divisor of another.

II. *Solution by J. L. Selfridge, IBM Research Center, Yorktown Heights, N. Y.* The published solution [1959, 424] contains an oversight. The proof is correct only for $m<12$. If $m \geqq 12$ and we select $n=[m/2]-1$ numbers, the remaining $m-n$ numbers need not contain 1 or $(k, 2k)$ as asserted by the solver. This is easily shown by selecting 1 and all numbers exactly divisible by an odd power of 2. For example, when $m=12$ select 1, 2, 6, 8, 10. The remaining numbers are odd or an odd multiple of 4, and no one is twice another. Following is believed to be a correct solution of the problem.

If $1, 2, \cdots, [m/2]$ are selected from $1, 2, \cdots, m$, then none of those remaining divides another of them. This shows that $n<[m/2]$. Now we show that for any $n \leqq [m/2]-1$, some a and b will remain unselected with $b=2^k a$. Suppose that no such a and b exist. Then we must select all but one from each sequence $r, 2r, \cdots, 2^s r$. These sequences are disjoint if r is odd, and we count those selected by noticing that selecting all but the largest in each sequence means selecting $1, 2, \cdots, [m/2]$.

Locus of Centers of a Family of Equilateral Triangles

E 1351 [1959, 141]. *Proposed by J. F. Darling, Woodstown, New Jersey*

Find the locus of the centers of the equilateral triangles whose sides a, b, c pass through three fixed points A, B, C respectively.

Solution by Helen M. Marston, Educational Testing Service, Princeton, N. J. Construct the three circles O_a, O_b, O_c for which the segments BC, CA, AB respectively subtend 120° arcs, the minor arc in each case being directed toward the inside of triangle ABC. From any point P_c on O_c draw P_cAP_b (line a) and P_cBP_a (line b). Angles P_cP_bC and P_cP_aC each equal 60° by construction, whence P_b, C, P_a are collinear (line c) and triangles $P_aP_bP_c$ represent one set of equilateral triangles abc. Let M_a, M_b, M_c be the midpoints of the minor arcs BC, CA, AB respectively. The center M of triangle $P_aP_bP_c$ is the intersection of P_aM_a and P_bM_b, which must intersect at 60°. Since the locus of M includes points M_a, M_b, M_c, these three points form an equilateral triangle and the locus of M is the circle circumscribing it.

If circles O_a', O_b', O_c' are constructed similarly but with minor arcs directed toward the outside of triangle ABC, we obtain a second set of equilateral triangles abc, a second set of minor-arc-midpoints M_a', M_b', M_c', and a circle circumscribing them for the locus of M'.

The required locus is therefore a pair of circles. It can be shown that they

have a common center G, which is the centroid of triangle ABC. In the limiting case where A, B, C are collinear, the two circles coincide. When A, B, C form an equilateral triangle, there is only one set of equilateral triangles abc and the locus of M is the single point G.

Also solved by Leon Bankoff, W. B. Carver, J. W. Clawson, Michael Goldberg, L. D. Goldstone, W. R. McEwen and Amos Nannini (jointly), D. C. B. Marsh, Beckham Martin, D. C. Stevens, and the proposer.

The loci of M and M' may be alternatively described as the circles having centers at the centroid of triangle ABC and passing through the isogonic centers R and R' of triangle ABC, or as the circumcircles of triangles $O_a' O_b' O_c'$ and $O_aO_bO_c$.

Goldberg, Marsh, and Martin used cartesian coordinates; Carver, Clawson, and the proposer used conjugate coordinates. Goldstone pointed out the connection between the problem and Problem 4301 [1950, 565]. Martin considered the more general problem of finding the locus of the centroids of all triangles of a given species whose sides a, b, c pass through three fixed points A, B, C respectively.

The Case of the Careless Typesetter

E 1352 [1959, 141]. *Proposed by C. W. Trigg, Los Angeles City College*

In setting the type for the multiplication $(abc)(bca)(cab)=234235286$, $a>b>c$, in which the unit's digit is 6, the remaining digits of the product became pied. Restore them to their proper order.

Solution by W. R. Talbot, Lincoln University, Missouri. Let $n=(abc)$ and let N be the desired product. To establish limits on n, we notice that if $c=1$, then the largest n, namely 981, gives $N=159{,}080{,}922$, which is too small. Therefore $988>n>981$. But the sum of the digits of N is $35\equiv 2 \bmod 3$; whence n and its permutations are congruent to 2 mod 3. Then n is 986 or 983, of which only 983 has a product of digits ending in 6. It is now readily found that

$$(983)(839)(398) = 328{,}245{,}326.$$

Also solved by A. N. Aheart, Merrill Barnebey, D. A. Breault, W. E. Buker, J. M. Calloway, G. B. Charlesworth, P. L. Chessin, R. J. Cormier, Eleanor G. Dawley, Monte Dernham, J. E. Faulkner, J. F. Foley, Michael Goldberg, L. D. Goldstone, H. W. Gould, Robert Gregorac, E. E. Griffee, P. G. Hodge, Jr., J. H. Hodges, J. M. Howell, A. R. Hyde, F. S. Innis, Jr., Sister Kenneth Kolmer, W. R. McEwen, D. C. B. Marsh, Helen M. Marston, J. H. Means, Franklin Mohr, D. L. Muench, J. B. Muskat, D. J. Persico, J. P. Phillips, J. L. Pietenpol, C. F. Pinzka, B. E. Rhoades, J. A. Rodriguez, J. L. Selfridge, D. C. Stevens, W. B. Stovall, Jr., Harry Weingarten, Mildred West, Charles Wexler, Dale Woods, Ronald Wyllys, and the proposer.

Eccentricity of an Elliptical Section of a Cylinder

E 1353 [1959, 141]. *Proposed by L. A. Kenna, University of Arizona*

What is the eccentricity of an ellipse formed by a plane cutting the axis of a right circular cylinder at an angle of θ?

Solution by A. N. Aheart, West Virginia State College. Denoting the lengths of the major and minor axes by $2a$ and $2b$, we have

$$e = (1 - b^2/a^2)^{1/2} = (1 - \sin^2 \theta)^{1/2} = |\cos \theta|.$$

Also solved by R. G. Albert, E. F. Allen, J. W. Baldwin, Leon Bankoff, Merrill Barnebey, H. F. Bechtell, A. P. Boblétt, D. A. Breault, C. S. Carlson, G. B. Charlesworth, P. L. Chessin, A. G. Clark, R. J. Cormier, C. H. Cunkle, Alfred Diarnbey, Michael Difon, C. W. Dodge, Ernest Enns, G. W. Erwin, Jr., J. E. Faulkner, Susan L. Friedman, Todd Gitlin, George Glauberman, Michael Goldberg, L. D. Goldstone, F. D. Grogan, J. D. Haggard, R. Henry, Richard Holt, A. F. Horadam, R. H. Hou, A. R. Hyde, G. S. Innis, Jr., M. S. Itzkowitz, J. D. E. Konhauser, W. E. Lawrence, W. R. McEwen, D. C. B. Marsh, Beckham Martin, C. S. Ogilvy's freshmen calculus class, D. J. Persico, J. L. Pietenpol, C. F. Pinzka, B. E. Rhoades, L. A. Ringenberg, D. A. Robinson, J. A. Schumaker, Ruth G. Smith, D. C. Stevens, W. B. Stovall, Jr., W. R. Talbot, C. W. Trigg, T. C. Wales, Charles Wexler, L. H. Williams, R. H. Wilson, Jr., Alan Wofford, Dale Woods, J. W. Young, and the proposer.

Editorial Note. The eccentricity of the conic section formed by a plane cutting at an angle θ the axis of a right circular cone of generating angle ϕ is $|\cos\theta|/\cos\phi$.

A Property of the Coefficients of $(1+x)^{p-2}$

E 1354 [1959, 141]. *Proposed by P. L. Chessin, University of Maryland*

If $(1+x)^{p-2}=1+a_1x+a_2x^2+\cdots+a_{p-2}x^{p-2}$, where p is a prime, then a_1+2, a_2-3, a_3+4, $\cdots$ are all multiples of p.

Solution by C. F. Pinzka, University of Cincinnati. Multiplying corresponding members of the k congruences $p-i\equiv -i \pmod p$, $i=2, 3, \cdots, k+1$, gives

$$(p-2)!(p-k-2)! \equiv (-1)^k(k+1)! \pmod p.$$

For $k<p$ we have $(k!, p)=1$ and thus

$$(p-2)!/(p-k-2)!k! \equiv (-1)^k(k+1) \pmod p,$$

demonstrating that $a_k+(-1)^{k+1}(k+1)$ is divisible by p.

Also solved by A. N. Aheart, R. G. Albert, J. W. Baldwin, H. F. Bechtell, R. F. Brown and Anna Endelman and Joel Levy (jointly), J. M. Calloway, G. B. Charlesworth, A. E. Danese, Alfred Diarnbey, H. B. Emerson, J. E. Faulkner, N. J. Fine, Todd Gitlin, George Glauberman, Michael Goldberg, L. D. Goldstone, Joseph Hammer, Leonard Hauer, Richard Holt, J. H. Hodges, Irving Katz, Jack Klugerman, Sidney Kravitz, W. E. Lawrence, Robert McGuire, D. C. B. Marsh, Andrzej Mgkowski, Valerie Miké, Mary P. Moseley, D. L. Muench, Stewart Nagler, K. K. Norton, C. S. Ogilvy, D. J. Persico, Benjamin Sapolsky, L. J. Schneider, Arnold Singer, D. C. Stevens, H. R. Stevens, Gerald Stoller, W. R. Talbot, W. A. Veech, T. C. Wales, Charles Wexler, Dale Woods, Ronald Wyllys, J. W. Young, David Zeitlin, and the proposer.

A Detail of a Larger Work

E 1355 [1959, 141]. *Proposed by A. J. Goldman, National Bureau of Standards*

If $y(x)>0$ and $y=\log[1+(\log x)/x+y/x]$ for all sufficiently large x, prove that $y\sim(\log x)/x$ as $x\to\infty$. (This is a step whose details are left to the reader in a paper by S. Chowla and F. C. Auluck, *Some properties of a function considered by Ramanujan*, J. Indian Math. Soc., vol. 4, 1940, pp. 169–173.)

Solution by David Zeitlin, Remington Rand UNIVAC, St. Paul, Minn. From

$$1+(\log x)/x+y/x=e^y>1+y$$

we have

$$0 < y < (\log x)/(x - 1),$$

whence $y(x) \to 0$ as $x \to \infty$. Therefore

$$\lim_{x\to\infty} xy/(\log x) = \lim_{y\to\infty} y/(e^y - 1) = 1.$$

Also solved by D. A. Breault, R. F. Brown and Anna Endelman and Joel Levy (jointly), N. J. Fine, George Glauberman, Walter James, D. C. B. Marsh, Morris Morduchow, K. K. Norton, C. F. Pinzka, D. C. Stevens, Peter Treuenfels, and W. A. Veech.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4863. *Proposed by W. F. Stinespring, Institute for Advanced Study*

For what constant k (if any) can the inequality

$$|A + B| \leqq k(|A| + |B|)$$

be established for all n by n real symmetric matrices ($n \geqq 2$)? Here the notation $|A|$ means the positive square root of A^2, and the relation $\leqq$ between two matrices means that their difference is nonpositive definite.

4864. *Proposed by D. J. Newman, Massachusetts Institute of Technology and AVCO Research and Development*

Let a coin be tossed repeatedly, and let p be the probability that at some point the number of heads exceeds twice the number of tails. Prove

$$p = \tfrac{1}{2}(\sqrt{5} - 1).$$

4865. *Proposed by L. Lewin, Enfield, England*

Defining the inverse tangent integral of the second order by

$$Ti_2(x) = \int_0^x \frac{\tan^{-1}(x)}{x}\,dx = \frac{x}{1^2} - \frac{x^3}{3^2} + \frac{x^5}{5^2}\cdots,$$

prove that $\quad 6Ti_2(1) - 4Ti_2(1/2) - 2Ti_2(1/3) - Ti_2(3/4) = \pi \log 2.$

4866. *Proposed by F. D. Parker, University of Alaska*

Show that a semi-magic matrix A (the sums of the rows and columns are all equal) can be decomposed into a sum $B+C$ such that for integral K,

$$(B+C)^K = B^K + C^K.$$

4867. *Proposed by Ky Fan, Oak Ridge National Laboratory*

Let f be a measurable function such that $0<f(x)\leqq 1/2$ on $[0, 1]$. Prove

$$\frac{\exp \int_0^1 \log f(x)dx}{\int_0^1 f(x)dx} \leqq \frac{\exp \int_0^1 \log (1-f(x))dx}{\int_0^1 (1-f(x))dx}.$$

4868. *Proposed by Ronald Pyke, Columbia University*

Show that for all $\beta \leqq e^{-1}$ and for all real x the function f defined by $f(0, \beta)=1$ and

$$f(x,\beta) = x\sum_{j=0}^{\infty} \beta^j \frac{(j+x)^{j-1}}{j!} \qquad (x \neq 0)$$

is finite and satisfies the relationship

$$f(x, \beta) = e^{x\beta f(1,\beta)}.$$

SOLUTIONS

Matrices Having a Given Multiplication Table

3099 [1924, 455]. *Proposed by S. A. Corey.*

Let H, I, J, and K be the complex quaternion units,

$$\tfrac{1}{2}\{1-j-\theta(i+k)\}, \qquad \tfrac{1}{2}\{1+j+\theta(i-k)\},$$
$$\tfrac{1}{2}\{1+j-\theta(i-k)\}, \qquad \tfrac{1}{2}\{1-j+\theta(i+k)\},$$

respectively, which have the multiplication table

	H	I	J	K
H	H	I	H	$-I$
I	H	I	$-H$	I
J	J	$-K$	J	K
K	$-J$	K	J	K.

Prove that there exists a set of four matrices, involving no imaginaries, which have the same multiplication table.

Solution by D. C. B. Marsh, Colorado School of Mines. Using known correspondences of $a+bi+cj+dk$ and $x+\theta y$ to

$$\begin{pmatrix} a & b & c & d \\ -b & a & -d & c \\ -c & d & a & -b \\ -d & -c & b & a \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} x & -y \\ y & x \end{pmatrix},$$

respectively, we can establish composite 8 by 8 matrices with real components which have the required multiplication table. One set follows:

$$H = \tfrac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & -1 & -1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 & -1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}, \qquad I = \tfrac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & -1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & -1 & -1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & -1 & 0 & 1 \end{pmatrix},$$

$$J = \tfrac{1}{2}\begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & 1 & -1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & -1 & 0 & -1 \\ 0 & -1 & 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & -1 & 0 & 1 & 0 \\ -1 & 0 & -1 & 0 & 0 & -1 & 0 & 1 \end{pmatrix}, \qquad K = \tfrac{1}{2}\begin{pmatrix} 1 & 0 & -1 & 0 & 0 & -1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & -1 & 0 \\ -1 & 0 & -1 & 0 & 0 & 1 & 0 & -1 \\ 0 & 1 & 0 & -1 & 1 & 0 & 1 & 0 \\ -1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \end{pmatrix}.$$

Linear Independence

4797 [1958, 452; 1959, 322]. *Proposed by D. J. Newman, Massachusetts Institute of Technology and A VCO Research and Development*

Prove that all expressions like $7\sqrt{19}/4-3\sqrt{7}+8\sqrt{6}/5$ are irrational. More specifically, prove that the square roots of the square-free integers are linearly independent over the rationals.

II. *Editorial Note.* P. T. Bateman has raised the objection to the solution printed in the April issue that the c_i appearing in formula (3) may be all zero and that a proof to the contrary is not given. A similar oversight seems to exist in each of the other contributions to this problem, so that we have no solution available at this time.

Bateman also notes that the result of the problem is contained in Theorem 2 and its Corollary in A. S. Besicovitch, *Linear independence of fractional powers of integers*, Proceedings of the London Mathematical Society, XV (1940), pp. 3–6.

Reducibility of a Polynomial

4815 [1958, 713]. *Proposed by N. S. Mendelsohn, University of Manitoba*

Let $a_1, \cdots, a_n$ be distinct integers. Discuss the reducibility of the polynomial $f(x)=(x-a_1)\cdots(x-a_n)+1$ over the field of rationals.

Remarks by Hans Rademacher, University of Pennsylvania. This problem is 50 years old and due to I. Schur. A solution is printed in Pólya-Szegö, *Aufgaben und Lehrsätze*, vol. 2, VIII, pp. 346–7. Many generalizations have been discussed in the meantime. References can be found in Brauer and Ehrlich, *On the irreducibility of certain polynomials*, Bull. Amer. Math. Soc., vol. 52, 1946, pp. 844–856.

Also solved by P. T. Bateman, J. H. Hodges, D. C. B. Marsh, W. F. Trench, and the proposer. The result is: the polynomial is irreducible except when $n=2$ and a_1 and a_2 differ by 2, and when $n=4$ and a_1, a_2, a_3, a_4 are in arithmetic progression with common difference 1.

An Integral Equation

4816 [1958, 779]. *Proposed by M. S. Klamkin, AVCO Research and Development, Wilmington, Mass.*

Solve the integral equation

$$\int_0^\infty t^3\phi(x-t)dt = a\left\{\int_0^\infty t^2\phi(x-t)dt\right\}^b,$$

where a and b are independent of x.

Solution by Robert Weinstock, University of Notre Dame. Introducing the change of variable $u=t-x$ and defining F by

$$F(x) = \int_0^\infty t^2\phi(x-t)dt = \int_{-x}^\infty (u+x)^2\phi(-u)du, \tag{1}$$

we may differentiate the integral equation and obtain

$$3\int_{-x}^\infty (u+x)^2\phi(-u)du = ab\left\{\int_{-x}^\infty (u+x)^2\phi(-u)du\right\}^{b-1}\cdot 2\int_{-x}^\infty (u+x)\phi(-u)du,$$

that is

$$3 = ab[F(x)]^{b-2}F'(x) = ab(d/dx)\{(b-1)^{-1}[F(x)]^{b-1}\},$$

provided $b\neq 1$; the case $b=1$ is handled separately below. We introduce $m=b/(1-b)$ and integrate to obtain

$$F(x) = [c-(3/am)x]^{-m-1}, \tag{2}$$

where c is an arbitrary constant.

Differentiating (1) three times and using (2) we obtain

$$\phi(x) = \tfrac{1}{2}F'''(x) = \tfrac{1}{2}(3/am)^3(m+1)(m+2)(m+3)[c-(3/am)x]^{-m-4}.$$

For the existence of the integrals involved we must have $m>0$ (whence $0<b<1$), $a>0$, and $x<amc/3$.

In case $b=1$, the differential equation for F reads $F'(x)=(3/a)F(x)$, whence $F(x)=2(a/3)^3Be^{3x/a}$, where B is an arbitrary constant. We then have $\phi(x)=\frac{1}{2}F'''(x)=Be^{3x/a}$, which satisfies the integral equation for $b=1$, provided $a>0$, for all x.

If $a=0$, there is clearly only the trivial solution $\phi(x)=0$.

Also solved by H. E. Fettis, Emil Grosswald, C. C. Yalavigi and the proposer.

Entire Functions

4817 [1958, 779]. *Proposed by I. S. Gál and Stanley Kaplan, Cornell University*

Determine all entire functions $f(z)$ such that $|f(z)|=1$ whenever $|z|=1$.

Solution by C. Goffman and M. Henriksen, Purdue University. Write $f(z)=z^ng(z)$, where n is a nonnegative integer, g is entire, and $g(0)\neq 0$. Since $|z|=1$ implies $|g(z)|=1$, either g is constant, or the open mapping g sends the unit circle onto itself. In the latter case, the Schwarz reflection principle tells us that if z_1 and z_2 are collinear with the origin and $|z_1z_2|=1$, then

$$|g(z_1)g(z_2)| = 1.$$

(*cf.*, *e.g.*, Bieberbach, *Conformal Mapping*, New York, 1953, p. 76). It follows that g is bounded on a neighborhood of infinity. So, by Liouville's theorem, g is a constant.

Hence $f(z)=az^n$, where $|a|=1$.

Also solved by I. N. Baker, P. T. Bateman, Harley Flanders, D. S. Greenstein, Benoit Lachapelle, W. F. Trench, and the proposers.

A Property of the Miquel Circle of a Complete Quadrilateral

4818 [1958, 779]. *Proposed by Hüseyin Demir, Zonguldak, Turkey*

Let d_i be the sides of a complete quadrilateral, and A_{ij} be the vertex on d_i, d_j. Let t_i be the triangle formed by the sides other than d_i, and (O_i) denote the circumcircle of t_i. Denote the Simson line of a point S_i of (O_i) with respect to t_i by D_i.

Then prove that, if D_i and d_i are parallel for all i, (1) the line S_iO_p passes through the vertex A_{qr} $(i,\ p\neq q,\ r)$, and (2) the points S_i all lie on the Miquel circle (O).

Solution by the proposer. (1) Let the projections of S_i and O_p on d_r be denoted by U_{ir}, V_{pr}. Then, using directed angles, we have

$$\begin{aligned} \measuredangle V_{pr}O_pA_{qr} &= \measuredangle A_{ir}A_{qi}A_{qr} && \text{(from the circle } (O_p)), \\ &= \measuredangle U_{ir}U_{iq}A_{qr} && \text{(since } D_i \text{ is parallel to } d_i), \\ &= \measuredangle U_{ir}S_iA_{qr}. && \text{(from the circle } S_iU_{ir}A_{qr}U_{iq}). \end{aligned}$$

Now, since O_pV_{pr} is parallel to S_iU_{ir}, we get the required collinearity of S_i, O_p, A_{qr}.

(2) Since the line of centers O_pO_q is perpendicular to the radical axis FA_{pq}, where F is the Miquel point, we have successively

$$\begin{aligned} \measuredangle\, O_pO_rO_q &= \measuredangle\, A_{qi}FA_{pi}, \\ &= \measuredangle\, A_{qr}A_{pq}A_{pi} && \text{(from the circle } (O_r)), \\ &= \measuredangle\, A_{qr}A_{pq}A_{pr}, \\ &= \measuredangle\, A_{qr}S_iA_{pr} && \text{(from the circle } (O_i)), \\ &= \measuredangle\, O_pS_iO_q, && \text{(from property (1))}, \end{aligned}$$

and S_i lies on the Miquel circle (O).

Also solved by A. E. Landry.

Sum of Five Distinct Squares

4819 [1958, 779]. *Proposed by P. T. Bateman, University of Illinois*

Suppose k is a given nonnegative integer. Show that every sufficient large positive integer is a sum of five squares none of which is less than k^2 and no two of which are equal.

Editorial Note. After having submitted an independent proof, the proposer finds stronger results obtained by E. M. Wright, using Vinogradov's method. [Quart. J. Math. Oxford Ser. (2), vol. 4, 1933, pp. 37–51.] Wright showed that every large positive integer can be expressed as a sum of five squares whose ratios are roughly prescribed in advance. This certainly implies the present theorem.

Also solved by L. Carlitz, using a result of Walfisz'.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

Mathematics in Business. By Lloyd L. Lowenstein. Wiley, New York, 1958. xv+364 pp., \$4.95.

"This text is for use in a first course in the mathematics of business. The intention is to give the student a firm foundation for further courses in the mathematics of finance as well as accounting, business statistics, insurance, and other business subjects." The book should be successful in accomplishing the stated aims of the author. Professor Lowenstein has a gift of writing in clear and simple, as well as concise, English. Apparently the students for whom this book

was written are those without exceptionally strong backgrounds in high school mathematics, and although the degree of rigor attained is superior to that in many similar books, the approach is not too subtle for his audience.

Chapters I and II review arithmetic and beginning high school algebra. This material is not an integral part of a course for those with normal preparation. Logarithms and the Binomial Theorem are treated in appendices whereas progressions are taken up when needed in the body of the text. The business mathematics chapters cover the topics: percentage; profit and loss; simple interest and discount; compound interest; depreciation; annuities with applications to amortizations, sinking funds, and bonds. The book is not intended to be a treatise on the theory of interest and Professor Lowenstein has achieved a splendid balance between theory and practice. More attention is given to simple interest methods than in some books.

Unusual features of the book are the listing of the answers to the odd-numbered exercises at the end of each chapter and very effective use of footnotes for clarification of theory. The text includes 61 pages of the usual financial tables. The logarithm table is only a five-place table but the logarithms of interest rates are given to seven decimal places.

C. L. SEEBECK, JR.
University of Alabama

Experimental Designs in Industry. Edited by Victor Chew. Wiley, New York, 1958. 268 pp. $6.00.

This is a collection of some of the papers given at a symposium held November 5–9, 1956, conducted by the Institute of Statistics of the University of North Carolina, State College Section, and sponsored by the Mathematics Division of the Air Force Office of Scientific Research.

The papers are of a technical, statistical nature but use relatively elementary mathematics. Up-to-dateness is the keynote. Parallel treatments of experimental design in general, factorial designs, and designs for exploring response surfaces are given via theoretical discussions, examples of applications, and extensive bibliographies. Ample detail is given so that the reader can understand the differences between experimental problems in industry and agriculture (the origin of many of the designs and the vocabulary) and so that the special techniques discussed here can be used by the working statistician.

I. RICHARD SAVAGE
University of Minnesota

A Modern Approach to Intermediate Algebra. By Henry A. Patin. Putnam's, New York, 1958. 224 pp. $3.75.

According to the preface, this text is a one-semester modern approach to intermediate algebra, with two major purposes. The first is "to acquaint the student with a logical system as it appears in mathematics." The preface also states that the topics and their order of presentation are fairly traditional and

that the presentation, although not rigorous, is coherent.

Thus, the major new contribution which this book purports to make is the development of algebra as a logical system. However, most of this is contained on pages 6 through 19, where the postulates for the integers, then the postulates for the rationals are stated and the basic rules of ordinary algebra are "proved." The treatment is very terse and abstract and is begun without warning, motivation, or any attempt to relate the laws to the student's past experience. The only hint that something is afoot is the assertion, "We will now state our fundamental assumptions." The next thing the reader sees is

$$\text{``}\ \boxed{1.1 \qquad a+b=b+a}\ \text{''}$$

Unlike certain recent texts on the collegiate level, the problems make no contribution toward preparing the student for the next abstract formulation, or in helpng him actually devise it; they are mere practice in identifying the laws. Here is one: "Show what laws have been used in $-3-2=(-3)+(-2) = -1(3+2)$." While a person acquainted with modern abstract algebra might like this presentation in chapter 1, the novice would probably find Chapters 1 and 2 of Birkhoff and MacLane more accessible.

Chapter 1 contains the author's major effort at infusing modern ideas into his work. Thereafter, the book grows more traditional, chapter by chapter. By the time the one on coordinate geometry was written, modern ideas were abandoned completely. For example, the introduction of the concept of function is shockingly inadequate for a text published in AD 1958: "An algebraic expression in $x-3/2x+7/2$ is called an algebraic function of x." There is no definition beyond this.

There are few imaginative or challenging problems. Most could have been copied directly from the easier parts of traditional texts. Since one of this book's claims to the modern spirit is that excessive manipulation is avoided, the lists of problems contain only those involving easy manipulations. Consequently the book would not satisfy the teacher of a traditional course, who would consider it watered down.

The final chapter is a four and one-half page discussion of algebra as a deductive system. This might be clear enough for a mature reader, but is certainly impenetrable to most 11th-grade high school students.

All prospective authors of modern texts should read this work, since it indicates how much thought must be given to a pioneering work, by its many evidences of having been poorly thought out: little or no motivation, off-the-cuff lists of problems, language unsuited to the level of 11th-grade students, pretentiously abstruse proofs, introduction of concepts such as *variable* and *function* without actual definitions, etc.

Another point strongly made by this book is that the writing of a modern text should *not* be undertaken unless the author has made contemporary think-

ing in all branches of elementary mathematics a part of his working knowledge.

One final comment. This book could be used in the second semester of a traditional 9th-grade course.

The type, paper and layout are superior; for this the publishers should be commended.

ED WALTERS
William Penn High School
York, Pennsylvania

Calculus. By Walter Leighton. College Mathematics Series. Allyn and Bacon, Boston, 1958. x+373 pages. $6.95.

The publication of this new volume in the College Mathematics Series follows by approximately a year that of the textbook *Calculus with Analytic Geometry* by Johnson and Kiokemeister in the same series. The Johnson and Kiokemeister text pleased many instructors, but had the disadvantage of being too difficult for most students. It contained perhaps an excessive emphasis on rigor. The present text goes too far in the other direction, and the author omits proofs which he should include. No proofs are given for the standard limit theorems, or for the existence of the definite integral. The proof that $\lim_{x\to 0} (\sin x)/x = 1$ is based upon the assumption that $\cos x$ is continuous at $x = 0$, and the author does not justify this assumption. The proof given for the Chain Rule is the one which most modern instructors try to avoid, and the lack of generality of this proof is pointed out by the author in a Critique. The author does not empashize that the derivative of a function $f(x)$ is taken with respect to x. Although the definition of a derivative is given on page 14, the idea that we differentiate $f(x)$ with respect to the variable x is apparently not mentioned until page 58, after the discussion and proof of the Chain Rule and the subsequent introduction of the differential. In short, the able student will feel that there is material which he needs, but which the author has not given him. His discouragement will be increased when he consults the references listed at the end of Chapter 1, and finds them far beyond him. An excellent reference text for this course is the Johnson and Kiokemeister book referred to above.

In view of the number of proofs which are omitted, it seems unnecessary to include an outline of "the basic algebraic theory of the representation of a rational function in terms of partial fractions" (see pp. 183–186).

Much of the discussion is too scanty. The chapter on Solid Analytical Geometry contains no adequate discussion of any surfaces except the plane, or any curves except the straight line, and contains not a single problem on quadric surfaces. The discussion of partial derivatives of functions of functions is rushed, and the reader is given the impression that if he wishes to learn the subject adequately he should consult other textbooks. Infinite series are motivated by Taylor's formula with the remainder. No discussion is given of operations with series. The book contains a short chapter on differential equations.

The writing style is good, and the book will be easy for most students to read. The problems appear in general to be well chosen. Problem 7, page 38 is incorrectly stated, while most students will abandon Problem 5, page 50 before they see the point to it.

DICK WICK HALL
Harpur College,
State University of New York

A First Course in Statistics. By R. Loveday. Cambridge University Press, London, 1958. x+121 pp. $1.75.

This is a nonmathematical introduction to descriptive statistics. It does not go very far nor does it give a very full discussion of any of the topics it contains. It does have worked-out illustrations that might be useful for an untrained person operating a desk calculator. It would not be useful in a course in statistics having any mathematical prerequisite. In the words of the author, "It may . . . prove helpful to students . . . who require an elementary introduction to the subject before embarking upon a more mathematical treatment."

WILLIAM G. MADOW
Stanford Research Institute

Some Aspects of Multivariate Analysis. By S. N. Roy. Wiley, New York, 1957. viii+214 pp. $8.00.

In the words of the author, this monograph does not attempt to cover the entire area of multivariate analysis but, rather, to deal with some important developments with which he has been associated during recent years. In particular, he omits a general discussion of such topics as factor analysis, classification problems and the multivariate counterpart of analysis of variance. Some of this omitted material will be treated in a future monograph.

The first two-thirds of the book deals mainly with obtaining confidence bounds for certain parametric functions. The testing of hypotheses in so far as it is developed is mainly a means to this end. The parametric functions under consideration are a set of natural measures of departure from the usual null hypothesis, there being a single such function in some cases and a set of such functions in more complicated cases.

The last third of the book consists of appendixes concerning matric theory, quadratic forms, linear transformations, jacobians and integration theory. The last two pages contain fifty-five references.

The author is to be commended for the excellence of this monograph. The printing is also very good. It is somewhat regrettable however that it is published on $8\frac{1}{2}'' \times 11''$ paper rather than a more standard size.

PAUL M. HUMMEL
University of Alabama

Scientific Programming in Business and Industry. By Andrew Vazsonyi. Wiley, New York, 1958. xix+474 pp. $13.50.

In this book the author states in the preface "what he (the reader) *can* expect to get is a thorough understanding of what scientific programming is and what it means to the business world." The book is addressed to the business man and the writer claims success in presenting the material to business men and confidence that this book, seriously studied, is suitable for business men. While the reviewer agrees that there are some business men who may read this book he feels that it is by far a better book for a college graduate well grounded in statistical theory and mathematics than it is for the average business man of his acquaintance.

The book contains chapters on mathematical models, three on linear programming, one each on convex programming, dynamic programming and game theory. Several chapters on various kinds of applications are also included. Among these are statistical inventory control, transportation allocation, and three chapters on scheduling. The treatment is expository and the author goes to some length to point out problems not solved by the devices at hand. While matrices are avoided generally the author does not explain notations for sums nor his nonstandard use of $x_{1,1}$ for x_{11} for example. Hence the book will not be simple for anyone not familiar with certain concepts of linear equations and finite dimensional linear spaces. The author is to be commended for the care with which he states assumptions.

The book should be valuable for college and high school teachers of mathematics who need to have some inkling of the applications of mathematics. It should be pointed out that the kind of applications given have been carried out mainly by industrial giants and not by small business men. Scientific programming is still in its infancy as far as both theory and practice go.

PRESTON C. HAMMER
University of Wisconsin

Applied Mathematics for Engineers and Physicists. (2nd ed.) By Louis A. Pipes. McGraw-Hill, New York, 1958. xi+723 pp. $9.50.

This well-written text is for the student of engineering or physics. It contains an amazing number of topics in a clear, concise style, thus making it a possible introductory source book for mathematical methods applicable to the engineering sciences. It does not treat topics depending on probability. It leaves the proof of basic mathematical theorems and the elaboration of special cases to mathematical analysis and advanced treatments. Much of it is at a slightly higher level than that of the student just out of a meager calculus course. There is considerable new material not found in the first edition.

The first chapter covers infinite series up through uniform convergence on a slightly higher level than the beginning calculus. Theorems are well stated. Dis-

cussion and applications bring out clearly their significance. The use of operators on numerous occasions is started here with the exponential operator form for Taylor's series. Each chapter closes with exercises and references.

A short chapter giving the elementary functions of complex numbers prepares for the development in the following chapter of Fourier series and the Fourier integral in complex form.

The fourth chapter treats linear algebraic equations, determinants, and matrices. The elementary properties of determinants are covered very rapidly. A numerical method due to F. Chio for evaluating large determinants is given. The work on matrices is far more complete than expected. Silvester's theorem permits the elegant solution of a system of linear differential equations in terms of the exponential function of a matrix. The confluent cases, where this method becomes especially valuable, is left to the literature.

Following a chapter on Newton's and Graeffe's method for solving equations comes one on linear differential equations with constant coefficients. Chapters 7, 8, and 9 are on "Oscillations of Linear Lumped Electrical Circuits," "Vibrations of Elastic Systems with a Finite Number of Degrees of Freedom," and "The Differential Equations of the Theory of Structures." Here is a profusion of applications of the earlier methods and some extensions. For example, one finds the analysis of a general network in matrix notation using Laplace transforms, Lagrange's equations, stability conditions, and Rayleigh's method for calculating frequencies.

For systems having repeated components and for numerical work there is a treatment of finite differences and difference equations. The work on partial differentiation has a discussion of maxima and minima, including Lagrange multipliers, and the evaluation of some definite integrals. Material on gamma, beta, and error functions prepare for Bessel functions and Legrendre Polynomials. Legendre functions of the second kind are omitted.

In "Vector Analysis" there is a clear discussion of curvilinear coordinates. Gauss's theorem and Stokes's theorem are used to find divergence and curl in curvilinear systems. The fundamental equations of hydrodynamics and of heat flow in solids are developed. The chapter closes with an introduction to the basic tensors and their applications.

The next three chapters cover the wave equation, Laplace's equation, and the equation of heat conduction.

The first chapter on complex variables goes through Laurent expansions, use of residues, and the argument principle. The other carries mapping through Schwartz's transformation and the applications of conjugate functions to potential and hydrodynamic problems.

Chapter 21 gives a more extended treatment of Laplace transforms. The p-multiplied form is used so as "to retain the operational forms of Heavyside." A heuristic treatment of the Fourier-Mellin theorem permits finding inverse transforms by residues. There are applications to the impulsive functions, partial differential equations, and integral equations. An appendix gives the basic

working theorems and a ten-page table of the transforms "most frequently occurring in the transient solution of electrical problems."

Some methods of treating nonlinear oscillatory problems are given in the last chapter. Some of the topics are "Forced Vibrations of Nonlinear Systems," "The Method of Kryloff and Bogolieuboff," and "Relaxation Oscillations." This 61-page chapter closes with 20 references, the most recent having a 1956 date.

It is a very useful book.

EARL LAFON
University of Oklahoma

Polynomial Expansions of Analytic Functions. By R. P. Boas, Jr. and R. C Buck. Springer-Verlag, Berlin, 1958. viii+77 pp. 19.80 DM.

This little book differs from most items in the Ergebnisse series in that, rather than being a survey of a field, it is a semi-expository presentation of the authors' approach to a topic.

By a basic set $\{p_n\}$ of polynomials is meant a Hamel basis of the vector space of all polynomials. This book deals with classes of analytic functions f which have expansions of the form $f(z)=\sum a_n p_n(z)$, where the sequence $\{a_n\}$ is defined in a prescribed manner. One such procedure, exploited by Whittaker and others, consists simply of substitution in the power series for f, so that $a_n=\sum \pi_{kn} f^{(k)}(0)/k!$, where $z^k=\sum \pi_{kn} p_n(z)$, the last summations all being finite since $\{p_n\}$ is a basic set. The authors' point of view allows other possibilities for the coefficients. The main idea is that of kernel expansion, which we describe briefly.

Given $\{p_n\}$, let $\{u_n\}$ be a sequence of functions, and define the kernels $K(z, w)=\sum p_n(z)u_n(w)$. Let T: $F \rightarrow f$ be a function-to-function-transformation defined by $f(z)=1/(2\pi i)\int_\Gamma K(z, w)F(w)dw$, Γ being a suitably chosen path. Then $f(z)=\sum a_n p_n(z)$, where $a_n=1/(2\pi i)\int_\Gamma u_n(w)F(w)dw$. The simplest example is that for which $p_n=z^n$ itself. Two classical choices for u_n are $u_n=w^n$ and $u_n=w^n/n!$, so that the respective kernels are $(1-zw)^{-1}$ and e^{zw}. With proper choice of F, these kernels yield the Taylor series and the Pólya representation.

The method is applied to a wide class of basic sets which the authors call generalized Appell polynomials. Many of the polynomial sets in the literature are special cases. Particular attention is given to Appell, Sheffer, and Brenke-Huff polynomials. Expansion theorems are obtained involving Borel and Mittag-Leffler summability as well as convergence. The form of the typical theorem is to characterize the class of functions for which the particular type of polynomial expansion converges to the function in the specified sense.

While the book is by no means light reading, it should be within the scope of a large segment of the mathematical public. In particular, it should make an excellent seminar topic.

CASPER GOFFMAN
Purdue University

Introductory Mathematical Analysis. By Edgar D. Eaves and Robert L. Wilson. Allyn and Bacon, Boston, 1958. xv+424 pp. $5.95.

This book is intended to serve the needs of freshmen enrolled in a terminal course meeting three hours a week for one academic year. The students are expected to have completed at least one year of high school mathematics; however, because so much is usually forgotten before the students begin their college mathematics, the book opens with a chapter of 37 pages providing a rather thorough review of the fundamental operations of arithmetic and algebra. The following seven chapters are devoted to an exposition of the concepts of function, instantaneous rates, differentiation and integration, with additional topics in algebra introduced when they are needed. Then follow chapters on trigonometry, simple and compound interest, the exponential and logarithmic functions and their derivatives, sequences, progressions and annuities, and a brief chapter on probability.

The text is well written in a style that is both lucid and thorough. The authors amply demonstrate their teaching skill by anticipating students' difficulties and by the care with which definitions and theorems are formulated. In the reviewer's estimation the authors can be commended for their good judgment in the selection of applications likely to interest students at this level and for achieving a standard of rigor considerably higher than in most textbooks of this category. The exercises, however, are nearly all quite simple, adequate for the run-of-the-mill student, but not sufficiently challenging to the abler ones.

J. M. Feld
Queens College

Calculus for Electronics. By A. E. Richmond. McGraw-Hill, New York, 1958. viii+407 pp. $6.00.

This book is designed for use as a mathematics text in technical institutes and training-in-industry programs. As the title indicates, it is primarily concerned with applications. Very few of the fundamental concepts of the calculus are described in terms sufficiently precise to be understandable. Many of its mathematical assertions are too vague to be called statements; a goodly number are false if taken literally. In summary, it is not a mathematics book.

If it is possible at all for a student to learn the basic techniques of the calculus from this book, he can do so only by working his way through the large number of illustrative examples and exercises provided by the author. His fear of the calculus may be allayed by the fact that most of these examples are selected from the subject matter of electronics. Can a student learn the calculus well enough to apply it only by learning how to do type-problems? This reviewer doubts it.

M. Henriksen
Purdue University

Studies in the Mathematical Theory of Inventory and Production. By Kenneth J. Arrow, Samuel Karlin, Herbert Scarf. Stanford University Press, Stanford, 1958. x+340 pp. $8.75.

This book is a collection of papers exceptionally well inter-related. In part I, after giving a historical background, the nature and structure of the inventory problem are discussed. Part I ends with extensive summaries of the remaining fourteen chapters. Part II considers the finding of optimal policies in deterministic inventory processes. Part III discusses the same for stochastic inventory processes. Part IV concerns operating characteristics of inventory policies.

The mathematical tools used range from statistics through the calculus of variations, including differential equations and programming.

As the authors state in Chapter 2, they are interested only in existence theorems, etc. as long as they "are useful tools toward arriving at effective solutions," that is "effectively computable solutions."

This book is a welcome addition to the literature of applications of mathematics in business.

ALBERT NEWHOUSE
University of Houston

An Introduction to the Theory of Integration. By A. C. Zaanen. Interscience, New York, 1958. ix+254 pp. $7.25.

The author is to be commended for succeeding in the difficult task of writing a good book, of readable length, on modern integration. One of its most welcome and noteworthy features, which adds greatly to its value and usefulness, is its inclusion of certain related topics from classical and functional analysis, in which the theory has applications.

In developing the subject, the author tends to strike a middle position between the completely measure-theoretic approach and the more abstract treatment in which the integral is introduced as a linear functional. He treats the basic construction and extension problem in the following fashion. The existence of "elementary" integrals of the Daniell type is established by the methods of measure theory. The problem of extending an arbitrary elementary integral, defined on a linear lattice of real valued functions, is then reduced by the use of ordinate sets, to the problem of extending measures on semi-rings to the σ-rings they generate; and this latter problem is solved by the classical technique of outer measures. As a matter of personal taste, this reviewer would have preferred a treatment independent of ordinate sets.

After proving the principal properties of the extended integral and discussing the various relations between integrals derived abstractly from linear functionals and those arising more concretely from measures, the author proves Fubini's theorem in the form given by Stone. He then gives some applications of the theorem by establishing some results on integration by parts, the gamma function, and fractional integration.

Numerous other topics from analysis and functional analysis, including chapters on ergodic theory, unitary transformations on L_2, the Radon-Nikodym theorem, and Lebesgue's theorem on the differentiation of an indefinite integral, to mention a few, are also in the book. In addition, the book contains an excellent collection of exercises.

R. A. KUNZE
Massachusetts Institute of Technology

Zahlwort und Ziffer, Bd. II. By Karl Menninger. Vandenhoeck and Ruprecht, Gottingen, 1958. 314 pp. $6.00 (D.M. 24.80) Paperbound.

This is the second volume comprising the second part of the author's original treatise, *A Cultural History of Number*. The material of this book can be found scattered in other books and journals. The present treatment is a thorough, scholarly, and very readable coherent logical-historical development. The history of the development of number symbols and computation, both as things and as written symbolic manipulations, becomes an exciting drama, which is made all the more vidid by hundreds of diagrams, photographs, and drawings.

The treatise begins with the use of fingers, both to represent numbers up to 100,000, and to do simple operations on numbers. This is followed by a chapter on the way common people represented number by sticks, ropes, and other material things, leading to the use of letter symbols and finally unique numerical symbols. The way in which abaci and counting boards paved the way to the various algorisms performed with written symbols completes the story for the Western world. A final chapter treats the development of number language and number writing in China and Japan.

Although written in German, it is simple, yet elegant in style, and can be read easily by a second-year student of the language. Volumes I and II, bound together, are now available for $9.50 (D.M. 40.00).

HOWARD F. FEHR
Teachers College, Columbia University

BRIEF MENTION

Planning of Experiments. By D. R. Cox. Wiley, New York, 1958. vii+308 pp. $7.50.

While this book is not, in any sense, a mathematical or statistical book, it certainly provides worthwhile reading for the mathematician and statistician, as well as being an interesting reference for nonmathematically minded colleagues, who are always with us.

Theories of Figures of Celestial Bodies. By Wenceslas S. Jardetzky. Interscience, New York, 1958. xi+186 pp. $6.50.

A mathematical treatment of the shape of a rotating body.

Theoretical Mechanics: An Introduction to Mathematical Physics. By J. S. Ames and F. D. Murnaghan. Dover, New York, 1958. ix+462 pp. $2.00.

A republication of the 1929 volume by Ames and Murnaghan.

An Introduction to Fourier Methods and the LaPlace Transformation. By Philip Franklin. Dover, New York, 1958. x+289 pp. $1.75.

A corrected edition of the 1949 book *Fourier Methods* by the same author.

Introduction to Bessel Functions. By Frank Bowman. Dover, New York, 1958. x+135 pp. $1.35.

A reprinting of Bowman's book published in the late 1930's.

The Theory of the Potential. By William Duncan MacMillan. Dover, New York, 1958. xi+469 pp. $2.25.

A welcome republication of MacMillan's 1930 volume.

The Conduction of Heat in Solids. By H. S. Carslaw and J. C. Jaeger. Oxford University Press, New York 16, N. Y., 1959. viii+510 pp. $13.45.

This revised edition of the well-known standard by Carslaw and Jaeger will be welcomed by all. Call it to the attention of your engineering and physics faculty.

Dynamics and Nonlinear Mechanics. By E. Leimanis and N. Minorsky. Volume II, Surveys in Applied Mathematics. Wiley, New York, 1958. xii+206 pp. $7.75.

This book is divided into two portions, the first of which deals with recent advances in the dynamics of rigid bodies, celestial mechanics, and mathematical exterior ballistics, by Leimanis. The second portion by Minorsky deals with the theory of oscillations, including relaxation theory. Both portions conclude with up-to-date bibliographies.

Dictionary of Astronomy and Astronautics. By A. Spitz and F. Gaynor. Philosophical Library, New York, 1959. vi+439 pp. $6.00.

The understandable definitions in this volume are not limited to a few lines, but often run to half or even a full page discussion.

Vector Analysis with an Introduction to Tensor Analysis. By A. P. Wills. Dover, New York, 1958. xxxii+285 pp. $1.75.

A republication of Wills' 1931 text.

The Fundamental Principles of Quantum Mechanics with Elementary Applications. By Edwin C. Kemble. Dover, New York, 1958. xviii+611 pp. $2.95.

A reprinting of Kemble's 1937 expansion of an earlier work.

Quantum Electrodynamics. Julian Schwinger, Ed. Dover, New York, 1958. xvii+424 pp. pp. $2.45.

A collection of papers which appeared in various journals during the first half of the present century. Well selected, but unfortunately some of the print is too small for comfortable reading.

Statics and the Dynamics of a Particle. By William Duncan MacMillan. Dover, New York, 1958. xviii+430 pp. $2.00.

A reprint of MacMillan's 1927 book, which was originally designed as a text book in mechanics.

A Compendium of Mathematics and Physics. By Dorothy S. Meyler and O. G. Sutton. Van Nostrand, Princeton, N. J., 1958. x+384 pp. $5.00.

A collection of formulae from elementary mathematics, physics, and statistics.

Linear Groups with an Exposition of the Galois Field Theory. By Leonard Eugene Dickson. Dover, New York, 1959. xvi+312 pp. $1.95.

Since Dickson's book was written in 1900, Finite or Galois Field Theory has undergone extensive development. Many of the proofs given here could be materially shortened and the modern approach would be somewhat different. In spite of the work by Wedderburn, Chevalley, Dieudonné, and Artin, this work of Dickson's is still eminently readable and very much worthwhile. It could well be required reading for every beginning graduate student.

The Fourier Integral and Certain of Its Applications. By Norbert Wiener. Dover, New York, 1959. xi+201 pp. $1.50.

This reprint of the Cambridge University Press 1933 publication will be of particular interest to those who wish to study *Cybernetics* or *Time Series* by the same author, since much of the related mathematics is given here.

College Mathematics for Freshmen (2nd ed.). By P. K. Smith and H. F. Schroeder. Van Nostrand, Princeton, N. J., 1959. x+314 pp. $4.25.

This revision of the authors' 1948 work retains much of the flavor of the original book. The title is somewhat misleading, since the content is clearly remedial mathematics.

Lectures on the Theory of Elliptic Functions. By Harris Hancock. Dover, New York, 1958. xxiii+498 pp. $2.55.

A reprint of Hancock's 1909 book by the same title.

Elliptic Integrals. By Harris Hancock. Dover, New York, 1958. 104 pp. $1.25.

A reprint of Hancock's 1916 monograph.

The Foundations of Euclidean Geometry. By Henry George Forder. Dover, New York, 1958. xii+349 pp. $2.00.

The current interest in high school geometry has probably spurred the reprinting of this 1927 work.

The Elements of Non-Euclidean Geometry. By D. M. Y. Sommerville. Dover, New York, 1958. xvi+274 pp. $1.50.

A reprint of Sommerville's 1914 lectures.

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

IBM COMBINATORIAL PROBLEMS INSTITUTE

An institute on Combinatorial Problems was sponsored by IBM Corporation at its Lamb Estate Research Center, near Ossining, New York, from July 6 to August 14,

1959. The institute was devoted to studies and seminars in mathematical programming, combinatorial design problems, game theory, and switching theory. The following accepted IBM's invitation to attend: Professor A. W. Tucker, Chairman, Princeton University, Dr. G. B. Dantzig, The RAND Corporation, Professor M. M. Flood, University of Michigan, Dr. D. R. Fulkerson, The RAND Corporation, Professor R. E. Gomory, Princeton University, Dr. A. J. Hoffman, General Electric Company, Dr. E. F. Moore, Bell Telephone Laboratories, Mr. J. Riordan, Bell Telephone Laboratories, Professor H. J. Ryser, Ohio State University, Professor C. B. Tompkins, UCLA.

The results of the Institute will be presented at a symposium, to be held at Princeton University in the fall of 1959, jointly sponsored by the Society for Industrial and Applied Mathematics and IBM Corporation.

NSF CONTINUES PROGRAMS

The National Science Foundation announces that it will continue to support the following programs during the summer of 1960 and the academic year 1960–61:

Undergraduate Research Participation
Research Participation for Teacher Training
Science Training for Secondary School Students

Colleges, universities, and other educational institutions desiring to consider participation in these programs may receive full information by addressing: Special Projects in Science Education, Scientific Personnel and Education, NSF, Washington 25, D. C.

PRELIMINARY ACTUARIAL EXAMINATION PRIZE AWARDS

The winners of the prize awards offered by the Society of Actuaries to the nine undergraduates ranking highest on the score of Part 2 of the 1959 Preliminary Actuarial Examination are as follows:

First prize of $200: Harold M. Stark, California Institute of Technology.

Additional prizes of $100 each: Stephen L. Alder, Harvard University; William G. Brown, University of Toronto; Richard M. Dudley, Harvard University; David D. Grossman, Harvard University; Bertrand I. Halperin, Harvard University; Ralph E. Miller, Harvard University; Steven A. Orszag, Forest Hills (N. Y.) High School; Barry Wolk, University of Manitoba.

The Society of Actuaries has authorized a similar set of nine prizes for the 1960 examination on Part 2.

The 1960 Preliminary Actuarial Examinations will be administered by the Society of Actuaries at centers throughout the United States and Canada on May 11, 1960 and on November 16, 1960. The closing date for the May examinations is April 1, 1960, and for the November examinations, it is October 1, 1960. Further information concerning these examinations may be obtained from The Society of Actuaries, 208 South LaSalle Street, Chicago 4, Illinois.

AFFILIATE MEMBERSHIP IN IRE COMPUTER GROUP

Members of the Mathematical Association of America are cordially invited to affiliate with the Professional Group on Electronic Computers of The Institute of Radio Engineers. The Professional Group on Electronic Computers is an association of IRE members and affiliates with professional interest in the fields of analog and digital electronic computers. The group is concerned with the advancement of the electronic computer field and serves to aid in promoting close cooperation and exchange of technical information among its members.

The fee is $6.50 per year and includes the quarterly Transactions of this professional group. Application forms may be obtained from The Institute of Radio Engineers, 1

East 79th Street, New York 21, New York. Write attention of Mr. L. G. Cumming, Technical Secretary.

NEW JOURNAL

The Department of Commerce, through its Office of Technical Services (OTS), has started a new journal, TECHNICAL TRANSLATIONS. This journal appears twice a month and contains abstracts of foreign scientific publications (including mathematics) for which translations (in many cases from the Russian) are available through OTS. Further details may be obtained from Office of Technical Services, U. S. Department of Commerce, Washington 25, D. C.

PERSONAL ITEMS

Fresno State College: Professor Roy Dubisch will be on leave during the academic year 1959–60, and will be a Lecturer at the University of California, Berkeley; Professor G. D. Alkire will be Department Chairman (acting) during the absence of Professor Dubisch; Dr. W. L. Allen, Litton Industries, Santa Monica, California, Professor D. L. Boyer, New Mexico College of Agriculture and Mechanic Arts, and Dr. W. D. James, Stanford University, have been appointed Assistant Professors.

Seattle University: Dr. B. R. Toskey has been promoted to Assistant Professor; Dr. S. A. Husain, Purdue University, has been appointed Assistant Professor.

Associate Professor B. H. Arnold, Oregon State College, has been promoted to Professor.

Mr. P. L. Chessin, Westinghouse Electric Corporation, Baltimore, Maryland, has accepted a position as Applied Science Representative with the International Business Machines Corporation, Washington, D. C.

Mr. D. M. Estes, Baylor University, has accepted a position as Staff Assistant with the Southwestern Bell Telephone Company, Dallas, Texas.

Dr. T. S. Ferguson, University of California, Los Angeles, is on sabbatical leave and will spend the academic year at Princeton University.

Mr. F. W. Gibson, Douglas Aircraft Corporation, Santa Monica, California, has been appointed Graduate Research Mathematician with the Management Sciences Research Project, University of California, Los Angeles.

Dr. Seymour Ginsburg, National Cash Register Company, Hawthorne, California, has accepted a position as a member of the technical staff in the Information Processing Research Department of the Research Laboratories of the Hughes Aircraft Company.

Mr. R. B. Hall, Arizona State College, has accepted a position as Research Engineer with North American Aviation, Downey, California.

Associate Professor W. B. Houston, Jr., Morehouse College, has been appointed Assistant Professor at Carleton College.

Assistant Professor T. R. Jenkins, Washington State College, has accepted a position as Research Scientist with the Lockheed Aircraft Corporation, Missiles and Space Division, Sunnyvale, California.

Mr. J. C. Jones, University of Kentucky, has been appointed Teacher of Mathematics at the Cincinnati Public Schools.

Mr. P. J. Knopp, Harvard University, has been appointed Instructor at Spring Hill College.

Professor Stephen Kulik, Utah State University, has been appointed Professor at Long Beach State College.

Dr. Milton Lees, Institute for Advanced Study, has been appointed Visiting Assistant Mathematician at Brookhaven National Laboratories, Upton, Long Island, New York.

Assistant Professor Kenneth Loewen, Tabor College, has accepted a Graduate School Fellowship at The Pennsylvania State University for next year.

Mr. L. I. Lowell, Hughes Aircraft Company, Tucson, Arizona, has accepted a position as Research Engineer at the Missile Division of North American Aviation, Inc., Downey, California.

Dr. C. A. McCarthy, Yale University, has been appointed C. L. E. Moore Instructor in Mathematics at Massachusetts Institute of Technology.

Dr. Amin Muwafi, University of Florida, has been appointed Supervisor at the Ministry of Education, Amman, Jordan.

Mr. W. V. Neisius, Systematics, Inc., New York, has been promoted from General Manager to Vice-President.

Mr. C. C. Nielsen, Utah State Agricultural College, has accepted a position as Mathematician with the Hercules Powder Company, Magna, Utah.

Mr. Hugh Noland, Montana State College, has been appointed Assistant Professor at Chico State College.

Mr. G. W. Reitwiesner, Aberdeen Proving Ground, Maryland, has joined the Applied Mathematics Division of the National Bureau of Standards, Washington, D. C.

Dr. J. E. Wilkins, Jr., Nuclear Development Corporation of America, White Plains New York, has been promoted to Manager, Research and Development Operations. '

Associate Professor Fred Brafman, University of Oklahoma, died February 4, 1959. He was a member of the Association for seven years.

Associate Professor Myrtle C. Brown, North Texas State College, died March 8, 1959. She was a member of the Association for thirty-six years.

Professor Emeritus E. R. Smith, Florida State University, died March 21, 1959. He was a member of the Association for thirty-six years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

Professor H. M. Gehman, Secretary Treasurer, announces that the following 121 persons have been elected to membership by the Board of Governors on applications duly certified.

Emil Amelotti, M.S. (Illinois) Chairman of Dept., Villanova University

Capt. Edward Anlian, M.S. (S.U. of Iowa) Instr., United States Air Force Academy

Paul Anthony, M.A.B. (New York) Professor, New York City Community College

Pasquale J. Arpaia, B.S. (St. John's) Teacher, Woodland Junior High School, East Meadow, New York

Richard A. Baker, Student, Portland State College

Joe R. Ballard, B.A. (North Texas S.C.) Mathematician, National Bureau of Standards

William C. Beck, M.S. (S.U. of Iowa) Instr., Wisconsin State College

Victor Bik, B.S. (DePaul) Teacher, St. Stanislaus High School, Cleveland, Ohio

Mrs. Lois Blondeau, Student, Wisconsin State College

Alfred Blumstein, M.A. (Buffalo) Research Physicist, Cornell Aeronautical Lab.

Edward E. Bosman, M.A. (Columbia) Head of Dept., Barrett School, Cresco, Pennsylvania

Helen M. Bowden, M.A. (Wisconsin) Teacher, Bay Ridge High School, Brooklyn, New York

Alfred G. Boyd, B.S. in E.E. (North Dakota) Manager, Marketing Planning, General Electric

M. Philbrick Bridgess, A.M. (Harvard) Head of Dept., Roxbury Latin School, Massachusetts

Brother Alexis H. Guilbault, M.S. (Montreal) Instr., Prevost High School, Fall River, Massachusetts; Grad. Student, Columbia University

Brother Joseph Heisler, B.S. (St. Edward's) Grad. Student, University of Notre Dame

John Brown, Student, McGill University

Kenneth N. Brown, Student, Los Angeles State College

Arthur F. Brownell, Jr., B.S. in Ed. (Northern Illinois) Grad. Asst., Mankato State College

Robert L. Campbell, M.Ed. (Louisiana S.U.) Instr., University of Hawaii

Walter J. Carpenter, M.A. (North Carolina) Asso. Professor, North Georgia College

James W. Cederberg, Student, University of Kansas

John C. Chitwood, Jr., M.A. (Texas) Asst. Professor, University of Kansas City

Ambrose R. Clarke, Ed.M. (St. Lawrence) Head of Dept., Batavia High School, New York

Wallace A. Cole, M.S. (Wisconsin) Asst. Professor, Michigan College of Mining and Technology

Martha J. Colquitt, A.B. (Mercer) Teacher, A. L. Miller Junior High School, Macon, Georgia

ROBERT E. COMLEY, M.S. (N.Y.S.T.C., Albany) Instr., Hudson Valley Technical Institute, Troy, New York
ROBERT M. COTTON, Student, Swarthmore College
JOSE B. COURTIS, Actuary (Facultad de Ciencias Economicas) IBM World Trade Corp., Buenos Aires, Argentina
NOEL E. CUFF, M.S. (Kentucky) Asso. Engineer, The Martin Co.
CAROL CUMMINGS, Student, Texas Technological College
ARLEY T. CURLEE, M.A. (North Carolina) Professor and Head of Dept., Salem College
EVAN L. DAVIS, JR., Student, University of Colorado
KENNETH J. DAVIS, Student, Wofford College
ERNEST J. ECKERT, M.A. (Southern California) Instr., Los Angeles State College
DON M. ESTES, Student, Baylor University
CHARLES F. FEDERSPIEL, M.A. (Michigan) 1509 St. Mary's Street, Raleigh, North Carolina
JOHN A. FIGUEROA, Student, Los Angeles State College
DONALD E. FITZGERALD, A.M. (Yale) Instr., Long Beach City College
EILEEN G. FLAHERTY, A.B. (Salve Regina) Instr., Salve Regina College
MICHAEL A. GERAGHTY, Ph.D. (Notre Dame) Jr. Research Instr., University of Notre Dame
MRS. ALVA M. GOLDSMITH, B.A. (Smith) Teacher, Charles E. Gorton High School, Yonkers, New York
MRS. RICHARD H. GRAMANN, B.S. (Illinois) Teacher, Arlington Township High School, Illinois
HARVEY W. GREENE, M.A. (Missouri) Mathematician, Jersey Production Research Co.
JOHN R. HALEY, Student, University of Tulsa
J. HARRISON HANCOCK, M.S. (Brown) Mathematician, Naval Research Lab.
HARRY R. HANNUM, A.B.A. (Pennsylvania) Research Manager, The Philadelphia Inquirer, Pennsylvania
HARVEY A. HARTENSTEIN, Student, Long Beach State College
LEONARD J. HASSLER, B.S. (Yale) Master, Pomfret School, Connecticut
JANE M. HILL, M.A. (Ohio S.U.) Teacher, Washington, D.C. Public Schools
NICK HOLONYAK, JR., Ph.D. (Illinois) Physicist, General Electric, Syracuse, New York
JOHN H. HUNTZINGER, M.S. (Temple) Head of Dept., Northeast High School, Philadelphia, Pennsylvania
JURI KALVISTE, B.S. in E.E. (Washington) Research Engineer, Boeing Airplane Co.
ARTHUR KLEIN, Ed.M. (Boston T.C.) Master, Boston Latin School
LYNN C. KURTZ, Student, South Dakota School of Mines and Technology
JOHN P. LASCHENSKI, Student, College of the Holy Cross
MARTHA LENTZ, Student, Wisconsin State College
HOWARD LEVI, Ph.D. (Columbia) Asso. Professor, Columbia University
F. A. LIMOUZIN, Professor Emeritus, Ecole Normales, France
MARY E. LOGAN, Student, University of Kentucky
ROBERT C. LYNESS, B.A. (Oxford) Inspector of Schools, London, England
LAVAL MATHIEU, Student, University of Montreal
MOTHER M. CARMELITA MCINTYRE, Ed.M. (St. Louis) Instr., Springfield Junior College
LEONARD B. MCKEE, B.S. (California) Senior Research Chemist, Shell Chemical Corp.
KARL MICHAELIS, Dipl.-Phys. (Hansische) Teacher, Heeresoffizierschule II, Hamburg, Germany
JOHN W. MILSOM, B.A. (Pennsylvania S.U.) Grad. Student, Southwestern Louisiana Institute
DAVID B. MOSER, Student, Emory University
PEGGY J. MULLINS, Student, University of Alabama
FRANCIS X. MULVIHILL, A.B. (Boston Coll.) Lt. (jg), Naval Amphibious Base, Virginia
SANDRA G. NESS, Student, Emory University
NATHAN O. NILES, M.S. (St. Lawrence) Asso. Professor, Naval Academy
HELEN PAISNER, B.A. (Hunter) Lecturer, Barnard College; Grad. Student, Columbia University
STEPHEN D. PASTOR, Student, Oberlin College
RUSI K. N. PATELL, M.A. (Columbia) Research Asst., Columbia University
GEORGE B. PECK, M.S. (Illinois) Asst. Professor, Arizona State University
WILLIAM PENCHUCK, B.Ed. (Alberta) Vice Principal, Ft. Assinboine School, Alberta, Canada
RICHARD W. PIKE, Student, Wisconsin State College
HOWARD L. PROUSE, M.S. (Wayne S.T.C.) Supervisor, Math. Dept., Mankato State College
NORMAN J. PULLMAN, M.A. (Harvard) Grad. Asst., Syracuse University
CHARLES W. RAY, B.A. (San Francisco S.C.) Grad. Student, San Francisco State College
WILLIAM H. RICHARDSON, Student, Chico State College
H. ARNALL ROBERTS, B.E. (Vanderbilt) Grad. Student, Emory University
PAUL ROETHEL, Student, Wisconsin State College
JAMES H. ROLF, Student, Eastern Kentucky State College
EDGAR A. ROSE, Ed.M. (Rochester) Teacher, Monroe High School, Rochester, New York
MRS. DOROTHY S. RUTLEDGE, B.A. (Birmingham-Southern) Grad. Student, Emory University
JOSEPH E. RYUS, B.A. (California, Berkeley) Librarian, University of California
JOHN C. SALLIS, Student, University of Arkansas
JOHN F. H. SCHLUEP, M.A. (Columbia) Head of Dept., Cato Meridian Central School, New York
DAVID I. SCHNEIDER, Student, Oberlin College
HILBERT SCHULTZ, Student, Wisconsin State College
NATHAN T. SEELY, JR., M.A. (Pennsylvania) Acting Chairman of Dept., Agricultural and Technical College of North Carolina
CARL J. SINKE, Ph.D. (Purdue) Asst. Professor, Calvin College
SISTER CATHERINE JOSEPHINE, Ph.D. (Boston) Asso. Professor, Emmanuel College
PARKY SKELTON, Student, Emory University
W. RICHARD SLINKMAN, M.S. (Kansas S.C., Pittsburg) Instr., University of Colorado
JAMES M. SLOSS, B.S. (Pomona) Research Asst., University of California, Berkeley
CHARLES J. SMITH, A.B. (Washington) Grad. Student, Washington University
GEORGE M. SNEED, Student, Emory University
BERNARD SOHMER, Ph.D. (New York) Instr., College of the City of New York
FRANK W. STARKS, III, Student, J. M. Atherton High School, Louisville, Kentucky
WILLIAM F. STEELE, Asst. Professor, Heidelberg College
ROBERT M. STONE, M.Ed. (Howard Payne) Asst. Professor, Howard Payne College
EDWARD E. SUNDELL, B.A. (S.C. of Washington) Math. Analyst, General Electric Co.
JAMES J. SWISHER, JR., M.B.S. (Colorado) Instr., College of the Sequoias
DONALD G. TEMPLETON, Student, University of Chattanooga
MRS. JUANITA S. TOLSON, M.S. (Howard) Chairman of Dept., Roosevelt High School, Washington, D.C.
WILLIAM F. TYNDALL, B.S. (Franklin & Marshall) Grad. Student, Brown University
HAROLD R. VANDENBURGH, B.A. (North Texas S.C.) Mathematician, Project Matterhorn, Princeton University
ROBERT G. VAN METER, A.M. (Duke) Asst. Professor, Geneva College
JOHN R. VEATCH, Student, Whitman College
WILLIAM M. WAGNER, M.S. (Iowa S.C.) Engineer-Computer, North American Aviation
NORMAN H. WALTON, Student, San Jose State College
ULYSSES V. WARD, M.A. (Wayne State) Instr., Howard University
RONSON J. WARNE, M.S. (New York) Asst., University of Tennessee
JOHN WEISSMAN, M.A. (Temple) Instr., Rutgers University
MARVIN J. WINER, Student, University of Buffalo
BURTON M. WOODWARD, Student, University of Florida
WILLIAM S. WUNCH, Ph.D. (Stanford) Asso. Professor, Arizona State University
FREDERIC A. WYATT, B.A. (Union) Senior Consultant, Richardson, Bellows, Henry & Co.
LILLIAN ZARLING, M.A. (Minnesota) Asso. Professor, Wisconsin State College

THE APRIL MEETING OF THE KENTUCKY SECTION

The annual meeting of the Kentucky Section of the Mathematical Association of America was held at Centre College, Danville, Kentucky, on April 25, 1959. Professor W. J. Robinson, Chairman of the Section, presided at both the morning and afternoon session. Fifty-three persons attended the meeting including forty-three members of the Association.

At the business meeting the following officers were elected for the coming year: Chairman, Professor W. C. Royster, University of Kentucky; Secretary-Treasurer, Professor V. F. Cowling, University of Kentucky; Traveling Lecturer, Professor T. J. Pignani, University of Kentucky.

The following papers were presented:

1. *The numerical solution of the Van der Pol equation*, by Professor W. Krogdahl, University of Kentucky, introduced by the Secretary.

Limit cycle solutions of the Van der Pol equation were obtained by numerical integration for integral values of the parameter from 1 to 10. The characteristics of these solutions were compared with the characteristics of solutions for large parameter values as determined by asymptotic formulae.

2. *Singularities of analytic functions*, by Mr. J. P. King, University of Kentucky.

The author uses the transformation $z=t+kt^2$ to obtain new criteria that the point $z=1$ be a singular point for $f(z)=\sum a_n z^n$.

3. *Some remarks on grading*, by Professor A. G. Anderson, Western Kentucky State College.

A scheme whereby much of the arbitrariness in grading can be eliminated is the subject of the paper. The method is not designed for experienced teachers or advanced courses but should enable a new department member to conform to an institution's grading policy while at the same time incorporating ideas of his own.

4. *A class of odd typically real functions in an ellipse*, by Professor W. C. Royster, University of Kentucky.

Let T_0 denote the class of odd functions $(*) f(z)=z+\sum_{n=1}^{\infty} a_{2n+1}T_{2n+1}(z)$, $T_n(z)=\cos(n\cos^{-1}z)$, with the properties that (a) $f(z)$ is regular in the ellipse with foci ± 1 and semi-axes $a>b>0$, (b) $f(z)$ is real on the real axis and (c) if z lies in E then $\operatorname{Im}\{z\}\cdot\operatorname{Im}\{f(z)\}\geq 0$ and $\operatorname{Re}\{z\}\cdot\operatorname{Re}\{f(z)\}\geq 0$. It is shown that if $f(z)$ given by $(*)$ belongs to the class T_0 then $|a_{2n+1}R^{2n}+a_{2n-1}R^{-2n}|\leq 2$, $n=1, 2, \cdots$.

5. *A problem in musical composition*, by Professor W. H. Spragens, University of Louisville.

In this paper the author studies problems of intervals as related to the twelve tone scale.

6. *Discrete analytic functions*, by Mr. W. T. Sledd, University of Kentucky.

An expository talk.

7. *Report on the Visiting Lecturer Program*, by Professor S. E. Pence, University of Kentucky.

8. *Remarks on MacIntyre's method of solving a Pfaffian*, by Professor T. J. Pignani, University of Kentucky.

An expository talk concerned with MacIntyre's (Proc. Edinburgh Math. Soc. vol. 4, 1934–1936) method of solving the exact differential equation $P(x, y)dx+Q(x, y)dy=0$.

9. *Some problems in constructive function theory*, by Professor George Piranian, University of Michigan. (Invited one-hour address).

V. F. Cowling, *Secretary*

THE APRIL MEETING OF THE TEXAS SECTION

The annual spring meeting of the Texas Section of the Mathematical Association of America was held April 17–18, 1959 at the University of Texas, Austin, Texas. There were 150 in attendance, including 100 members of the Association.

The following officers were elected for the coming year: Chairman, Professor W. T. Guy, Jr., University of Texas; Vice-Chairman, Professor P. R. Culwell, San Antonio College; Secretary-Treasurer, Professor C. R. Sherer, Texas Christian University. The next meeting will be held at San Antonio College, San Antonio, Texas.

Professor H. J. Ettlinger gave a detailed report of the High School Mathematics Contest which was given on a trial basis. Through Professor Ettlinger's expert leadership, the results were very satisfactory and the section decided to continue the contest next year.

The following program was presented:

1. *Diagonal forms of odd prime degree over finite fields*, by Professor J. F. Gray, St. Mary's University of San Antonio, introduced by the Secretary.

Consider the problem of representing zero by a non-trivial linear combination of pth powers in a finite field k. By extending a theorem of D. J. Lewis, it is proved that p pth powers suffice when p is an odd prime. If, in addition, $p \geq 5$, then $p-1$ pth powers are shown to suffice and, for $p \geq 7$, $p-2$ pth powers are sufficient. In general, where p and p_0 are odd primes and $p \geq p_0$, there exists an integer $t(p_0)$ such that over a finite field k, $a_1x_1^p + a_2x_2^p + \cdots + a_{p-t(p_0)}x_{p-t(p_0)}^p = 0 (a_i \in k)$ has a nontrivial solution in k. The values $t(3)=0$ and $t(5)=1$ are best possible. That $p_0 - t(p_0) > \log_2 p_0 + 1$ for an infinity of primes p_0 would follow from an undoubted, but unverified, conjecture of Dickson's which implies the existence of an infinity of prime pairs (p, q) with $q = 2p+1$.

2. *Some mathematical aspects of the problem of stability*, by Professor W. S. McCulley, Agricultural and Mechanical College of Texas.

Consideration of an electro-mechanical system described by a second order linear ordinary differential equation with constant coefficients indicated that negative real parts for the roots of auxiliary equation represent stability. For feedback systems, information about stability may be obtained by Routh-Hurwitz criterion, Bode plots, Nyquist criterion, root locus method, describing functions, or closed loop pole-zero location.

3. *The accumulation of round-off error in the numerical integration of ordinary differential equations*, by Professor Peter Henrici, University of California, Los Angeles. (By invitation.)

The paper studies the dependence of the accumulated round-off error (r.o.e.) in the numerical integration of an ordinary differential equation by a general multi-step method in the sense of Dahlquist (Math. Scand., vol. 4, 1956, pp. 33–53) on the local r.o.e. Under the assumption that the local r.o.e. are independent random variables with known distributions, a formula for the variance of the accumulated r.o.e. is derived. The results are tested by numerical experiments in which a given differential equation is solved by the same method under varying initial conditions.

4. *An overall look at extended analytic geometry*, by Professor R. S. Underwood, Texas Technological College.

When X and Y are particular single-valued functions of n variables, a locus on the XY-plane (occasionally vacuous) is implicitly associated with any equation in the variables. Extended analytic geometry provides ways of finding these loci rapidly by use of algebra, plane analytic geometry, and in some cases, calculus. The system yields solutions, or proves inconsistence, for various Diophantine and simultaneous equations, including all linear-versus-quadratic pairs. Trick results appear, as well as properties of equations that are independent of plotting rules. Criteria for quadric surfaces shorter than the conventional ones are developed.

5. *Extended quadrics of Darboux*, by Professor Dale Maness, Howard Payne College.

The equation of the most general non-composite quadric hypersurface having contact of second order at a general point with an analytic hypersurface in n-space is found. The classical tangents of Daroux are found to be specializations of cubic $(n-2)$-dimensional cones to be known as the "cones of Darboux." The classical directions of Darboux are generalized for hypersurfaces and are to be known as the "extended directions of Darboux." A one parameter family of hyperquadrics to be known as the "extended quadrics of Darboux" is defined and specializes to give the Darboux family of quadric surfaces of three space.

6. *The unitary equivalence of 3×3 matrices*, by Professor Carl Pearcy, Rice Institute, introduced by the Secretary.

Consider the collection $A^{n_1}A^{n_2}\cdots A^{n_j}$, where $n_1, n_2, \cdots n_j$ is any finite sequence of non-negative integers, and A is any $n\times n$ complex matrix. Specht showed that the traces of the above collection form a complete set of unitary invariants for $n\times n$ complex matrices. Murnaghan showed that for $n=2$ a subset of three of the above traces suffices, and the author has shown that for $n=3$ a subset of eleven traces suffices.

7. *Numerical integration with weight functions X and X^2*, by Professor R. E. Greenwood, University of Texas.

The use of the correspondence $\int_a^b x^k f(x)dx \sim \sum_{i=1}^n x_{i,n}^k f(x_{i,n})$ is advantageous in evaluating first moments of means $(k=1)$ and second moments $(k=2)$ for certain empirically determined functions. The set of needed abscissas $\{x_{i,n}\}$, $i=1, \cdots, n$, may be called a set of generalized theorem-of-the-mean points. It is shown that for some values of n one or more of these points may fall outside of the interval (a, b) thereby precluding effective use of the numerical approximation correspondence. Other values of n yield usable results. This work is a portion of an article by Greenwood, Carnahan and Nolley which is scheduled to appear in a forthcoming issue of *Mathematical Tables and Other Aids to Computation.*

8. *Inversion of Jacobian matrices*, by Professor Louis Brand, University of Houston.

In a transformation of coordinates $x^i=f^i(y^1, \cdots, y^n)$, $i=1, \cdots, n$, the n^2 derivatives $\partial y^i/\partial x^j$ may be computed from the fact that the Jacobian matrices $(\partial x^i/\partial y^j)$ and $(\partial y^i/\partial x^j)$ are reciprocals when written so that i is a row index. The inversion of $(\partial x^i/\partial y^j)$ is particularly simple when the transformation is from one orthogonal system to another. Then $(\partial x^i/\partial y^j)$ can be expressed as the product MD of an orthogonal matrix M and a diagonal matrix D, whose elements are the lengths of its column vectors; and $(\partial y^i/\partial x^j)=(MD)^{-1}=D^{-1}M'$ where M' is the transpose of M. The calculation may also be made by observing that the vectors $e_{mi}=(\partial x^1/\partial y^i, \cdots, \partial x^n/\partial y^i)$, $e_m^j=(\partial y^j/\partial x^1, \cdots, \partial y^j/\partial x^n)$ satisfy $e_i\cdot e^j=\delta_i^j$ and hence form reciprocal sets. Thus $e_m^j=g^{jr}e_{mr}$; or since $g^{jk}=0(j\neq K)$, $g^{jj}=1/g_{jj}$ for orthogonal transformations, $e_m^j=g^{jj}e_{mj}=e_{mj}/g_{jj}=e_{mj}/(e_j\cdot e_j)$ with summation suspended.

9. *Sub-structure vs. super-structure in presenting real algebra on the secondary and elementary collegiate level*, by Professor J. F. Gray, St. Mary's University of San Antonio.

In the current concern to present the real number system, both on the secondary and on the elementary collegiate level, from a truly mathematical point of view, two approaches stand out. The first proceeds from structure to substructure, beginning with a set of postulates, assumptions, or properties of the real number system essentially equivalent to the axioms for an ordered field and later descending to study the various sub-systems of interest. The alternative approach proceeds from structure to super-structure, ascending from a consideration of the natural numbers to the rational and real fields. Reasons and supporting experiences are cited to strongly suggest that the first approach is far more desirable.

10. *Equilibrium position of a certain flexible cable*, by Professor Guy Johnson, Rice Institute.

11. *Concerning matrix norms*, by Professor Ben Fitzpatrick, Jr., University of Texas.

12. *Peculiar sort of power series*, by Professor G. R. MacLane, Rice Institute.

13. *Coverings*, by Mr. Terence Reed, Rice Institute, introduced by the Secretary.

14. *Advanced calculus for undergraduate majors*, by Dr. Fred Bamforth, Texas Christian University, introduced by the Secretary.

C. R. SHERER, *Secretary*

THE JUNE MEETING OF THE PACIFIC NORTHWEST SECTION

The twelfth annual meeting of the Pacific Northwest Section of the Mathematical Association of America was held at the University of Oregon, Eugene, Oregon on June 19, 1959 in conjunction with the 558th meeting of the American Mathematical Society and jointly with the Pacific Northwest Section of the Society for Industrial and Applied Mathematics. Professor Kenneth Bush, chairman of the Section, presided over the meetings. One hundred and eleven persons were in attendance, including 69 members of the Association.

At the Business Meeting the following officers were elected: Chairman, Professor A. E. Livingston, University of Washington; Vice-Chairman, Professor T. G. Ostrom, University of Montana; Secretary-Treasurer, Professor K. S. Ghent, University of Oregon.

Professor Allendoerfer reported briefly on plans of the Mathematical Association of America.

The program was as follows:

1. *Set theory and numbers*, by Professor J. L. Kelley, University of California, Berkeley (by invitation).

The purpose of the paper is to construct the real number system within intuitive set theory. The void set is 0, and for each set x we let x' be the set whose only members are x and the members of x. (*i.e.*, $y \in x'$ iff $y \in x$ or $y = x$). An inductive class Δ is a class such that $0 \in \Delta$ and if $x \in \Delta$ then $x' \in \Delta$. A set n is a natural number iff n belongs to each inductive class. The Peano postulates are easily proved. Using the properties of the natural numbers as constructed, addition and multiplication are defined as cardinal addition and multiplication. The development is simple and intuitively natural.

2. *Boundary value problems in ordinary differential equations*, by Professor René De Vogelaere, University of California, Berkeley, introduced by the Secretary.

A general method of numerical analysis, called the *discretization method* is defined using as illustration the problem of integrating a function over an interval. The definition, inspired by work of Kantorovich, Kelley, Lemaître and Crout, contains as special cases a large number of the methods used for the solution of differential equations, ordinary and partial, as well as integral equations; in particular, difference, Galerkin and least-square methods. The compactness of presentation that this definition permits is illustrated for the solution of boundary value problems of differential equations. Advantage of restricting the functions involved to non-equidistant points over the prevailing practice (difference methods) is stressed.

3. *An elementary approach to Lebesgue integration*, by Professor H. H. Schaefer, State College of Washington.

This is a brief elementary exposition of a basically well-known method to obtain the concept of Lebesgue integral for, say, real functions on a compact interval $I \subset R^n$. Here "elementary" means that no use is made of the notions of measure and measurable function; familiarity with the Riemann integral and with the completion of a metric space is assumed. The (didactical) stress is laid upon the fact that with this material, along with the basic tools of calculus and using a concept equivalent to (Lebesgue) measure 0, the most important properties of the L-integral can be deduced including the convergence theorems of B. Levi, Fatou, and Lebesgue. The method extends easily to the Lebesgue-Stieltjes integral and shows rather clearly the relation to the corresponding Riemann integrals.

4. *Some mathematical aspects of the theory of life contingencies*, by Professor S. A. Jennings, University of British Columbia.

The theory of life contingencies offers a variety of interesting and important applications of elementary calculus which many beginning students might find more meaningful than the conventional discussions of velocity, acceleration, centroids and moments of inertia. To illustrate this contention, the speaker discussed how, starting with ${}_tp_x$, the probability that a person aged x will survive to age $x+t$, the force of mortality may be defined as a logarithmic derivative, and Gompertz' and Makeham's laws of mortality derived by solving a differential equation. Of more interest from the point of view of calculus applications, however, are problems involving integration. The speaker mentioned several examples of this type.

5. *Mathematics goes to market*, by Professor R. E. Gaskell, Oregon State College.

Interest of the government in mathematics and mathematicians developed from military necessity between 1940–1945, and this has carried over to industry since then. There are, as a result, growing numbers of mathematicians employed in industry, in fact the growth could be described as explosive. The dramatic effect of high-speed computing machinery is easily visible, but fewer people—mathematicians as well as non-mathematicians—realize the no less dramatic effect of new developments in mathematics itself on industry. A need for more mathematicians is becoming evident, but beyond this more mathematics will be required of all of us—producers, consumers, and even those few who insist that they are merely observers.

K. S. GHENT, *Secretary*

CALENDAR OF FUTURE MEETINGS

Forty-third Annual Meeting, Conrad Hilton Hotel, Chicago, Illinois, January 28–30, 1960.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary:

ALLEGHENY MOUNTAIN, Grove City College, Grove City, Pennsylvania, April 30, 1960.

ILLINOIS, Illinois Wesleyan University, Bloomington, May 13–14, 1960.

INDIANA

IOWA, State University of Iowa, Iowa City, October 16, 1959.

KANSAS, Kansas State College of Pittsburg, April 30, 1960.

KENTUCKY, University of Kentucky, Lexington, April, 1960.

LOUISIANA-MISSISSIPPI

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, American University, Washington, D. C., December 5, 1959.

METROPOLITAN NEW YORK

MICHIGAN, University of Michigan, Ann Arbor, March 26, 1960.

MINNESOTA

MISSOURI, Central Missouri State College, Warrensburg, April 30, 1960.

NEBRASKA, University of Nebraska, Lincoln, April 23, 1960.

NEW JERSEY, Princeton University, November 7, 1959.

NORTHEASTERN, Boston College, Chestnut Hill, Massachusetts, November 28, 1959.

NORTHERN CALIFORNIA, University of California, Berkeley, January 16, 1960.

OHIO, Kent State University, May 7, 1960.

OKLAHOMA, Oklahoma City University, October 23, 1959.

PACIFIC NORTHWEST, State University of Montana, Missoula, June 17, 1960.

PHILADELPHIA, University of Delaware, Newark, November 28, 1959.

ROCKY MOUNTAIN, United States Air Force Academy, Colorado Springs, May 6–7, 1960.

SOUTHEASTERN, University of South Carolina, Columbia, April 1–2, 1960.

SOUTHERN CALIFORNIA, Los Angeles State College, March 12, 1960.

SOUTHWESTERN, Air Force Missile Development Center, Holloman Air Force Base, New Mexico, April, 1960.

TEXAS, San Antonio College, April, 1960.

UPPER NEW YORK STATE, University of Rochester, May 7, 1960.

WISCONSIN, Mount Mary College, Milwaukee, May 7, 1960.

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ON CORES AND PRIME IMPLICANTS OF TRUTH FUNCTIONS

W. V. QUINE, Harvard University

What is called a truth-functional formula in (alternational) *normal form* is built up of sentence letters 'p', 'q', etc., or their negations '$\bar{p}$', '$\bar{q}$', etc., or both, by using only the notations of conjunction and alternation, and in such a way as to subject alternations never to conjunction but only vice versa: thus '$pq \vee \bar{p}r\bar{s} \vee \bar{r}$'. This paper is concerned with the problem of reducing an arbitrary truth-functional formula to a shortest equivalent in normal form. Part of the content of my last previous paper on the subject* will be presented anew in an improved way, and a further theorem will be established. I shall not assume familiarity with my previous papers.

Let us sharpen our terminology. Sentence letters and their negations are called *literals*. Literals and conjunctions of them are called *fundamental formulas*, provided that none contains the same letter twice. Fundamental formulas and alternations of them are called *normal*, and are said to have those fundamental formulas as their *clauses*. On these definitions, a formula is convertible to nornal form only if it is not self-contradictory; but there is no serious loss in setting aside the self-contradictory cases.

A *prime implicant* of a formula Φ is a fundamental formula that logically implies Φ but ceases to when deprived of any one literal. A normal formula will be called *uniliterally redundant* if it is equivalent to what remains of itself on dropping some one occurrence of a literal. Obviously then a normal formula is uniliterally redundant if and only if not all its clauses are prime implicants of it. Consequently, in particular,

(I) *Any shortest normal equivalent of a formula Φ will be an alternation of prime implicants of Φ.*

Ignoring tautologies along with self-contradictions, as will be convenient hereafter, one finds that

(II) *The prime implicants of a formula exist and are finite in number.*

To see that they exist, consider a formula Φ and any assignment of truth values to the letters of Φ that verifies Φ—say truth to 'p', 'q', and 's' and falsity to 'r'. The assignment determines a fundamental formula—'$pq\bar{r}s$', in this example—that implies Φ; and it, or part of it, is a prime implicant of Φ. To see further that the prime implicants of Φ are finite in number, we just reflect that they are bound to contain no letters foreign to Φ, since any such letter could be dropped without affecting the implication.

(III) *A formula is equivalent to the alternation of all its prime implicants.*

* A way to simplify truth functions, this MONTHLY, vol. 62, 1955, pp. 627–631. (Cited hereafter as WSTF.)

For, a formula Φ is implied by each of its prime implicants and hence by their alternation; and conversely every truth assignment that verifies Φ verifies some Φ-implying fundamental formula (*cf.* '$pq\bar{r}s$' above) and therewith some prime implicant (viz., that fundamental formula or part of it) and therewith the alternation of the prime implicants.

In view of (I) and (III), a formula can be transformed into a shortest normal equivalent in two stages: (A) transform it into the alternation of all its prime implicants and then (B) delete from that alternation the largest possible combination of jointly superfluous clauses. (A) can be accomplished by a technique due to Samson and Mills,* the explanation of which calls for two more definitions. If two fundamental formulas ϕ and ψ are opposed in exactly one letter (so that ϕ contains 'p' affirmatively, say, and ψ contains '$\bar{p}$'), then ϕ and ψ will be said to have as their *consensus* the formula which we get from the conjunction $\phi\psi$ by deleting the two opposed literals and any repetitions.§ If ϕ and ψ are fundamental formulas opposed in no letter, and all letters (hence all literals) of ψ are in ϕ, then ϕ will be said to (notationally) *subsume* ψ.†

Preparatory to the business of (A), we may suppose Φ put into normal form by familiar logical procedures. Now the discovery of Samson and Mills is that Φ, thus prepared, goes over into the alternation of its prime implicants if we persevere in these two equivalence transformations:

(i) If a clause subsumes another, drop the former.

(ii) Adjoin, as an additional clause, the consensus of two clauses (unless it subsumes a clause already present).

That (i)–(ii) are bound eventually to convert Φ into the alternation of just its prime implicants, and all of them (ignoring differences of order in a conjunction), is proved by proving the following four theorems.

(IV) *A normal formula remains susceptible to* (ii) *as long as some prime implicant of it is not a clause of it* (to within a permutation of a conjunction).

(V) *A normal formula remains susceptible to* (i) *or* (ii) *as long as some clause of it is not a prime implicant of it.*

(VI) *Normal formulas go into normal formulas under* (i) *and* (ii).

* Edward W. Samson and Burton E. Mills, Circuit minimization: algebra and algorithms for new Boolean canonical expressions, AFCRC Technical Report 54-21, April, 1954.

§ This definition of consensus is more liberal than that in WSTF, in that it counts ϕ as consensus of $\alpha\phi$ and $\bar{\alpha}$ (and of $\bar{\alpha}\phi$ and α). (i) below is consequently simpler than its counterpart in WSTF, and correspondingly for the proof of (IV) below. This liberalization of the definition is due essentially to Kurt Bing, On simplifying truth-functional formulas, J. Symbolic Logic, vol. 21, 1956, pp. 253 f; but note that I avoid his "void formula."

† My terminology has proved confusing. When ϕ subsumes ψ in my intended notational sense of having among its literals all those of ψ, then, precisely, ψ subsumes ϕ in a certain logical sense: ψ has among its verifying truth-value assignments all those of ϕ.

(VII)* *No normal formula is susceptible to* (i)–(ii) *without end.*

Proof of (IV). Let Φ be a normal formula and χ a prime implicant of it that differs (in more than order) from all clauses. Then also, being prime, χ subsumes no clause. So there is at least one fundamental formula, χ anyway, that has the three properties of (a) subsuming χ, (b) subsuming no clause of Φ, and (c) exhibiting no letters foreign to Φ (*cf.* end of proof of (II)). Moreover, there is an obvious limit to how long a fundamental formula having property (c) can be. So there is at least one fundamental formula, call it ψ, that has the three properties (a)–(c) and is exceeded in length by no fundamental formula having those properties. Yet ψ does not exhibit all letters of Φ; for, if it did, then, having also property (b) as it does, ψ would oppose every clause of Φ in one or another letter, and so not imply Φ; whereas actually ψ must imply Φ, having property (a). So ψ lacks some letter of Φ, say 'p'. Since ψ is a longest fundamental formula with the properties (a)–(c), $p\psi$ must lack one of those properties and so must $\bar{p}\psi$; yet obviously not (a) or (c); so (b). So there are clauses ϕ_1 and ϕ_2 of Φ such that $p\psi$ subsumes ϕ_1 and $\bar{p}\psi$ subsumes ϕ_2. But ψ, having the property (b), subsumes neither; so 'p' and '$\bar{p}$' must occur respectively in ϕ_1 and ϕ_2. Still ϕ_1 and ϕ_2 are not just 'p' and '$\bar{p}$', or Φ would be tautologous; nor are ϕ_1 and ϕ_2 opposed in letters other than 'p', since their further literals are common to ψ. So ϕ_1 and ϕ_2 have a consensus, say ϕ. Moreover ϕ subsumes no clause of Φ; for ψ subsumes ϕ, and ψ has the property (b). So ϕ can be added by (ii).

Proof of (V). Let Φ be a normal formula and ϕ any clause of it that is not a prime implicant. Then ϕ, implying Φ, subsumes some prime implicant ψ. If ψ is a clause of Φ (to within a permutation of a conjunction), we can apply (i) to drop ϕ; and otherwise we can apply (ii) in view of (IV).

Proof of (VI). Obvious.

Proof of (VII). Elimination of a clause ϕ by (i) depends on there surviving some clause ϕ' that ϕ subsumes; elimination of ϕ' in turn depends on there still surviving some clause ϕ'' that ϕ' (and hence ϕ) subsumes; and so on. Therefore, in view of the parenthetical part of (ii), no clause ϕ once dropped by (i) can ever be restored by (ii). But neither, in view of the parenthetical part of (ii), can a clause already present be reintroduced in duplicate by (ii). To sum up, (ii) can never introduce the same clause twice. But there are only finitely many different fundamental formulas for (ii) to introduce, since no letters foreign to Φ are drawn on. So the use of (ii) must terminate. Also the use of (i), being subtractive, obviously must terminate.

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* This theorem and its proof, which I neglected to include in WSTF, are given here on the appreciated advice of the referee. Note that the parenthetical proviso in (ii), which may look dispensable in view of (i), is needed to assure (VII)—as is indeed remarked in effect in WSTF at the top of page 628.

So much for task (A), the conversion of a formula into the alternation of all its prime implicants. Toward the continuation task (B), that of determining what largest combination of the resulting clauses can be dropped as jointly superfluous, systematic methods have been developed by Ghazala.*

But a shortcoming of the whole (A)–(B) approach is that it depends on exhausting the prime implicants; for, as Rolf K. Müller remarked to me in 1955, their number can even exceed the total number of possible truth-value assignments when there are many variables. According to Fridshal,† this can happen after five variables. He cites a formula in nine variables which, by his computations, has 1698 prime implicants—whereas the total number of possible truth-value assignments is only 512. So a general technique is needed for finding an adequate minimum alternation of prime implicants without handling all prime implicants on the way.

A normal formula may be called *clausally redundant* if it is equivalent to what remains of itself on dropping one of its clauses of alternation; and *irredundant*, simply, if it is neither clausally nor uniliterally redundant. Obviously any shortest normal equivalent of a formula will be irredundant. Happily we can render a normal formula irredundant without regard to the totality of its prime implicants, simply ridding the formula of superfluous clauses and superfluous literals one by one. To see whether a clause ψ is superfluous in $\Phi \vee \psi$, we just check whether ψ implies Φ. Such implication cannot in general be recognized by mere notational subsumption, as can implication between fundamental formulas, but it can be checked very simply: we have merely to mark the literals of ψ as true and see whether Φ thereupon reduces to a tautology. Again, to see whether a literal ζ is superfluous in $\Phi \vee \zeta\psi$, we just check, in the same swift way, whether ψ implies $\Phi \vee \zeta$. For

(VIII) *ψ implies $\Phi \vee \zeta$ if and only if $\Phi \vee \zeta\psi$ is equivalent to $\Phi \vee \psi$.*

For, the conditional $\psi \supset . \Phi \vee \zeta$ is verifiably equivalent to the biconditional $\Phi \vee \zeta\psi . \equiv . \Phi \vee \psi$, and hence tautologous if and only if the biconditional is tautologous.

Unhappily, however, such reduction of a formula to *an* irredundant equivalent need not deliver a shortest. What it delivers may well lack some of the clauses of every shortest normal equivalent, and contain clauses shared by no shortest normal equivalent.

At the same time there commonly will be, given a formula Φ, certain clauses that are bound to appear in every irredundant equivalent of Φ, and hence in any shortest. The alternation of such clauses I call the *core* of Φ. Commonly also there will be, at the opposite extreme, prime implicants of Φ that are bound never to occur as clauses of irredundant equivalents of Φ. For a prime implicant

* M. J. Ghazala, Irredundant disjunctive and conjunctive forms of a Boolean function, I. B. M. Journal of Research and Development, vol. 1, 1957, pp. 171–176.

† R. Fridshal, The Quine algorithm, Summaries of Talks at the Summer Institute of Symbolic Logic (mimeographed), Cornell University, 1957, pp. 211 f.

of Φ to be thus absolutely superfluous, it is obviously sufficient that it imply the core of Φ; but I do not know whether this is necessary.

I shall explain a test whereby, without exhausting the prime implicants of a formula, we can identify its core. This done, we obviously can then quickly decide also, of a prime implicant, whether it implies the core (and is thus absolutely superfluous). Query: Given an irredundant formula, can all its absent prime implicants that are *not* absolutely superfluous be reached by the operation (ii) of iterated consensus-taking without interim retention of any absolutely superfluous ones? This, if true, would allow us to shortcut the (A)–(B) approach to the extent of leaving some of the prime implicants ungenerated in some cases; but I do not know it to be true, and anyway the saving would be limited to formulas with cores.

Guesses aside, there are cases where identifying the core of an irredundant formula settles the question of shortest normal equivalent without further ado. Thus if every clause belongs to the core, the formula is already as short as possible. If every clause but one belongs to the core, then again our simplification is good enough; the only possible improvement would be the negligible one of finding some shorter clause to take the place of the odd one.

But in the general case a technique of core identification has, alas, no evident bearing on our central problem: that of finding shortest normal equivalents without exhausting prime implicants. It seems worth communicating mainly for what it may contribute to one's understanding of the general workings of irredundant formulas.

Criterion, given an irredundant formula Φ, of whether a clause ϕ thereof belongs to the core: From Φ delete ϕ, also each clause that opposes ϕ in more than one letter, and finally all literals whose letters are in ϕ; what remains will be a tautology if and only if ϕ does not belong to the core of Φ. *Examples*: 'pq' does not belong to the core of '$pq \vee qs \vee q\bar{r} \vee \bar{q}r\bar{s}$', for '$s \vee \bar{r} \vee r\bar{s}$' is tautologous; whereas 'qs' belongs to the core, for '$p \vee \bar{r}$' is not tautologous.

That the criterion is geared only to irredundant formulas is no limitation, for we have seen how to make a formula irredundant. That the criterion enables us to spot as core clauses only clauses already present is again no limitation, for, by definition, in any irredundant formula all the core clauses are present. Actually unilateral irredundancy would suffice here without clausal; but it is more efficient to start from a fully irredundant formula, since there are then fewer clauses to examine.

It remains to justify the criterion, by proving that a clause of an irredundant formula belongs to the core if and only if it meets the stated test. In other words, what is wanted is the theorem:

(IX) *A clause ϕ of an irredundant Φ belongs to the core of Φ if and only if there is an assignment of truth values to letters not in ϕ that falsifies all clauses of Φ, other than ϕ, that oppose ϕ in at most one letter.* (I assume as usual that Φ is not simply a tautology $\alpha \vee \bar{\alpha}$.)

*Proof.** Let Φ be any irredundant formula, ϕ any clause of Φ, and Ψ the alternation of all prime implicants of Φ but ϕ. Then ϕ is in the core of Φ if and only if Ψ is not equivalent to $\phi \vee \Psi$ (and thus to Φ); hence if and only if ϕ does not imply Ψ. So what is to be proved is (a) that if ϕ does not imply Ψ then some assignment to letters not in ϕ falsifies all clauses of Φ, other than ϕ, that oppose ϕ in at most one letter, and (b) that if ϕ implies Ψ then each assignment to letters not in ϕ is compatible with some clause of Φ, besides ϕ, that opposes ϕ in at most one letter.

Proof of (a). Since ϕ does not imply Ψ, some assignment A to the letters of Φ verifies ϕ and falsifies Ψ. Let A' be the part of A that has to do with letters not in ϕ, and let ψ be any clause of Φ other than ϕ that opposes ϕ in at most one letter; to show that A' falsifies ψ. *Case* 1. ψ opposes ϕ in no letter. Then, since A verifies ϕ, A falsifies no literals of ψ whose letters are in ϕ; but A does falsify ψ, for ψ is clearly a clause of Ψ, and A falsifies Ψ; so A' falsifies ψ. *Case* 2. ψ has just one literal, call it $\bar{\zeta}$, opposed to a literal of ϕ. But ψ and ϕ are not simply $\bar{\zeta}$ and ζ, or Φ would be a tautology. So ψ and ϕ have a consensus, and it, since it implies $\psi \vee \phi$ and therefore Φ, subsumes a prime implicant χ of Φ. Since χ lacks ζ, χ is not ϕ; hence χ is a clause of Ψ; and hence A falsifies χ. But all literals of χ whose letters are in ϕ are identical with literals of ϕ, and hence verified by A; so A' falsifies at least one literal of χ. But any such literal of χ is a literal of ψ. So A' falsifies ψ.

Proof of (b). Here the assumption is that ϕ implies Ψ; *i.e.*, that Φ and Ψ are equivalent. Let A' be any assignment to the letters not in ϕ. Let Φ' and Ψ' be what Φ and Ψ reduce to under A'; thus Φ' and Ψ' are equivalent. It will be sufficient to show that Φ' has a clause, other than ϕ, that opposes ϕ in at most one letter. Now ϕ, being a clause still of Φ', implies Φ' and therefore Ψ'. Hence ϕ must subsume some clause ψ' of Ψ'; for ϕ subsumes every clause of Ψ' not opposed to ϕ, there being no further letters. This ψ' is the residue in Ψ' of some clause ψ of Ψ. Since ϕ is not a clause of Ψ, ψ is not ϕ; moreover, since any clause of Ψ is a prime implicant of Φ, ψ does not even subsume ϕ. So ψ lacks a literal ζ of ϕ; ϕ subsumes $\zeta\psi'$. Now assign falsity to ζ and truth to the rest of the literals of ϕ. This assignment verifies ψ', and therewith Ψ'; hence also Φ', which is equivalent to Ψ'. Therefore this assignment verifies some clause of Φ'. But a clause of Φ', to be thus verified, must consist solely of literals of ϕ other than ζ, plus perhaps $\bar{\zeta}$; hence it is other than ϕ and opposes ϕ in at most one letter.

* Previously proved in one of my lectures on "Simplifying truth functions," College of Engineering, University of Michigan, June, 1958.

THE VOTING PROBLEM*

RICHARD STEARNS, Princeton University

1. A pair of binary relations, P and I, are said to *define a pattern* S *on the elements of a set* A if and only if the three following conditions are satisfied: i) P is antireflexive and antisymmetric, ii) I is reflexive and symmetric, iii) for all a and b in A, aIb if and only if neither aPb or bPa. When aIb, one says that a is *indifferent* to b and when aPb, one says that a is *preferred over* b.

A pattern is *strong* if aIb implies $a=b$ and is *ordered* if both P and I are transitive.

A set of strong ordered patterns defined on the elements of a finite set A by various pairs of binary relations is called a *set of voters* and *generates* a new pair of relations on the set according to the following rule: aPb if and only if a is preferred over b in a majority of the voters and aIb if and only if neither aPb or bPa. It is obvious that the relations P and I so defined satisfy conditions i, ii, and iii above and thus P and I define a pattern S. This pattern S is called the pattern *generated* by the set of voters.

It was shown in a paper by McGarvey† that a set of voters can always be found to generate an arbitrary pattern. This was done with a construction which required $n(n-1)$ voters for a pattern of n elements. The voting problem considered here is that of finding the smallest number of voters required to generate an arbitrary pattern of n elements. First, it will be shown that this number is no larger than $n+1$ when n is odd or $n+2$ when n is even.

2. Using the notation of McGarvey, a strong ordered pattern S defined on the elements of $A=\{a_i: i=1, \cdots, n\}$ will be represented by the following: $S=a_1, \cdots, a_n$ where a_iPa_j if and only if a_i is to the left of a_j.

If $S_i=a_{i,1}, \cdots, a_{i,r_1}$ and all the S_i are disjoint, then define $S_1, \cdots, S_m = a_{1,1}, \cdots, a_{1,r_1}, \cdots, a_{m,1}, \cdots, a_{m,r_m}$.

3. Suppose S is an arbitrary pattern on $2k$ elements, a_i $(i=1, \cdots, 2k)$. Define for $1 \leqq i \leqq k$:

$$\begin{aligned} A_{1,i} &= \{a_j: a_jPa_{2i-1} \text{ and } a_jPa_{2i}\}, \\ A_{2,i} &= \{a_j: \sim(a_jPa_{2i-1}),\ a_jPa_{2i}, \text{ and } j \neq 2i-1\}, \\ A_{3,i} &= \{a_j: \sim(a_jPa_{2i-1}),\ \sim(a_jPa_{2i}), \text{ and } 2i-1 \neq j \neq 2i\}, \\ A_{4,i} &= \{a_j: a_jPa_{2i-1},\ \sim(a_jPa_{2i}), \text{ and } j \neq 2i\}. \end{aligned}$$

For a fixed i, each of the elements of $A-\{a_{2i-1}, a_{2i}\}$ falls into one and only one

* This paper is the result of some work started by the author as an undergraduate research assistant at Carleton College under a grant from the National Science Foundation to the Mathematics Department.

† David C. McGarvey, A theorem on the construction of voting paradoxes, Econometrica, vol. 21, 1953, pp. 608–610.

of the $A_{j,i}$, depending on its relation to a_{2i-1} and a_{2i}. Also, a_{2i-1} and a_{2i} do not belong to any $A_{j,i}$.

Let $T_{j,i}$ be the strong ordered pattern defined on $A_{j,i}$ by letting $a_r P a_s$ if and only if $r<s$. Let $\overline{T}_{j,i}$ be the pattern on $A_{j,i}$ with the reverse order.

Furthermore, define $Q_i=a_{2i-1}, a_{2i}$ if $a_{2i-1}Pa_{2i}$ in S; otherwise let $Q_i=a_{2i}, a_{2i-1}$. Finally, let $R_i=Q_i$ if $\sim(a_{2i-1}Ia_{2i})$; but if $a_{2i-1}Ia_{2i}$, let $R_i=a_{2i-1}, a_{2i}$, the reverse of Q_i.

Now consider the following set of voters:

$$V_{2i-1} = T_{1,i}, a_{2i-1}, T_{2,i}, a_{2i}, T_{3,i}, T_{4,i} \qquad (1 \leqq i \leqq k),$$
$$V_{2i} = a_{2i}, \overline{T}_{4,i}, a_{2i-1}, \overline{T}_{3,i}, \overline{T}_{2,i}, \overline{T}_{1,i} \qquad (1 \leqq i \leqq k),$$
$$V_{2k+1} = Q_k, \cdots, Q_1, \qquad V_{2k+2} = R_1, \cdots, R_k.$$

A routine inspection of this set of voters shows that it generates the desired pattern S. For example, consider what happens when $a_{2r}Pa_{2s}$ in S. For $r\neq i\neq s$, the effects of V_{2i} and V_{2i-1} cancel each other with respect to these two elements. The crucial voters in this case are seen to be V_{2r-1}, V_{2r}, V_{2s-1}, and V_{2s}. Since a_{2r} is in either $A_{1,s}$ or $A_{2,s}$ (depending on its relation to a_{2s-1}) and a_{2s} in either $A_{3,r}$ or $A_{4,r}$, three of these four crucial voters (including V_{2r} and V_{2r-1}) will have $a_{2r}Pa_{2s}$ and only one will have the reverse relation. This two-vote difference generates the desired relation. If a_{2r} and a_{2s} were indifferent, V_{2r} and V_{2r-1} would cancel V_{2s} and V_{2s-1} and the vote would be a tie. The proper relationship between a_{2r-1} and a_{2r} is established by the last two voters because of the way Q_r and R_r are ordered. Other pairs of elements check in a similar manner. *S has been generated with $2k+2$ voters.*

4. Now assume that S has $2k+1$ elements. If possible, choose an element to be called a_{2k+1} such that either (Case 1) $B=\{x: xPa_{2k+1}\}$ has an even number, $2r$, of elements or (Case 2) B has $2r+1$ elements and there is a $b\in B$ and a $c\in(A-B-\{a_{2k+1}\})$ such that $\sim(cPb)$. In case one, name the other elements so that $a_i\in B$ $(i=1, \cdots, 2r)$. In case two, name them so that $a_i\in B$ $(i=1, \cdots, 2r+1)$, $b=a_{2r+1}$, and $c=a_{2r+2}$.

Now define:

$$V_{2i-1} = T_{1,i}, a_{2i-1}, T_{2,i}, a_{2i}, T_{3,i}, T_{4,i} \qquad (1 \leqq i \leqq k),$$
$$V_{2i} = a_{2i}, \overline{T}_{4,i}, a_{2i-1}, \overline{T}_{3,i}, \overline{T}_{2,i}, \overline{T}_{1,i} \qquad (1 \leqq i \leqq k),$$
$$V_{2k+1} = a_{2k+1}, Q_k, \cdots, Q_1,$$
$$V_{2k+2} = R_1, \cdots, R_r, a_{2k+1}, R_{r+1}, \cdots, R_k \text{ (Case 1)}$$
$$V_{2k+2} = R_1, \cdots, R_r, a_{2r+1}, a_{2k+1}, a_{2r+2}, R_{r+2}, \cdots, R_k. \text{ (Case 2)}$$

A check of the various pairs of elements shows that this set generates S. Without a_{2k+1}, these voters are the same as those for $2k$ elements. The proper relation between a_{2k+1} and a_{2i} or a_{2i+1} is established by V_{2i-1}, V_{2i}, V_{2k+1}, and V_{2k+2}, the other pairs canceling.

Now assume that no point of A can be found for a_{2k+1} that satisfies either of

the two cases. *Then for all x, y, and z in A, either xPy, yPz, and zPx in S or the reverse relations hold.*

Proof. Assume that xIy in S for some distinct x and y. If Case 1 cannot be applied, there is an odd number of elements preferred over x and hence some z such that zPx. If yPz, apply Case 2 by letting $x=c$, $y=b$, and $z=a_{2k+1}$. Otherwise, apply Case 2 by letting $x=a_{2k+1}$, $y=c$, and $z=b$.

Now assume that S is strong. If xPy, yPz, and xPz for some x, y, and z and Case 1 does not apply, Case 2 applies by letting $x=b$, $y=a_{2k+1}$, and $z=c$. Thus the theorem holds.

A check of all the possible ways four elements can be related in a strong pattern reveals that the condition of the above theorem cannot hold for patterns on four or more elements. Thus the only pattern which does not fit into one of the two cases is the three element pattern on $\{a, b, c\}$ with aPb, bPc, and cPa. Since this can be generated with three voters (a, b, c; c, a, b; and b, c, a), *the arbitrary pattern on $2k+1$ elements has been generated with $2k+2$ voters or less.*

5. How good are these results? It would not be surprising if fewer voters are required for large n because large subsets of the elements in the pairs of voters have been ordered merely to cancel each other rather than to contribute constructively to the generation of the pattern. Nevertheless, the result is known to be exact for $n=3$, 4, and 5. It will now be shown that, *for large n, the number of voters required is greater than $.55n/(\log n)$.*

Proof. The number of patterns with n elements is $3^{\binom{n}{2}}$, there being three possible relations between each pair of elements. The number of patterns that can be generated with v voters on n elements is surely less than $(n!)^v$, the number of ordered sets of voters.

Any pattern that can be generated by exactly k voters can also be generated with exactly $k+2$ voters by adding two voters which cancel each other. If all the patterns on n elements can be generated with v or less voters (and consequently with exactly v or $v-1$ voters), the number of possible patterns that can be generated with exactly v or $v-1$ voters cannot be smaller than the number of patterns. Thus: $3^{\binom{n}{2}} \leqq (n!)^{v-1}+(n!)^v < 2(n!)^v$.

Now it is well known that $n! = pn^n e^{-n}(2\pi n)^{1/2}$ where p goes to one as n gets large. Substituting into the inequality and taking logarithms:

$$\frac{n(n-1)}{2}\log 3 < \log 2 + v[\log p + n\log n - n + \tfrac{1}{2}\log n + \tfrac{1}{2}\log 2\pi],$$

$$\frac{n-1}{2}\log 3 < \frac{\log 2}{n} + v\left[\frac{\log p}{n} + \log n - 1 + \frac{\log n}{2n} + \frac{\log 2\pi}{2n}\right].$$

For large n, $\frac{1}{2}(n-1)\log 3 < v(\log n - 1)$ or $v > .55n/(\log n)$.

6. Thus, for large n, the number of voters required to generate an arbitrary pattern lies between $n+2$ and $.55n/(\log n)$.

THE RELATION BETWEEN A COMPACT LINEAR OPERATOR AND ITS CONJUGATE

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1. Introduction. We present a systematic account of known theorems relating compact linear operators and their conjugates. Examples are given showing that all the theorems which are "possible" in a certain broad sense are already known. The general method is that of [6].

In what follows, X and Y are normed linear spaces. $[X, Y]$ is the set of bounded linear operators with domain X and range in Y. T denotes a linear operator and $R(T)$ is its range. T is compact if, for each bounded sequence (x_n) in X, (Tx_n) has a convergent subsequence. $[X, Y]_c$ stands for the set of compact linear operators with domain X and range in Y. We say that T has an inverse if $Tx=0$ implies $x=0$, *i.e.*, if T sets up a 1-1 mapping of X onto $R(T)$. The inverse mapping T^{-1} is also linear. X' is the space of bounded linear functionals on X, normed in the usual way. If $T\in[X, Y]$, the operator T' is defined as follows: $T'y'=x'$, where $x'\in X'$ is defined by $x'(x)=y'(Tx)$, all $x\in X$. $T'\in[Y', X']$ and $\|T'\|=\|T\|$.

Motivated by the known theorems relating a bounded operator T and its conjugate, we classify various possibilities for T by:

I: $R(T)=Y$ (indicated by writing $T\in I$).

II: $R(T)\neq Y$ but $\overline{R(T)}=Y$ (written $T\in$ II, and so on for succeeding cases).

III: $\overline{R(T)}\neq Y$.

1: T^{-1} exists and is bounded.

2: T^{-1} exists but is not bounded.

3: T^{-1} does not exist.

If $T\in$ II and $T\in 1$, we combine this by writing $T\in \mathrm{II}_1$. Thus there are nine possibilities for T. Similarly, T' has nine classifications. Thus, the pair (T, T') has 81 classifications. We call these 81 classifications the states of the pair (T, T'). If, for example, $T\in\mathrm{II}_1$ and $T'\in\mathrm{III}_3$, we say that the pair is in state $(\mathrm{II}_1, \mathrm{III}_3)$.

Taylor and Halberg [6] found the possible states for the pair (T, T') when $T\in[X, Y]$. They organized their results schematically as shown in Figure 1. Referring to this figure, if a box is crossed out, this means the corresponding state is impossible for any pair (T, T'), regardless of the choice of X and Y. If, on the other hand, a box is not crossed out, it is "possible." Some "possible" states become impossible when X and Y are suitably restricted. This is symbolically noted in the square representing the state. The key below the figure gives the meanings of the symbols.

Remark. A generalization to unbounded operators of the Taylor-Halberg

state diagram for bounded operators has been made by Goldberg [4]. The details are to appear in the Pacific Journal of Mathematics and the results are:

1) If $T: X \to Y$ is an operator whose graph is closed in the product topology of $X \times Y$ and the domain of T is dense in X, then the Taylor-Halberg state diagram remains true. (It is remarkable that such a vast extension of the operator class does not result in the opening up of a single new square in the state diagram.)

2) If $T: X \to Y$ is *any* linear operator with dense domain (Linear is being used to mean $T(ax+by) = aTx+bTy$; there are no topological implications.), the state diagram is the same as the Taylor-Halberg state diagram except that all the X and Y_r symbols are to be erased.

State Diagram for Bounded Linear Operators ($[X, Y]$)

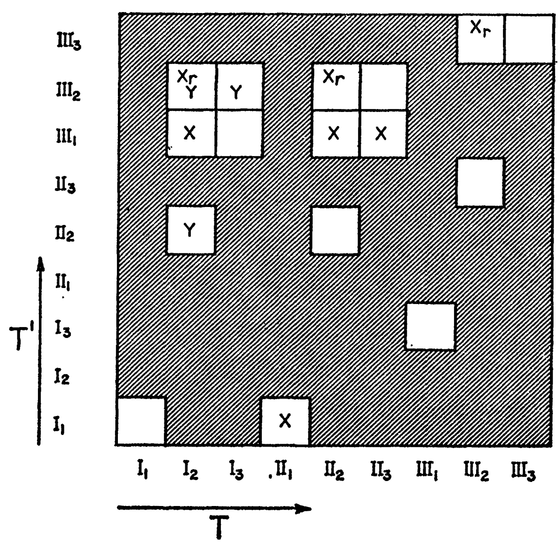

FIG. 1

Key: X: cannot occur if X is complete; Y: cannot occur if Y is complete; X_r: cannot occur if X is reflexive.

2. Derivation of the state diagram for compact operators. An operator is bounded if and only if it sends bounded sequences into bounded sequences. Convergent sequences are bounded. Therefore every compact operator is a bounded operator so $[X, Y]_c$ is a subset of $[X, Y]$. Hence the state diagram for $[X, Y]_c$ can be thought of as a restriction (fewer open squares) of the diagram for $[X, Y]$. The restriction of the state diagram for $[X, Y]$ to the state diagram for $[X, Y]_c$ follows from two lemmas below. Because of the form of Lemma 1, the state diagram will be established under the assumption that X and Y are

infinite dimensional. The simple case in which X and/or Y is finite dimensional is considered separately.

LEMMA 1. *If T is compact and $T \in 1$, then* dim $R(T)$ *and* dim X *are finite and equal* ([5], p. 115).

From Lemma 1, if X is infinite dimensional and T is compact, state 1 is impossible for T. This means that the first, fourth, and seventh columns of the $[X, Y]$ state diagram correspond to impossible states in the $[X, Y]_c$ state diagram. Similarly, Y' is infinite dimensional as a consequence of our assumption that Y is infinite dimensional, so the first, fourth and seventh rows are impossible states in the $[X, Y]_c$ state diagram. Thus we see from Lemma 1 that seven of the sixteen possible states for $[X, Y]$ are impossible for $[X, Y]_c$. To see the full strength of Lemma 1, observe that forty-five squares, more than half the total, are shown to be impossible by Lemma 1, without invoking any other theorems.

State Diagram for Compact Linear Operators ($[X, Y]_c$)

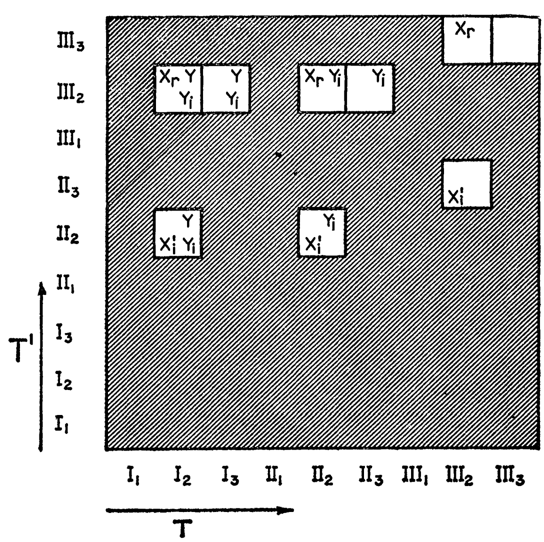

FIG. 2

Key: Y: impossible if Y is complete; X_r: impossible if X is reflexive; $Y_i(X_i')$: impossible if $Y(X')$ is inseparable.

The following result of Banach shows that the state diagram for compact operators is further restricted for certain choices of X' or Y.

LEMMA 2. *If T is compact, then $R(T)$ is separable* ([1], p. 96).

If T is not in III and Y is inseparable, then $\overline{R(T)}$ is inseparable. Hence $R(T)$ is inseparable and Lemma 2 shows that T is not compact. Thus, if T is compact, states I and II are impossible and the first six columns are deleted.

Similarly, if X' is inseparable the first six rows are impossible states. The resultant state diagram is shown in Figure 2. The examples of Section 3 show that this is the final form.

Remark. In constructing the state diagram for X and/or Y finite dimensional, it is simplest to consider in turn each of the three cases listed below. The results given are immediate. If neither X nor Y is (0), the two states shown in each case always exist. No others exist.

1. dim X = dim Y; X and Y are both finite dimensional. (III_3, III_3), (I_1, I_1).
2. dim X is greater than dim Y; Y is finite dimensional. (III_3, III_3), (I_3, III_1).
3. dim Y is greater than dim X; X is finite dimensional. (III_3, III_3), (III_1, I_3).

Obviously II and 2 are impossible for T and T' so the state diagram (Fig. 3) is drawn with only the 16 squares listed.

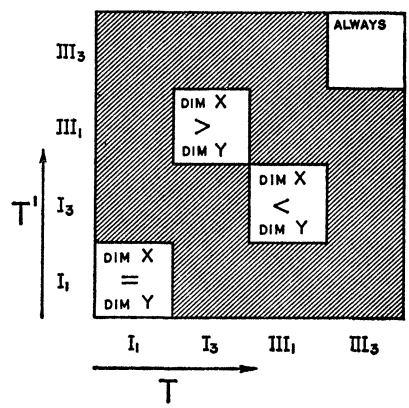

FIG. 3

Key: The four open squares correspond to existing states if and only if the conditions within the open squares are fulfilled.

3. All the states shown as possible exist "maximally." The examples to follow show that every state shown as "possible" actually exists. The examples are "maximal" in the sense that (1) X and Y are each complete, or even reflexive, and (2) X' and Y are inseparable, unless this is already forbidden by the $[X, Y]_c$ state diagram. Making X complete and Y reflexive is suggested by the occurrence of these restrictions in the $[X, Y]$ state diagram, for they do not appear as restrictions in the $[X, Y]_c$ state diagram.

(III_3, III_3): All the operators with finite-dimensional ranges are in this state for any infinite-dimensional (X, Y) pair. An (X, Y) pair such that X and Y are reflexive, and X' and Y are inseparable, is: $X = Y = l^2(Q)$ where the cardinality of Q is greater than $\aleph_0$. Note: For any set Q, $l^2(Q)$ is defined as the set of those

scalar-valued functions with domain Q such that (1) at most a countable number of the coordinates are nonzero and (2), $\sum|x_q|^2$ is finite. x_q is the qth coordinate of a typical function x. The norm of x is the square root of the sum in (2).

$(\text{II}_2, \text{II}_2)$: $X = Y = l^2$; let (u_k), $k = 1, 2, \cdots$ be a countable orthonormal basis in l^2. Define T by $Tu_k = 2^{1-k}u_k$. T is compact because it satisfies the criterion (see, *e.g.*, [3], Th. 7, Cor.) $\sum|t_{ij}|^2$ is finite, where (t_{ij}) is the matrix corresponding to T. $T' = T$; hence the state must be, according to the diagram, $(\text{II}_2, \text{II}_2)$ or $(\text{III}_3, \text{III}_3)$. But $\overline{R(T)} = Y$ because every element with at most a finite number of nonzero coordinates is in $R(T)$.

$(\text{II}_3, \text{III}_2)$: $X = Y = l^2$; define T by $Tu_k = 2^{1-k}u_{k-1}$, $k = 2, 3, \cdots$ and $Tu_1 = 0$. The arguments of the preceding example can be used to show T is compact and in $(\text{II}_3, \text{III}_2)$. To make this example "maximal," we must modify it so that X is inseparable. Let X_0 be the direct sum of l^2 and $l^2(Q)$, where $l^2(Q)$ is a nonseparable Hilbert space. Define T_0 as the direct sum of T_1 and T by setting $T_1 = 0$ on $l^2(Q)$. This T_0, with the pair (X_0, Y), is the desired example.

This device can be used to make X' or Y inseparable in all the following state examples, unless the state diagram already forbids this.

$(\text{III}_2, \text{II}_3)$: Use the conjugate of the operator in the preceding example.

$(\text{II}_2, \text{III}_2)$: $X = l^1$, $Y = l^2$; define T by $Tu_k = 2^{1-k}u_k$, as in example $(\text{II}_2, \text{II}_2)$. It is shown in [6], p. 104 that the state is $(\text{II}_2, \text{III}_2)$. Compactness is shown as follows: Let T_0 be the same as T except that the domain of T_0 is l^2. Let I_0 be the canonical imbedding of l^1 in l^2 defined by $I_0x = x$. It is readily verified that $\|I_0\| = 1$. Notice that $T = T_0I_0$ and that T_0 is compact by the previously used criterion. Hence T is compact.

$(\text{III}_2, \text{III}_3)$: $X = l^1$, $Y = l^2$; $Tu_k = 2^{-k}u_{k+1}$, $k = 1, 2, \cdots$; T is shown to be compact by the method of the previous example.

$(\text{I}_2, \text{II}_2)$, $(\text{I}_2, \text{III}_2)$, $(\text{I}_3, \text{III}_2)$: Examples of these states are obtained by modifying the previously given examples of the states $(\text{II}_2, \text{II}_2)$, $(\text{II}_2, \text{III}_2)$, $(\text{II}_3, \text{III}_2)$, respectively, using the procedure given in [6].

4. The state diagram for weakly compact operators is the same as that for bounded operators. The weak topology on X is the weakest (coarsest, smallest) topology making every element of X' continuous. A linear operator $T \in [X, Y]$ is weakly compact if the weak closure of $T(S_X)$ is compact in the weak topology on Y (S_X is the unit sphere in X). The set of weakly compact operators is designated by $[X, Y]_{wc}$. Observe that $[X, Y]_c$ is a subset of $[X, Y]_{wc}$ and that $[X, Y]_{wc}$ is itself a subset of $[X, Y]$. To show the second inclusion, note that a weakly compact set is weakly bounded hence, by the uniform boundedness principle it is norm-bounded. The first inclusion is true because $T \in [X. Y]_c$ means that the norm closure of $T(S_X)$ is norm-compact and hence weakly com-

pact. Now note that the norm closure of a manifold equals the weak closure, therefore the weak closure of $T(S_X)$ is weakly compact.

We now show the equality of state diagrams for $[X, Y]$ and $[X, Y]_{wc}$. We need the following result:

THEOREM. *Y is norm reflexive if and only if S_Y is weakly compact* (see, e.g., [2], Ch. IV, Sec. 5, n⁰ 2, Prop. 6).

This theorem shows us that $[X, Y]=[X, Y]_{wc}$ if Y is reflexive. Thus the examples in [6] which show a state is possible and have Y reflexive also show the state is possible for $[X, Y]_{wc}$. The only possible states in the $[X, Y]$ diagram not included by this are $(\mathrm{I}_2, \mathrm{II}_2)$, $(\mathrm{I}_2, \mathrm{III}_2)$, $(\mathrm{I}_3, \mathrm{III}_2)$. But if T is compact, T is weakly compact so the examples of these three states given in the $[X, Y]_c$ diagram suffice for the $[X, Y]_{wc}$ diagram.

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A CONSTRUCTION OF THE RATIONAL NUMBERS

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1. Introduction. Constructing the real numbers from the rationals is necessary for an understanding of what the real numbers are. Not so the construction of the rationals starting from the natural numbers. Yet the possibility of such a construction is an interesting, classical, and elementary fact. Knowing this fact by having seen the construction is worthwhile, provided that the cost is not too great: the proof should either be quick and easy, as Landau estimated it to be ([1], Preface for the Teacher), or it should at the same time teach something else of value, as Thurston has made it do [2]. The new method given here is probably not easier than the well-known ones, but it uses algebraic methods which hopefully lend it new interest. Such a treatment would go well in an abstract algebra course, if not in beginning analysis.

The usual motivation for signed integers is arithmetical: observing that sub-

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traction of natural numbers is only sometimes possible, one contrives artificial new numbers to serve as differences of natural numbers. The point of view we have adopted is rather geometrical: the integers arise as a group of transformations. My intuitive preference for this point of view, apparently also preferred by Alfred North Whitehead [**3**], derives perhaps from the directness with which most of the algebraic properties of addition fall out. Indeed, if we use functions from the natural numbers to the natural numbers to represent integers, and functions from the integers to the integers to represent rationals, the following things fall immediately into place: the operations of addition and multiplication, with their associative laws, identities and inverses, come spontaneously into being as instances of composition of functions; the usual recursive definition of addition is seen to be the assertion that the functions being "added" are endomorphisms of the Peano succession of natural numbers; the left distributive law, of which the recursive definition of multiplication is a special case, is the assertion that the functions being "multiplied" are endomorphisms of the additive group; the absence of a multiplicative inverse for 0 is the inevitable result of the function representing 0 not being one to one. Note also how the principle of mathematical induction becomes one instance of a more general phenomenon, the use of a generator for an algebraic system.

Assuming some knowledge of sets, functions and groups, as we do, is not only for the sake of brevity; without prior knowledge of these, this would be hard reading. What is, of course, studiously avoided is any reference to a previous knowledge of arithmetic.

2. Successions. A pair (N, s), where N is a set and s a function mapping N into N, we shall call a *succession*. Subsuccessions are formed by restricting s to subsets of N which are closed under s. Such subsets we shall call *tails*: A is a tail when $s(A) \subset A$. We shall loosely write (A, s) for the resulting subsuccession. We call (N, s) a *Peano* succession when the following three axioms are satisfied:

INVERSION AXIOM. *s is one to one (s^{-1} is a function).*

INFINITY AXIOM. $s(N) \neq N$.

INDUCTION AXIOM. *(N, s) has a* generator, *that is, an element e of N such that N is the only tail containing e.*

A *homomorphism* of a succession (N_1, s_1) into a succession (N_2, s_2) is a function f mapping N_1 into N_2 such that $f(s_1(x)) = s_2(f(x))$ for every x in N_1. A homomorphism is an *isomorphism* if it is one to one and onto. All Peano successions are isomorphic, and they are regarded as models for the natural number system. We omit the proof of this uniqueness theorem, along with many other important things.

2.1. *If e is a generator for a succession (N, s), then* $s(N) \cup \{e\} = N$.

Proof. The set $s(N) \cup \{e\}$ is a tail containing e.

2.2. *If (N, s) is a Peano succession with generator e, then $s(N)=N-\{e\}$, and $(s(N), s)$ is a Peano succession with generator $s(e)$.*

Proof. Since $s(N)$ is a tail and $s(N)\neq N$, $e\notin s(N)$ by the induction axiom. Using this and 2.1, $s(N)=N-\{e\}$. If A is a tail contained in $s(N)$ and containing $s(e)$, then $A\cup\{e\}$ is a tail in N containing e. Hence $A\cup\{e\}=N$ and $A=s(N)$. Thus $s(e)$ is a generator for $(s(N), s)$, and the other axioms for $(s(N), s)$ are trivial.

3. Cyclic groups. We shall write groups additively. A group G is called *cyclic* if it has a *generator*, that is, an element 1 of G such that G is the only subgroup containing 1. (Compare this with the Induction Axiom for successions.)

3.1. *Every cyclic group is Abelian.*

Proof. The *centralizer* C_x of an element x of G is the set of all y such that $x+y=y+x$. It is easy to see that C_x is a subgroup, and that $x\in C_x$. In particular, for any generator 1 of G, $C_1=G$. Now the *center* C of G is by definition the intersection of all C_x, therefore a subgroup. Also $x\in C$ if and only if $C_x=G$. Having shown that $C_1=G$, we have $1\in C$, and therefore $C=G$. Thus G is Abelian.

We probe the structure of G by looking at the successions generated by its elements. For any y, the *translation* by y is the function T_y defined by $T_y(x)=y+x$ for all x. Note that T_0 is the identity function. Let 1 be a generator. In the succession (G, T_1) the intersection of all tails containing 0 is a tail, the smallest tail containing 0, called the *ray* through 1, and denoted G_1. By its definition the succession (G_1, T_1) satisfies the inversion and induction axioms. If for a generator 1 (G_1, T_1) is actually a Peano succession, we say that G is *free*. All free cyclic groups are isomorphic (as will be seen from Theorem 2), and they are models for the group of the integers. We write $-G_1$ for $\{x\colon -x\in G_1\}$.

3.2. *If 1 is a generator, then $G=G_1\cup(-G_1)$.*

Proof. Let $A=G_1\cup(-G_1)$, and let $H=\{x\colon T_x(A)\subset A\}$. Then H is a subgroup. For if $T_x(A)\subset A$ and $T_y(A)\subset A$, then $T_{x+y}(A)=T_x(T_y(A))\subset A$, and $T_{-x}(A)=-T_x(-A)\subset A$, because $-A=A$. Furthermore it is easy to show using 2.1 that $T_1(A)\subset A$, so that $1\in H$. Thus $H=G$. This shows that A is closed under addition. Since $A=-A$, A is a subgroup, and since $1\in A$, $A=G$.

3.3. *G_1 is closed under addition.*

Proof. We use induction in (G_1, T_1). The set M of those x in G_1 such that $T_x(G_1)\subset G_1$ is a tail. For if $x\in M$, then $T_{1+x}(G_1)=T_1(T_x(G_1))\subset T_1(G_1)\subset G_1$. Since obviously $0\in M$, $M=G_1$.

3.4. *If G is a free cyclic group with 1 a generator, then $G_1\cap(-G_1)=\{0\}$.*

Proof. Let $A=G_1-\{0\}$. We must show that A and $-A$ are disjoint. If

$x \in A \cap (-A)$, then x and $-x$ belong to A. By 2.1 $x = 1 + y$ for some y in G_1. Then $y + (-x) \in G_1$ by 3.3, and $1 + y + (-x) = x + (-x) = 0$. Thus $0 \in T_1(G_1)$, which according to 2.2 is impossible, since by hypothesis (G_1, T_1) is a Peano succession.

The partition of G into two closed-under-addition sets like this is tantamount to the introduction of an order in G, making G into a linearly ordered group: one arbitrarily chooses one of the generators, 1 or -1, to be the *positive* one (say 1 is chosen) and defines $x \leqq y$ to mean $y + (-x) \in G_1$. We shall ignore this possibility, but we shall need these lemmas in Section 6.

4. Skew-fields. A *skew-field* is a set F having more than one element, with two operations, addition and multiplication, such that F with addition is an Abelian group, multiplication is an associative operation with an identity 1 and inverses for all elements except 0 (the additive identity), and both the distributive laws hold: $(x+y) \cdot z = x \cdot z + y \cdot z$ and $z \cdot (x+y) = z \cdot x + z \cdot y$. A skew-field is *prime* if it has no proper subfields. It is a *field* if multiplication is commutative.

4.1. *Every prime skew-field is a field.*

Proof. The set C_x of all y such that $x \cdot y = y \cdot x$ is closed under all the operations of F. Since $1 \in C_x$, C_x is a subfield, whence $C_x = F$.

We probe the structure of a prime field F by looking at the subgroups generated by its elements. For any x in F, the intersection of all (additive) subgroups containing x is a subgroup F_x called the subgroup through x. In particular F_1 is called the *integer group* of F. The integer group is automatically cyclic; if it is free, we say that F has *characteristic zero.* All prime fields of characteristic zero are isomorphic, as will be seen from Theorem 4, and they are models for the rational number system.

5. From a Peano succession to a free cyclic group.

THEOREM 1. *Given a succession (N, s) satisfying the inversion and induction axioms, there exists a cyclic group I such that (N, s) is isomorphic to the ray (I_1, T_1) through a generator of I; I is free if and only if (N, s) is a Peano succession.*

The proof is carried out by the numbered steps which follow. It can be outlined thus. In 5.1 we show that (N, s) can be isomorphically represented by a succession consisting of its endomorphisms. Composition of endomorphisms gives us an addition, but the result may not be a group because the inverse of an endomorphism (which exists by 5.2) may not be an endomorphism, its domain not being all of N. Now look at the end product: I is isomorphically represented by its translations, and these act in I_1 (the image of N) as homomorphisms of tails. This suggests defining the elements of I to be homomorphisms of tails in N, which we do, except that homomorphisms which "agree" are identified.

In this section we denote composition of functions with values in N by the addition sign: $(f+g)(x)=f(g(x))$ for all x for which this makes sense. An *endomorphism* of (N, s) means a homomorphism of (N, s) into itself.

5.1. *To each x in N there corresponds a unique endomorphism f_x such that $f_x(e)=x$. Every endomorphism is one of these, f_e is the identity function on N, and $f_{s(x)}=s+f_x$ for all x in N.*

Proof. Induction on x: Let M be the set of those x such that such an endomorphism exists. Then $e\in M$ because for f_e you can use the identity, and M is a tail, because if f_x is already in hand, for $f_{s(x)}$ you can use $s+f_x$. Hence $M=N$. To prove the uniqueness, suppose f and g are endomorphisms such that $f(e)=g(e)=x$; then the set where f and g agree is a tail containing e, hence all of N, and $f=g$. If f is any endomorphism whatever, let $x=f(e)$; then by the uniqueness just proved $f=f_x$.

5.2. *All endomorphisms of (N, s) are one to one.*

Proof. Induction on x: Let M be the set of those x such that f_x is one to one. Then $e\in M$ because the identity is one to one, and M is a tail because, s being one to one, if f_x is one to one so is $f_{s(x)}=s+f_x$. Therefore $M=N$.

5.3. *If A is a tail and f an endomorphism, then $f(A)\subset A$.*

Proof. Induction on x: For any A let M be the set of those x such that $f_x(A)\subset A$. Then $e\in M$ because $f_e(A)=A$, and M is a tail because $s(A)\subset A$ so that if $f_x(A)\subset A$ then $f_{s(x)}(A)=s(f_x(A))\subset s(A)\subset A$. Thus $M=N$.

5.4. *If A and B are nonempty tails, then $A\cap B$ is a nonempty tail.*

Proof. $A\cap B$ is obviously a tail. Now let $x\in A$ and $y\in B$. Then by 5.3, $f_x(y)\in B$. Now the set of those z such that $f_x(z)\in A$ is a tail containing e ($f_x(e)=x\in A$), so all of N. In particular $f_x(y)\in A$, and $A\cap B$ is not empty.

Define a *translation* of (N, s) as a homomorphism of a nonempty tail of (N, s) into (N, s). Endomorphisms are translations, namely those with domain N. Addition of translations is composition, in accordance with our announced notational policy. The importance of 5.4 is to insure that a sum of translations shall have a nonempty domain. Indeed, if f and g are translations with domains D_f and D_g, then $g(D_g)$, the range of g, is a nonempty tail which by 5.4 intersects D_f. Hence $D_g\cap g^{-1}(D_f)$, which is the domain of $f+g$, is not empty.

Equality at infinity is a relation between translations, written $f=(\infty)g$, defined thus: $f=(\infty)g$ if and only if $f(x)=g(x)$ for all x in some nonempty tail. Since $\{x:f(x)=g(x)\}$ is a tail in any case, the existence of such an x is sufficient for $f=(\infty)g$ to hold.

5.5. *Equality at infinity is an equivalence.*

Proof. For any translation f, $f=(\infty)f$ because the domain of f is not empty. If

$f=(\infty)g$, then $g=(\infty)f$ trivially. If $f=(\infty)g$ and $g=(\infty)h$, then $f(x)=g(x)=h(x)$ for any x in the intersection of the nonempty tails $\{x: f(x)=g(x)\}$ and $\{x: g(x)=h(x)\}$. Such an x exists by 5.4.

5.6. *If* $f_1=(\infty)f_2$ *and* $g_1=(\infty)g_2$, *then* $f_1+g_1=(\infty)f_2+g_2$.

Proof. By repeated use of 5.4 there exists x in the intersection of the nonempty tails $\{x: g_1(x)=g_2(x)\}$, $\{x: f_1(g_1(x))=f_2(g_1(x))\}$, and the domains of f_1+g_1 and f_2+g_2. For such an x, $f_1(g_1(x))=f_2(g_1(x))=f_2(g_2(x))$.

5.7. *For any translation* f *there are endomorphisms* g *and* h *such that* $f=(\infty)g+h^{-1}$.

Proof. Let x belong to the domain of f, let $g=f_{f(x)}$ and let $h=f_x$.

Let the elements of I be the equivalence classes of translations which are equal at infinity. For any translation f let $i(f)$ be the equivalence class to which it belongs. Define addition in I by $i(f)+i(g)=i(f+g)$; this is possible by 5.6. Let $0=i(f_e)$ and $1=i(s)$.

5.8. *I is a cyclic group with* 1 *for a generator.*

Proof. Addition is associative for translations because it is composition; it is therefore associative in I. The identity is 0. By 5.7 and 5.2 any element of I contains a one to one translation, say f, whose inverse is therefore also a translation. Then $i(f^{-1})$ is $-i(f)$. Finally, to show that 1 is a generator, let G be a subgroup containing 1. Let $M=\{x: i(f_x)\in G\}$. Then $e\in M$ because $0\in G$, and M is a tail because $i(f_x)\in G$ and $1\in G$ imply $i(f_{s(x)})=i(s+f_x)=1+i(f_x)\in G$. Thus $M=N$ and $i(f)\in G$ for every endomorphism f. By 5.7, $i(f)\in G$ for every translation f, and $G=I$.

5.9. *Define* ϕ *mapping* N *into* I *by* $\phi(x)=i(f_x)$. *Then* ϕ *is an isomorphism of* (N, s) *with* (I_1, T_1).

Proof. We have shown in the proof of 5.8 that ϕ maps N into I_1. Now $\phi(e)=i(f_e)=0$, and for all x, $\phi(s(x))=1+i(f_x)=T_1(\phi(x))$, so that ϕ is a homomorphism, whose range is a tail containing 0, therefore all of I_1. To show that ϕ is one to one we must show that distinct endomorphisms are never equal at infinity. Induction: Let M be the set of x such that if $f(x)=g(x)$ for endomorphisms f and g, then $f=g$. The uniqueness part of 5.1 shows that $e\in M$. If $x\in M$, then $f(s(x))=g(s(x))$ gives $s(f(x))=s(g(x))$, whence $f(x)=g(x)$ and (since $x\in M$) $f=g$. Thus M is a tail, $M=N$, and ϕ is one to one.

We have completed the proof of Theorem 1. That I is free if and only if (N, s) is a Peano system follows immediately from the definition. Note that I is Abelian by 3.1.

The next theorem, while not essential, tidies things up by answering some

uniqueness questions: Does the isomorphism of N with I_1 uniquely determine I? Are the rays through different generators of a cyclic group always isomorphic?

THEOREM 2. *Let G be a cyclic group, 1 a generator of G. Let $N=G_1$ and $s=T_1$. Let I be the group constructed from (N, s) in Theorem* 1, *and ϕ the isomorphism of (N, s) to (I_1, T_1). Then ϕ can be extended to an isomorphism of G with I.*

Proof. The set of differences $x+(-y)$ with $x, y \in N$ form a subgroup containing $1=1+(-0)$; hence this set is G. Then given any $z=x+(-y)$ we have $z+y=x\in N$, so some element of N goes into N under T_z. Since T_z commutes with T_1, some restriction of T_z is a translation of (N, s), and since all such restrictions are equal at infinity, we can meaningfully define $\psi(z)$ as $i(T_z)$, and ψ maps G into I. It is easy to show that ψ has the required properties of extending ϕ and being an isomorphism.

6. From a free cyclic group to a field.

THEOREM 3. *Given a free cyclic group I, there exists a prime field R of characteristic zero such that I is isomorphic to the integer group of R.*

With one exception, the proof runs parallel to that of Theorem 1, and can be motivated in a similar way. In 6.1 we show that I can be isomorphically represented by a group consisting of its endomorphisms. Composition of endomorphisms gives us a multiplication which in effect makes I into a ring. The isomorphism of the theorem is actually an embedding of this ring in R. Since a field cannot have divisors of 0 (nonzero elements x and y such that $x\cdot y=0$), it is important to know that our ring of endomorphisms has none. This is the content of 6.2. The one place where the argument departs from imitating Theorem 1 is in the proof of 6.2: it is necessary here to use the fact that I is free, as can be seen from the example of the group of integers modulo 6 (for which the conclusion of Theorem 3 is false). Looking now at R, for any y, the *dilation* by y means the function D_y defined by $D_y(x)=y\cdot x$. R is isomorphically represented by its dilations, and these act in R_1 (the image of I) as homomorphisms of subgroups. This suggests defining the elements of R to be homomorphisms of subgroups in I, but again an identification of equivalent homomorphisms must be made.

Note the following change of notation: for functions f and g with values in I, $f+g$ now means the function defined by $(f+g)(x)=f(x)+g(x)$, while composition is indicated as multiplication: $(f\cdot g)(x)=f(g(x))$. As usual 1 is a generator of I.

6.1. *To each x in I there corresponds an endomorphism f_x such that $f_x(1)=x$. Every endomorphism is one of these, f_0 is the constant function mapping I into* 0, *f_1 is the identity function, and for all x and y, $f_{x+y}=f_x+f_y$, and $f_{-x}=-f_x$.*

Proof. Let H be the set of those x such that such an endomorphism exists. Then $1\in H$ because for f_1 you can use the identity, and H is a subgroup because if f_x and f_y are already in hand, for f_{x+y} and f_{-x} you can use f_x+f_y and $-f_x$ respec-

tively. Since 1 is a generator, $H=G$. To prove uniqueness, suppose f and g are endomorphisms such that $f(1)=g(1)=x$. Then the set where f and g agree is a subgroup containing 1, hence all of G, and $f=g$. If f is any endomorphism, let $x=f(1)$; then by the uniqueness just proved $f=f_x$.

Hereafter when a proof resembles this closely the corresponding proof in Section 5, we shall omit it.

6.2. *All endomorphisms of I except f_0 are one to one.*

Proof. To show that an endomorphism f_x is one to one, we need only show that $f_x(y)\neq 0$ for $y\neq 0$. By 3.2 and symmetry it is sufficient to treat x and y in $A=I_1-\{0\}$. Now A is closed under addition, for if $u, v\in A$, then $u+v\in I_1$ by 3.3, and $u+v$ cannot be 0 since this would require $u=0$ by 3.4, I being free. Let M be the set of those x in A such that $f_x(A)\subset A$. Then M is a tail in (A, T_1), because if $x\in M$ and $y\in A$, then $f_x(y)\in A$, whence $f_{1+x}(y)=y+f_x(y)\in A$, since A is closed under addition. By 2.2 (A, T_1) is a Peano succession whose generator is 1, and $1\in M$. Hence $M=A$. Since $f_x(y)\in A$ precludes $f_x(y)=0$, 6.2 is proved.

6.3. *If G is a subgroup and f an endomorphism, then $f(G)\subset G$.*

Proof. Like that of 5.3.

6.4. *If G and H are nontrivial subgroups (i.e., not $\{0\}$), then $G\cap H$ is a nontrivial subgroup.*

Proof. Let x and y be nonzero elements of G and H respectively. As in 5.4, using 6.3, $f_x(y)\in G\cap H$. By 6.2, $f_x(y)\neq 0$, so $G\cap H\neq\{0\}$.

Define a *dilation* of I as a homomorphism of a nontrivial subgroup of I into I. Endomorphisms are dilations, namely those with domain I. Addition of dilations has been defined as pointwise addition, and multiplication as composition. From 6.4 we are assured that if f and g are dilations, so are $f+g$ and $f\cdot g$, the argument for $f\cdot g$ being the same as that used in the corresponding situation in Section 5.

Equality in the neighborhood of 0 is a relation between dilations, written $f=(0)g$, defined thus: $f=(0)g$ if and only if $f(x)=g(x)$ for all x in some nontrivial subgroup. Again, the existence of some $x\neq 0$ such that $f(x)=g(x)$ is sufficient for $f=(0)g$.

6.5. *Equality in the neighborhood of* 0 *is an equivalence.*

Proof. Like 5.5.

6.6. *If $f_1=(0)f_2$ and $g_1=(0)g_2$, then $f_1+g_1=(0)f_2+g_2$ and $f_1\cdot g_1=(0)f_2\cdot g_2$.*

Proof. Like 5.6.

6.7. *For any dilation f there are endomorphisms g and h such that $f=(0)g\cdot h^{-1}$.*

Proof. Like 5.7.

Let the elements of R be the equivalence classes of dilations which are equal in the neighborhood of 0. For any dilation f, let $r(f)$ be the equivalence class to which it belongs. Define addition and multiplication in R by $r(f)+r(g)=r(f+g)$ and $r(f)\cdot r(g)=r(f\cdot g)$; this is possible by 6.6. In particular, let $0=r(f_0)$ and $1=r(f_1)$.

6.8. *R is a prime field.*

Proof. The Abelian group properties of R under addition come from the corresponding properties of I, via the corresponding properties for addition of dilations. Multiplication is associative because it comes from composition of dilations. The multiplicative identity is 1. By 6.7 any nonzero element of R contains a one to one dilation, say f, whose inverse is therefore a dilation; then $r(f^{-1})$ is $r(f)^{-1}$. The distributive laws for dilations go as follows: $(g+h)\cdot f(x) = g(f(x))+h(f(x)) = (g\cdot f+h\cdot f)(x)$, and $(f\cdot(g+h))(x) = f(g(x)+h(x)) = f(g(x)) + f(h(x)) = (f\cdot g+f\cdot h)(x)$ because f is a homomorphism. They carry over trivially to R. To show that R is prime, let F be a subfield, and let G be the set of x in I such that $r(f_x)\in F$. Then G is a subgroup of I containing 1, whence $G=I$. Thus $r(f)\in F$ for every endomorphism f, and by 6.7 $r(f)\in F$ for every dilation, that is $F=R$. By 4.1 R is a field.

6.9. *Define ϕ mapping I into R by $\phi(x)=r(f_x)$. Then ϕ is an isomorphism of I with the integer group of R. Hence R has characteristic zero.*

Proof. Like 5.9, except that proving ϕ is one to one is easier.

We have completed the proof of Theorem 3, and with it the construction of the rational numbers. Again, however, the following uniqueness theorem is worth having.

THEOREM 4. *Let F be a prime field of characteristic zero. Let I be the integer group of F, and let R be the field constructed from I in Theorem 3, ϕ the isomorphism of I with R_1. Then ϕ can be extended to an isomorphism of F with R.*

Proof. Like Theorem 2.

As a concluding remark we note that the process goes no further. A prime field has no endomorphisms except the identity, and no subfields except itself. What we have been exploiting in Peano successions and free cyclic groups is that they are "homogeneous" in the peculiar sense that each element is the beginning of a subsystem isomorphic with the whole parent system. Fields lack this flexibility.

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ON SETS OF ACQUAINTANCES AND STRANGERS AT ANY PARTY

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1. Introduction. In a recent issue of the MONTHLY, the following elementary problem [1] was posed:

*Prove that at a gathering of any six people, some three of them are either mutual acquaintances or complete strangers to each other.**

It is our purpose to prove a more general result when the number 6 is replaced by any positive integer N (see Theorem 1 below).

It is convenient to transform the problem into an equivalent problem concerning points and lines. The N persons involved are replaced by points A_k, $k=1, \cdots, N$, no three of which are collinear and if two persons are acquainted a line is drawn joining the corresponding pair of points. If the two persons are strangers then no line is drawn. Thus each collection of N people gives rise to a corresponding configuration of N points and L lines where $0 \leqq L \leqq N(N-1)/2$. If three people are mutually acquainted the corresponding figure is a triangle which we will call a *full triangle.* If three people are pairwise strangers the corresponding figure consists of three points with no lines joining any pair. We call such a figure an *empty triangle.* Any set of three points not the vertices of a full triangle, nor an empty triangle, will be called a *partial triangle.* Notice that a given point may simultaneously be a vertex of several triangles from each category. With these definitions we have

THEOREM 1. *Let E and F be the number of empty and full triangles respectively. Then in any configuration of N points*

$$E + F \geqq \begin{cases} \dfrac{u(u-1)(u-2)}{3}, & \text{if } N = 2u, \\[2ex] \dfrac{2u(u-1)(4u+1)}{3}, & \text{if } N = 4u+1, \\[2ex] \dfrac{2u(u+1)(4u-1)}{3}, & \text{if } N = 4u+3, \end{cases} \tag{1}$$

where u is a nonnegative integer, and this lower bound is sharp for each positive integer N.

We observe that for $N=6$, $u=3$ and then (1) gives $E+F \geqq 2$, which is a stronger result than the original problem suggested.

* The same problem in a different disguise appeared on the thirteenth William Lowell Putnam Examination in 1953: Six points are in general position in space (no three in a line, no four in a plane). The fifteen line segments joining them in pairs are drawn and then painted, some segments red, some blue. Prove that some triangle has all its sides the same color. In connection with other generalizations of this problem, see the article by Greenwood and Gleason [3].

2. The fundamental equations. Let p_j denote the number of points, each of which is a terminal point for exactly j of the lines. Then obviously

$$N = p_0 + p_1 + \cdots + p_L, \tag{2}$$

where $p_j \geqq 0$ for $j=0, 1, \cdots, L$. Since each of the L lines, joins two points a counting of the lines on each point gives $2L$; that is,

$$2L = p_1 + 2p_2 + \cdots + Lp_L. \tag{3}$$

Each line may be combined with each of the $N-2$ remaining points not on the line to form $N-2$ partial or full triangles. Hence there are $(N-2)L$ partial or full triangles, but in this counting some triangles have been counted more than once. Let us consider a particular point A at which j lines end. Any pair of these lines determines either a partial triangle with two sides, or a full triangle. The number of such partial triangles with two sides or full triangles with one vertex at A is $j(j-1)/2$, and for the entire configuration we have the sum

$$S = \sum_{j=2}^{L} \frac{j(j-1)}{2} p_j. \tag{4}$$

Now in the expression $(N-2)L$, a partial triangle with two sides has been counted twice and a full triangle has been counted three times. In the expression (4) each partial triangle with two sides is counted only once. However in (4) the full triangles are counted three times. Hence the expression

$$(N-2)L - \sum_{j=2}^{L} \frac{j(j-1)}{2} p_j \tag{5}$$

counts each partial triangle exactly once. Since the total number of triangles possible with N points is $N(N-1)(N-2)/6$ we have

LEMMA 1. *For any configuration of N points and L lines,*

$$E + F = \frac{N(N-1)(N-2)}{6} - (N-2)L + \sum_{j=2}^{L} \frac{j(j-1)}{2} p_j. \tag{6}$$

To minimize $E+F$, we consider the right side of (6). First for each fixed L we will minimize the sum S, and then we will determine a value for L which gives an absolute minimum for $E+F$.

LEMMA 2. *Let N and L be fixed with $0 \leqq L \leqq N(N-1)/2$. Then there is a set of nonnegative integers $(p_0, p_1, \cdots, p_L)$ which satisfies* (2) *and* (3) *and among such sets there is a unique one which makes S a minimum. For the minimizing set, at most two of the p_j are nonzero, and these two must be adjacent* (*subscripts differ by one*). *Further the minimizing set corresponds to a real configuration of lines and points.*

Proof. It is obvious that the system of equations (2) and (3) has at least one

solution in nonnegative integers because L lines can always be drawn. It follows immediately that some solution minimizes S. Let $(p_0, p_1, \cdots, p_L)$ be some solution in which two nonadjacent entries are positive. Indeed, let $p_j > 0$ and $p_k > 0$ with $j+d=k$, $d>1$. We construct a new solution $(p_0^*, p_1^*, \cdots, p_L^*)$ as follows. First suppose that $d>2$. Then set

$$
\begin{aligned}
&p_i^* = p_i, \quad \text{if} \quad i \neq j, j+1, k-1, k,\\
&p_j^* = p_j - 1, \qquad p_{k-1}^* = p_{k-1} + 1,\\
&p_{j+1}^* = p_{j+1} + 1, \qquad p_k^* = p_k - 1.
\end{aligned}
\tag{7}
$$

It is obvious that this new set of nonnegative integers also satisfies (2) and (3). Further an easy computation shows that

$$S^* = S - (k - j - 1) = S - (d - 1) < S, \tag{8}$$

where S^* denotes the expression (4), evaluated for the new solution.

If $d=2$, so that $p_j>0$ and $p_{j+2}>0$, then we set

$$
\begin{aligned}
&p_i^* = p_i, \quad \text{if} \quad i \neq j, j+1, j+2,\\
&p_j^* = p_j - 1, \qquad p_{j+1}^* = p_{j+1} + 2, \qquad p_{j+2}^* = p_{j+2} -
\end{aligned}
\tag{9}
$$

Again it is obvious that the set $(p_0^*, p_1^*, \cdots, p_L^*)$ satisfies (2) and (3) and that for this set

$$S^* = S - 1 < S. \tag{10}$$

Further, any solution of (2) and (3) corresponds to a realizable configuration, for it is easy to draw from each point the $2L$ half-lines as dictated by (3), assigning the proper number of half-lines to each point, and then join these half-lines pairwise to obtain L lines.

Thus starting with any solution, we can apply the two transformations described above stepwise until we arrive at a solution for which either $p_k>0$ and $p_{k+1}>0$ and the remaining p_j vanish, or a solution in which $p_k=N$ and the remaining p_j vanish. Finally the minimizing solution thus obtained does not depend on the initial solution chosen, because in any case where at most two of the p_j are nonzero the equation set (2) and (3) reduces to a pair of linear equations in at most two unknowns p_k and p_{k+1}. Solving these equations simultaneously gives

$$p_k = (k + 1)N - 2L. \tag{11}$$

But then the index k is uniquely determined by (11) and the fact that $0 < p_k \leqq N$. This completes the proof of Lemma 2. For example if $N=20$ and $L=44$ then by (11) $k=4$, $p_4=12$ and $p_5=8$ and this pair minimizes S for these values of L and N.

3. Determination of the absolute minimum. Let $\Delta(L)$ denote the change in the minimum of $E+F$ as L increases from L to $L+1$. Two cases must be considered. With L fixed we suppose that the solution which makes S a minimum has the form

$$\text{I. } p_k \geqq 2, \qquad p_{k+1} = N - p_k \geqq 0;$$
$$\text{II. } p_k = 1, \qquad p_{k+1} = N - 1.$$

When L is increased to $L+1$, the solutions which minimize S are respectively

$$\text{I. } \overset{*}{p}_k = p_k - 2, \qquad \overset{*}{p}_{k+1} = p_{k+1} + 2;$$
$$\text{II. } \overset{*}{p}_k = 0, \qquad \overset{*}{p}_{k+1} = N - 1, \qquad \overset{*}{p}_{k+2} = 1.$$

From the equation

$$\Delta(L) = -(N-2)(L+1) + (N-2)L + \sum_{j=2}^{L+1} (\overset{*}{p}_j - p_j)\frac{j(j-1)}{2}, \tag{12}$$

we find in Case I that

$$\Delta(L) = 2k + 2 - N, \tag{13}$$

and in Case II, that

$$\Delta(L) = 2k + 3 - N. \tag{14}$$

Now suppose that $N=2u$ is even. As L increases from 0 to $N(N-1)/2$, (13) and (14) show that $E+F$ first decreases, then is stationary, and then increases. Whence the minimum occurs when $k=u-1$ and whenever the conditions of Case I are satisfied. Thus the number of lines for a minimizing solution is not unique but is given by $2L=(u-1)p_{u-1}+up_u$, where p_{u-1} and p_u are the only nonzero elements among the p_j. However either p_{u-1} or p_u may be zero. Thus for L we have the limits $u^2-u\leqq L\leqq u^2$. To compute the minimum of $E+F$, we assume that $p_{u-1}=N$, whence $L=u^2-u$, $S=(u-1)(u-2)u$. Using these values in (6) gives the minimum announced in the theorem for N even.

Next suppose that $N=4u+1$. We observe that in Case I, $\Delta(L)$ is negative for $k=0, 1, \cdots, 2u-1$ and is positive thereafter. In Case II, $\Delta(L)=0$ for $k=2u-1$. Then $p_{2u-1}=1$, $p_{2u}=N-1$, and from (3), $2L=8u^2+2u-1$. But this is impossible because L must be an integer. Therefore the absolute minimum occurs in Case I, for the smallest index for which $\Delta(L)>0$. This gives $k=2u$, and $p_{2u}=N$, and hence $2L=2uN$, an even number. Then $S=2u(2u-1)(4u+1)/2$, and when these values are used in (6) we obtain $E+F=2u(u-1)(4u+1)/3$.

If $N=4u+3$, then Case II gives $\Delta(L)=0$ when $k=2u$. Thus one solution occurs when $p_{2u}=1$, $p_{2u+1}=N-1=4u+2$, $L=4u^2+5u+1$, and $S=8u^3+10u^2+u$. When these values are used in (6) we obtain $E+F=2u(4u-1)(u+1)/3$. A second minimizing solution occurs in this case, because $\Delta(L)=0$. For this second solution $p_{2u+1}=N-1$, $p_{2u+2}=1$, and $L=4u^2+5u+2$.

COROLLARY. *Let $E+F$ be a minimum for some configuration. Then*

$$u^2 - u \leqq L \leqq u^2, \qquad \text{if } N = 2u,$$

$$L = u(4u + 1), \qquad \text{if } N = 4u + 1,$$

$$(u + 1)(4u + 1) \leqq L \leqq (u + 1)(4u + 1) + 1, \qquad \text{if } N = 4u + 3.$$

Further if $N=4u+1$, each point lies on $2u$ lines. If $N=4u+3$, each point with one exception lies on $2u+1$ lines and the exceptional point lies either on $2u$ lines or on $2u+2$ lines.

4. Some open questions. It seems obvious that the methods used here should generalize to answer questions about the minimum number of full and empty quadrilaterals, and figures of a higher number of sides, but up to the present, I have not been able to carry through the computations successfully.

One could also ask if it is possible to have a configuration which minimizes $E+F$ in which either E or F is zero. In case $N=2u$ we can give an affirmative answer, but in the other cases, I have not been able to settle this question.

If C is a configuration, we construct C^*, a conjugate configuration, by drawing the line joining A_i and A_j in C^* if the line is missing in C, and by leaving the line out in C^* if they are joined by a line in C. Then empty triangles go into full triangles, partial triangles go into partial triangles, and full triangles go into empty ones in passing from C to C^*. It follows from this that if we wish to consider the problem of minimizing $E+F$ with $F=0$ or $E=0$ we need only examine one of the cases, say, $F=0$.

Suppose now that $N=2u$. We join points A_i and A_j if and only if i and j have different parity ($i-j$ is an odd integer). Then obviously there are u^2 lines in the resultant configuration. There are no full triangles because given any three points A_i, A_j, A_k, at least two must be of the same parity, and hence two of them are not joined by a line. The only empty triangles are those $A_iA_jA_k$ in which all three subscripts have the same parity. Therefore $E=u(u-1)(u-2)/3$. Thus in the configuration just described, $E+F$ is a minimum and $F=0$. The same type of configuration is not minimizing when N is odd.

Ramsey [4] has proved a very general existence theorem. Suppose that N points (or objects) are given and we form from these points all possible subsets consisting of k points, so that we have $\binom{N}{k}$ such subsets. These subsets are then distributed into r classes $C_1, \cdots, C_r$. If m is an integer, $m>k$, can we find m points such that all of the subsets of k of these m points fall in one of the classes C_i?

RAMSEY'S THEOREM. *There is a smallest integer $N_0=N_0(k, m, r)$ such that if $N\geqq N_0$, then for any distribution of the subsets of k points into r classes there is a set of m points for which all of its subsets of k points fall in one class.*

The function $N_0(k, m, r)$ is known as Ramsey's function, and the determination of this function, except in the simplest cases, is an open problem and a very

difficult one. The ideas which lie behind this theorem have been extended in a variety of directions, and one can find an account of these extensions and further references in an article by Erdös and Rado [2].*

The simplest case of Ramsey's theorem occurs when pairs of points are distributed into two classes ($k=2$, $m=2$) and we are to find $m=3$ points such that all pairs of these 3 points are in one of the classes. As the problem proposed by Bostwick shows, this will always occur when $N\geqq 6$, and it is easy to construct a distribution of the pairs when $N=5$, for which the property fails. This shows that $N_0(2, 3, 2)=6$.

If $k=2$, it is convenient to consider the pairs of points as connected by lines, and these lines painted with a suitable color corresponding to the particular class C_i ($i=1, \cdots, r$) in which it falls. When this is done the result is a chromatic graph. Greenwood and Gleason [3] have studied these chromatic graphs and have proved that N_0 $(2, 4, 2)=18$ and N_0 $(2, 3, 3)=17$. As far as the author is aware these three cases are the only nontrivial cases for which the Ramsey function has been evaluated, although various estimates have been obtained [2].

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* I am indebted to John Isbell for calling my attention to this reference. I am also indebted to the referee for calling my attention to the article by Greenwood and Gleason [3].

SPHÈRES ASSOCIÉES À UN TÉTRAÈDRE†

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Notations. Soient un tétraèdre $T\equiv ABCD$ inscrit à une sphère (O), de centre O; a, a', b, b', c, c' et A, B, C, D les mesures des arêtes BC, DA, CA, DB, AB, DC dont les milieux sont A_1, A_1', B_1, B_1', C_1, C_1', et celles des aires des faces BCD, CDA, DAB, ABC; (DA) la sphère tangente en D à la face BCD et passant par les sommets D et A; (D) le terne de sphères (DA), (DB), (DC); (A), (B), (C) les ternes de sphères analogues à (D) associées aux sommets A, B, C; $(BCD\text{-}DA)$

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la sphère tangente en D à l'arête DA et circonscrite à la face BCD; $(\mathfrak{D})$ le terne des sphères $[(DBC\text{-}DA), (DCA\text{-}DB), (DAB\text{-}DC)]$; $(\mathfrak{A})$, $(\mathfrak{B})$, $(\mathfrak{C})$ les ternes de sphères analogues à $(\mathfrak{D})$ et associées aux ternes d'arêtes (AB, AC, AD), (BA, BC, BD), (CA, CB, CD).

1. THÉORÈME. *Les axes radicaux des ternes de sphères* (A), (B), (C), (D) *se confondent avec ceux des ternes* $(\mathfrak{A})$, $(\mathfrak{B})$, $(\mathfrak{C})$, $(\mathfrak{D})$ *et forment un quadruple hyperholoïdique.*

Démonstration. Une inversion i, de pôle D, transforme les sommets A, B, C en des points A', B', C' et les sphères (DA), (DB), (DC) en des plans menés par A', B', C' parallélement aux faces opposées de T. Ces plans coupent la droite joignant D au barycentre g du triangle $A'B'C'$ en un point g' tel que $gg' = 2\,Dg$, et la droite Dg coincide avec l'axe radical du terne (D). Les sphères $(BCD\text{-}DA)$, $(CDA\text{-}DB)$, $(DAB\text{-}DC)$ se transforment en les plans parallèles aux arêtes du trièdre $D\text{-}ABC$ menés par $B'C'$, $C'A'$, $A'B'$ qui concourent sur Dg en un point g'' tel que $gg'' = Dg/2$. Les axes radicaux des ternes (D) et $(\mathfrak{D})$ sont donc confondus ainsi que, par analogie, ceux de (A) et $(\mathfrak{A})$, (B) et $(\mathfrak{B})$, (C) et $(\mathfrak{C})$. D'autre part, les plans (DA, Dg), (DB, Dg), (DC, Dg) passent par une médiane des triangles $DB'C'$, $DC'A'$, $DA'B'$, autrement dit, par une symédiane des triangles DBC, DCA, DAB, puisque $(BC, B'C')$, $(CA, C'A')$, $(AB, A'B')$ sont des couples antiparallèles pour ces triangles. La droite Dg rencontre donc les céviennes, par rapport à T, des points de *Lemoine* des faces DBC, DCA, DAB et par suite les céviennes des points de *Lemoine* des faces de T. Ces quatre céviennes étant rencontrées par la droite Dg et ses trois analogues, les huit droites appartiennent à un même hyperboloïde.

Autrement, l'axe radical du terne (D) rencontre le plan ABC au point D_2 de coordonnées barycentriques $(1/a'^2, 1/b'^2, 1/c'^2)$, pour le triangle ABC [1]. Or, les distances des points de l'axe radical du terne $(\mathfrak{D})$ aux plans DBC, DCA, DAB étant proportionnelles aux quantités $(1/A\cdot a'^2, 1/B\cdot b'^2, 1/C\cdot c'^2)$ [2], les axes radicaux de (D) et $(\mathfrak{D})$ coïncident. D'autre part, d'après les coordonnées de D_2 et des points analogues A_2, B_2, C_2, sur BCD, CDA, DAB, les droites AA_2, BB_2, CC_2, DD_2 sont hyperboloïdiques [3].

Remarques. L'axe radical $Dg \equiv DD_2$ des sphères (D) est le lieu des barycentres des sections antiparallèles de T relatives au trièdre $D\text{-}ABC$, sections parallèles au plan tangent en D à (O). Il rencontre respectivement les sphères (D), les sphères $(\mathfrak{D})$, aux transformés des points g', g'', g par i et situés à des distances de D proportionnelles à 1, 2, 3. Les centres des sphères (D), $(\mathfrak{D})$ coïncident avec six sommets d'un parallélpipède dont O et D sont les autres sommets opposés. Le trièdre de sommet D de ce parallélipipède est le supplémentaire du trièdre $D\text{-}ABC$. Le centre d'une sphère de (D) coïncide avec le symétrique de O par rapport au milieu de la distance des centres de deux sphères de $(\mathfrak{D})$ et vice versa.

COROLLAIRE. *Si le tétraèdre est isodynamique, les axes radicaux* AA_2, BB_2, CC_2, DD_2 *concourent au second point de Lemoine de* T.

Démonstration. A_2, B_2, C_2, D_2 coïncident avec les points de *Lemoine* des faces BCD, CDA, DAB, ABC [4].

2. THÉORÈME. *Les axes radicaux des ternes de sphères* $[(BA), (CA), (DA)]$, $[(CB), (DB), (AB)]$, $[(DC), (AC), (BC)]$, $[(AD), (BD), (CD)]$ *coïncident avec les droites qui joignent les sommets de T aux centres des cercles circonscrits aux faces opposées et forment un quadruple hyperboloïdique si le tétraèdre T est équifacial, orthocentrique ou bisymétrique.*

Démonstration. Les sphères (AD), (BC), (CD) étant tangentes au plan ABC en A, B, C le centre O_d du cercle ABC a même puissance pour ces cercles, et la droite DO_d est l'axe radical des trois sphères. Même conclusion pour les droites AO_a, BO_b, CO_c et les ternes de sphères correspondantes. D'autre part, les droites qui joignent les sommets d'un tétraèdre non dégénéré aux centres des cercles circonscrits aux faces opposées sont hyperboloïdiques si, et seulement si le tétraèdre fondamental T est équifacial, orthocentrique ou bisymétrique [5].

3. THÉORÈME. *Le centre radical de quatre sphères passant par trois sommets et tangentes à une même arête ou à celle qui lui est opposée est situé sur la bimédiane relative à ces deux arêtes opposées de T.*

Démonstration. Il est clair, par exemple, que le plan radical π des sphères $(BCD\text{-}DA)$, $(ABC\text{-}AD)$ passe par l'arête BC et le milieu A_1' de la tangente DA à ces sphères, et que celui des sphères $(DAB\text{-}BC)$, $(CDA\text{-}CB)$ passe par DA et A_1. Le centre radical E_1 de ces quatre sphères est donc sur la bimédiane A_1A_1' qu'il divise dans le rapport $-a'^2/a^2$, car, d'après le théorème classique relatif à la différence des puissances d'un point par rapport à deux sphères, on constate que π divise l'arête CA dans ce rapport $-a'^2/a^2$. Or, les plans passant par l'arête DB coupent CA et A_1A_1' suivant des divisions semblables puisque le points à l'infini se correspondent.

Autrement, A, B, C, D ayant pour puissances a'^2, a^2, a^2, a'^2, par rapport aux sphères $(BCD\text{-}DA)$, $(CDA\text{-}CB)$, $(DAB\text{-}BC)$, $(ABC\text{-}AD)$, le centre radical E_1 de ces sphères coincide avec le point de coordonnées barycentriques $(1/a'^2, 1/a^2, 1/a^2, 1/a'^2)$ [2]. De même, les centres radicaux E_2, E_3 des sphères $(BCD\text{-}CA)$, $(CDA\text{-}DB)$, $(DAB\text{-}AC)$, $(ABC\text{-}BD)$ et $(BCD\text{-}BA)$, $(CDA\text{-}AB)$, $(DAB\text{-}DC)$, $(ABC\text{-}CD)$, de coordonnées $(1/b^2, 1/b'^2, 1/b^2, 1/b'^2)$ et $(1/c^2, 1/c^2, 1/c'^2, 1/c'^2)$, sont sur les bimédianes B_1B_1' et C_1C_1' qu'ils divisent dans les rapports $-b'^2/b^2$ et $-c'^2/c^2$.

4. THÉORÈME. *Les centres radicaux des trois groupes de quatre sphères tangentes à une face de T et passant par les deux sommets d'une même arête ou de son opposée, sont trois points alignés situés sur les côtés du triangle déterminé par les centres radicaux des trois groupes de quatre sphères passant par trois sommets de T et tangentes à une même arête ou à son opposée* (J. Hecquet).

Démonstration analytique. Pour faciliter l'écriture, modifions les notations en considérant un tétraèdre $T \equiv A_iA_jA_kA_l$, d'arêtes $A_iA_j = a_{ij}$, $\cdots$. Soient (ij) la

sphère passant par A_i et A_j et tangente en A_i à la face $A_iA_kA_l$; (ikl) la sphère passant par A_i, A_k, A_l et tangente en A_i à l'arête A_iA_j. En se rapportant a un système de coordonnées barycentriques pour T, et posant

$$S = \sum a_{ij}^2 x_i x_j \text{ (sphère } A_iA_jA_kA_l), \qquad I = \sum x_i \text{ (plan de l'infini)},$$

la sphère (ikl) a pour équation

$$-S + I \cdot a_{ij}^2 x_j = 0,$$

et le centre radical e_1 des sphères (ikl), (jkl), (kij), (lij) de coordonnées $x_i = x_j = a_{kl}^2$, $x_k = x_l = a_{ij}^2$, coïncide avec le centre de la quadrique

$$Q_{ij} \equiv a_{kl}^2 u_i u_j + a_{ij}^2 u_k u_l = 0.$$

D'autre part, la sphère (ij) a pour équation

$$-S + I(a_{ik}^2 x_k + a_{il}^2 x_l) = 0,$$

et le centre radical f_1 des sphères (ij), (ji), (kl), (lk), qui a pour coordonnées

$$x_i = a_{jl}^2 - a_{jk}^2, \qquad x_j = a_{ik}^2 - a_{il}^2, \qquad x_k = a_{jl}^2 - a_{il}^2, \qquad x_l = a_{ik}^2 - a_{jk}^2,$$

coïncide avec le centre de la quadrique $q_{ij} \equiv Q_{ik} - Q_{il} = 0$, situé sur la droite des centres des quadriques Q_{ij}, Q_{il}. Les trois quadriques q_{ij}, q_{ik}, q_{il} qui correspondent aux sphères (ij), (ik), (il) satisfont à la condition

$$q_{ij} + q_{ik} + q_{il} = 0,$$

et appartiennent à un faisceau tangentiel; leurs centres f_1, f_2, f_3 sont donc colinéaires et situés sur les droite des centres e_1, e_2, e_3 de Q_{ij}, Q_{ik}, Q_{il}.

COROLLAIRE. *Le centre de la quadrique tangente aux quatres faces de T en leurs points de Lemoine est situé dans le plan $p \equiv (e_1, e_2, e_3)$.*

Démonstration. Le centre de cette quadrique d'équation

$$Q_{ij} + Q_{ik} + Q_{il} \equiv \sum a_{ij}^2 u_k u_l = 0$$

est dans le plan des centres de Q_{ij}, Q_{ik}, Q_{il}, lieu des centres des quadriques du réseau tangentiel déterminé par ces trois quadriques.

COROLLAIRE. *Si le tétraèdre T est équifacial, les centres radicaux e_1, e_2, e_3, f_1, f_2, f_3 sont confondus avec le barycentre de T.*

COROLLAIRE. *Le centre de la quadrique tangente aux quatre faces de T en leurs points de recontre avec les axes radicaux des ternes de sphères définis au théorème* (Sec. 1), *est situé dans le plan p.*

Démonstration. L'axe radical des sphères (ikl), (ijl), (ijk) a pour équations $a_{ij}^2 x_j = a_{ik}^2 x_k = a_{il}^2 x_l$ et sa trace sur $x_i = 0$ pour coordonnées

$$x_i = 0, \qquad x_j = 1/a_{ij}^2, \qquad x_k = 1/a_{ik}^2, \qquad x_l = 1/a_{il}^2.$$

Or, la quadrique $Q_{ij} = 0$ peut aussi s'écrire

$$Q'_{ij} \equiv u_i u_j / a_{ij}^2 + u_k u_l / a_{kl}^2 = 0,$$

et le centre de la quadrique

$$Q'_{ij} + Q'_{ik} + Q'_{il} \equiv \sum u_i u_j / a_{ij}^2 = 0,$$

tangente aux faces de T aux quatre points en cause, est situé dans le plan p.

5. Points de Brocard. Si l'on marque des points arbitraires a, b, c sur les côtés BC, CA, AB d'un triangle t, les cercles Abc, Bca, Cab concourent en un point D. Lorsque a, b, c coïncident respectivement avec les sommets de t, ce qui donne

$$c \equiv A,\ b \equiv C,\ a \equiv B \quad \text{ou} \quad c \equiv B,\ b \equiv A,\ a \equiv C,$$

les cercles Abc, Bca, Cab sont tangents en A, B, C à AB, BC, CA ou CA, AB, BC, et ces ternes de cercles concourent aux points de *Brocard* de t. Par analogie avec cette figure plane, après avoir constaté qu'une inversion dont le pôle est extérieur au plan ABC transforme celui-ci en une sphère et les droites AB, BC, CA en des cercles concourants, si l'on choisit arbitrairement un point α, α', β, β', γ, γ' sur chaque arête BC, DA, CA, DB, AB, DC d'un tétraèdre T, les sphères $A\beta\gamma\alpha'$, $B\gamma\alpha\beta'$, $C\alpha\beta\gamma'$, $D\alpha'\beta'\gamma'$ ont un point commun Q. (sphères de *Miquel*, [**6**]). Si quatre sphères prises dans l'un ou l'autre ou dans les deux groupes $[(A), (B), (C), (D)]$, $[(\mathfrak{A}), (\mathfrak{B}), (\mathfrak{C}), (\mathfrak{D})]$, sont des sphères de *Miquel*, un ou deux points α, α', β, β', γ, γ' coïncident nécessairement avec des sommets de T. On constate aisément que les sphères (DA), (AB), (BC), (CD) ne forment pas un quaterne de *Miquel*, pas plus que les sphères $(ABC\text{-}AD)$, $(BCD\text{-}BA)$, $(CDA\text{-}CB)$, $(DAB\text{-}DC)$. Mais dans l'hypothèse selon laquelle

$$\alpha \equiv C, \quad \alpha' \equiv A, \quad \beta \equiv A, \quad \beta' \equiv D, \quad \gamma \equiv B, \quad \gamma' \equiv D,$$

les sphères (DA), (AB), $(BCD\text{-}BA)$, $(CDA\text{-}CB)$ forment un quaterne de *Miquel*, car

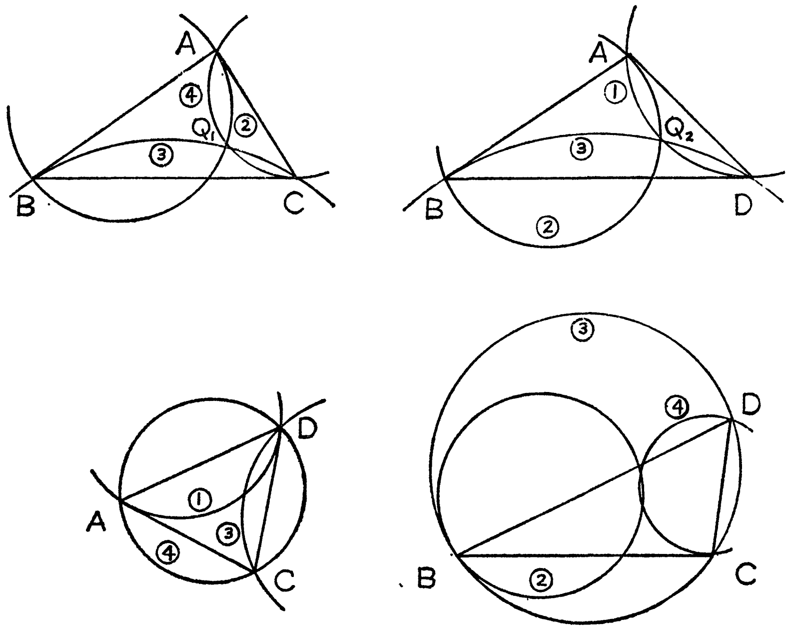
$$\left.\begin{array}{r}(DA)\\(AB)\\(BCD\text{-}BA)\\(CDA\text{-}CB)\end{array}\right\} \text{coupe les arêtes} \left\{\begin{array}{ccc}DA, & DB, & DC\\AD, & AB, & AC\\BA, & BD, & BC\\CB, & CA, & CD\end{array}\right\} \text{en} \left\{\begin{array}{lll}A, & D, & D,\\A, & B, & A,\\B, & D, & C,\\C, & A, & D.\end{array}\right.$$

De même

$$\left.\begin{array}{r}(AD)\\(BA)\\(ABC\text{-}CD)\\(BCD\text{-}DA)\end{array}\right\}\text{ coupe les arêtes }\left\{\begin{array}{ccc}AD, & AB, & AC\\BA, & BC, & BD\\CA, & CB, & CD\\DA, & DB, & DC\end{array}\right\}\text{ en }\left\{\begin{array}{ccc}D, & A, & A,\\A, & B, & B,\\A, & B, & C,\\D, & B, & C,\end{array}\right.$$

et ces sphères forment un quaterne de *Miquel* avec

$$\alpha \equiv B, \quad \alpha' \equiv D, \quad \beta \equiv A, \quad \beta' \equiv B, \quad \gamma \equiv A, \quad \gamma' \equiv C.$$



Dans chacune de ces hypothèses, les quatres sphères considérées concourent en un point Q comparable à l'un des points de ***Brocard*** d'un triangle. En associant ainsi les sphères des deux groupes, on obtient 24 quaternes de sphères de *Miquel* auxquels correspondent 24 points Q.

Construction d'un point Q. Si l'on pose $(DA) \equiv$ ①, $(AB) \equiv$ ②, $(BCD\text{-}BA) \equiv$ ③, $(CDA\text{-}CB) \equiv$ ④, l'examen des figures permet de constater que les sections des sphères [②, ③, ④] et [②, ③, ①] par les plans ABC et DAB déterminent, grâce à ③, des cercles qui se coupent en l'un des points Q_1 et Q_2 de ***Brocard*** des faces ABC et DAB. Le point Q commun aux quatre sphères en cause coïncide donc avec l'image commune de Q_1 et Q_2 par rapport aux plans des centres des sphères ②, ③, ④ et ②, ③, ①. Si l'on choisit deux faces de T, choix possible de six manières, et un point de ***Brocard*** dans l'une quelconque de ces deux faces, trois des quatre sphères sont déterminées et la quatrième peut être choisie de deux manières, ce qui porte bien à 24 le nombre des points Q.

Références

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MATHEMATICAL NOTES

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INTEGRAL TRANSFORMATIONS AND CLOSED-FORM EXPRESSIONS FOR SUMS OF INFINITE SERIES

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Wheelon [3] has presented an interesting application of the Laplace transformation in obtaining closed form expressions for the sums of certain infinite series. Since the volumes by Erdélyi et al. [2] include extensive tables of transformation pairs other than those of the Laplace type some of these others might be useful.

Consider, in particular, the Mellin transformation

$$g(s) = \int_0^\infty x^{s-1} f(x)dx.$$

If we set $s=n+1$, multiply by coefficients a_n, and sum on n; then, provided that it is permissible to interchange the operations of summation and integration (for example, see Bromwich [1]),

$$\sum_{n=0}^\infty a_n g(n+1) = \int_0^\infty \left\{ \sum_{n=0}^\infty a_n x^n \right\} f(x)dx.$$

Thus, if the terms of the desired series can be expressed as a product of the coefficients of a known power series and of Mellin transforms, the series can be

expressed as a definite integral and tables will perhaps lead to a closed form expression.

With certain specific choices for the a_n we can reduce the integral on the right to standard transformation integrals (sometimes a simple change of variable is also required) as, for example:

(1) $a_n = (-b)^n/n!$, Laplace transformation,

$$\sum_{n=0}^{\infty} (-b)^n g(n+1)/n! = \int_0^{\infty} e^{-bx} f(x)dx;$$

(2) $a_n = (-b^2)^n/(2n)!$, Fourier cosine transformation,

$$\sum_{n=0}^{\infty} (-b^2)^n g(n+1)/(2n)! = \int_0^{\infty} \cos bu\{2uf(u^2)\}du;$$

(3) $a_n = (-b^2)^n/2^{2n}(n!)^2$, Hankel transformation of order zero,

$$\sum_{n=0}^{\infty} (-b^2)^n g(n+1)/2^{2n}(n!)^2 = b^{-1/2}\int_0^{\infty} J_0(bu)(bu)^{1/2}\{2u^{1/2}f(u^2)\}du;$$

(4) $a_n = (-1)^n/b^n$, Stieltjes transformation,

$$\sum_{n=0}^{\infty} (-1)^n g(n+1)/b^n = b\int_0^{\infty} (b+x)^{-1}f(x)dx;$$

In each case the g-function is the Mellin transform of the f-function.

A specific example will illustrate the technique. Consider the sum

$$\sum_{n=0}^{\infty} (-2c)^n\Gamma\{(n-\nu+1)/2\}\Gamma\{(n+\nu+1)/2\}/\Gamma(n+1) = \sum_{n=0}^{\infty} (-c)^n g(n+1)/n!$$

with the terms factored as indicated and where the Mellin transform is chosen as $g(s) = 2^{s-1}\Gamma\{(s-\nu)/2\}\Gamma\{(s+\nu)/2\}$, so that from [2], 6.8(26), $f(x) = 2K_\nu(x)$, and the integral becomes

$$2\int_0^{\infty} e^{-cx}K_\nu(x)dx.$$

This is a Laplace integral, hence from [2], 4.16(24) the value of the sum for $|\nu| < 1$ is given as

$$\pi \csc(\nu\pi)(c^2-1)^{-1/2}\{[c+(c^2-1)^{1/2}]^{\nu} - [c+(c^2-1)^{1/2}]^{-\nu}\}.$$

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A NOTE ON FUNCTIONS CONTINUOUS ALMOST EVERYWHERE

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Let $f: R \to R$ be a single valued function of a real variable, R being the set of real numbers. f is termed a *CAE* if and only if f is continuous almost everywhere. We give now a necessary and sufficient condition for f to be a *CAE*.

THEOREM 1. *$f: R \to R$ is a CAE if and only if for O^* open, then $f^{-1}(O^*)$ can be written as a union $O \cup A$, where O is open and $m(A)=0$. $m(A)$ denotes the Lebesgue measure of A.*

Proof. Necessity. Let $f: R \to R$ be a *CAE*. Let O^* be open in R and $X=f^{-1}(O^*)$. Set $X=X_c \cup X_d$ where X_c is the set of points of continuity of f in X and X_d is the set of points of discontinuity of f in X. Let $x \in X_c$. Then there exists O_x such that $x \in O_x$ and $f(O_x) \subset O^*$. Then $O_x \subset f^{-1}(O^*)$. Hence $X=X_c \cup X_d \subset \bigcup_{x \in X_c} O_x \cup X_d \subset X$. Thus $X=\bigcup_{x \in X_c} O_x \cup X_d$, where $m(X_d)=0$ and $\bigcup_{x \in X_c} O_x$ is open.

Sufficiency. Let E be the set of points of discontinuity of f. Let $p \in E$. Then there exists an O_p^* open such that $f(p) \in O_p^*$ and $p \in O$ implies that $f(O) \not\subset O_p^*$. There exists a neighborhood $N(s; r)$ with rational center s and positive rational radius r such that $f(p) \in N(s; r) \subset O_p^*$. Then $p \in O$ implies $f(O) \not\subset N(s; r)$. But $f^{-1}(N(s; r))=O_{sr} \cup A_{sr}$, where O_{sr} is open and A_{sr} has measure zero. Now $p \in f^{-1}(N(s; r))$ and $p \notin O_{sr}$. Thus $p \in A_{sr}$. Hence $E \subset \bigcup_{r,s} A_{sr}$ and thus $m(E)=0$.

COROLLARY. *$f: R \to R$ is CAE if and only if F^* closed implies $f^{-1}(F^*)$ can be written as $F-A$, where F is closed and $m(A)=0$.*

Proof. Sufficiency. Let O^* be open. Then $f^{-1}(-O^*)=F-A$, where F is closed and $m(A)=0$. Taking complements we have $-f^{-1}(-O^*)=-F \cup A$ or $f^{-1}(O^*) = O \cup A$, where $O=-F$ and thus is open. This proves that f is a *CAE*.

Necessity. Let F^* be closed and f a *CAE*. Then $f^{-1}(F^*)=-f^{-1}(-F^*) = -(O \cup A)$, where O is open and $m(A)=0$. Thus $f^{-1}(F^*)=(-O) \cap (-A) = (-O)-A$. Since $-O$ is closed, the necessity is proved.

A *CAE* of a *CAE* is not in general a *CAE*. For consider the following example: Let

$$g(x) \equiv \begin{cases} 0 \text{ if } x \text{ is irrational,} \\ 1/p \text{ if } x = q/p \text{ where } q, p \text{ are relatively prime integers and } p \text{ positive.} \end{cases}$$

It is well known that g is a *CAE*. Now let

$$f(x) \equiv \begin{cases} 1 & \text{if } x = 1/n \text{ for } n = 1, 2, \cdots, \\ 0 & \text{otherwise.} \end{cases}$$

It is also obvious that f is a *CAE*. But the composite fg is 1 if x is rational and 0 if x is irrational. Thus fg is not a *CAE*.

Suppose f and g are both *CAE* and let $h=fg$. Then $h^{-1}(O^*)=g^{-1}(f^{-1}(O^*)) = g^{-1}(O^{\#} \cup A^{\#}) = g^{-1}(O^{\#}) \cup g^{-1}(A^{\#}) = O \cup A \cup g^{-1}(A^{\#})$, where O, $O^{\#}$, and O^* are

open and A and $A^{\#}$ are sets of measure zero. Then h is a CAE if and only if the set $O \cup A \cup g^{-1}(A^{\#})$ can be written as the union of an open set and a set of measure zero. The following theorems are then easy consequences of the above remarks.

THEOREM 2. *Let f and g be CAE, and $m(g(A))>0$ whenever $m(A)>0$. Then $h=fg$ is CAE.*

THEOREM 3. *Let f be continuous and g be CAE. Then $h=fg$ is CAE.*

We conclude with the remark that the set of CAE's is a ring which might have some interesting properties.

AN APPLICATION OF MATRICES TO LINEAR RECURSION RELATIONS*

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Matrix methods are helpful in solving certain problems stemming from linear recursion relations, such as that of finding an explicit expression for the nth term of the Fibonacci sequence, or of analyzing the vibration of a weighted string ([1] pp. 152–154). The procedure will be illustrated by means of a simple example.

Suppose that a sequence of complex numbers, $\{a_n\}$, is defined recursively as follows:

$$a_0 = a,\ a_1 = b,\ a_{n+2} = p \cdot a_{n+1} + q \cdot a_n,\ n \geqq 0\ (a, b, p, q \text{ complex}).$$

What is an explicit expression for a_n in terms of a, b, p, q, n?

We note that

$$\begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix} = \begin{bmatrix} p & q \\ 1 & 0 \end{bmatrix}\begin{bmatrix} a_n \\ a_{n-1} \end{bmatrix} = \begin{bmatrix} p & q \\ 1 & 0 \end{bmatrix}^2\begin{bmatrix} a_{n-1} \\ a_{n-2} \end{bmatrix} = \cdots = \begin{bmatrix} p & q \\ 1 & 0 \end{bmatrix}^n\begin{bmatrix} b \\ a \end{bmatrix}.$$

Let

$$R = \begin{bmatrix} p & q \\ 1 & 0 \end{bmatrix}.$$

The characteristic equation of R is $k^2 - pk - q = 0$.

Case 1. $p^2+4q \neq 0$. In this case, R has two distinct characteristic roots, k_1, k_2, say. Then there exists a matrix A such that

$$A^{-1}RA = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}.$$

It is clear that $(A^{-1}RA)^n = A^{-1}R^nA$. Therefore

* Presented at a meeting of the Northeastern Section of the Mathematical Association of America at Durham, N. H., on Nov. 25, 1955.

$$R^n = A\begin{bmatrix} k_1^n & 0 \\ 0 & k_2^n \end{bmatrix} A^{-1}.$$

Hence

$$\begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix} = A\begin{bmatrix} k_1^n & 0 \\ 0 & k_2^n \end{bmatrix} A^{-1}\begin{bmatrix} b \\ a \end{bmatrix}.$$

Since the elements of A are independent of n, we can conclude that $a_n = c_1 \cdot k_1^n + c_2 \cdot k_2^n$, where c_1, c_2 are complex numbers independent of n. We can determine the values of c_1, c_2 by setting $n=0$, 1: $c_1+c_2=a$, $c_1k_1+c_2k_2=b$. These two equations, together with the values of k_1, k_2 obtained from the characteristic equation, give the desired result.

Case 2. $p^2+4q=0$. In this case, R has a repeated characteristic root, k_0, say. There exists a matrix A such that

$$A^{-1}RA = \begin{bmatrix} k_0 & 0 \\ 1 & k_0 \end{bmatrix}.$$

Hence

$$\begin{bmatrix} a_{n+1} \\ a_n \end{bmatrix} = \begin{bmatrix} k_0^n & 0 \\ nk_0^{n-1} & k_0^n \end{bmatrix} A^{-1}\begin{bmatrix} b \\ a \end{bmatrix}.$$

By arguing as in Case 1, one obtains $a_n = c_1k_0^n + c_2 \cdot n \cdot k_0^{n-1}$, where $c_1 = a$ and $c_2 = b - a \cdot k_0$.

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THE DIMENSION OF THE MAGIC SQUARE VECTOR SPACE*

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It was stated in [1] that the set of all magic squares of order n forms a linear subspace of the algebra of semimagic squares. We became interested in determining the dimension of this space. Our solution, given below, involves the characteristic of the field in a rather interesting way.

Let F be a field. For matrices A, $B \in F_n$, define $(A|B)$ to be the trace of AB^T. Regarding F_n as an F-vector space, we wish to determine the dimension of the subspace M_n of those $M \in F_n$ for which

$$(1) \qquad (C_i | M) = (R_i | M) = (I | M) = (J | M) \qquad (i = 1, \cdots, n),$$

* This note is extracted from a Master's thesis written at the State University of Iowa. The present proof is due to suggestions of Professors S. K. Berberian and M. F. Smiley. The original proof, as given in the thesis, is somewhat less elegant but has the advantage of providing an explicit basis for M_n.

where $C_i(R_i)$ is the matrix of F_n whose ith column (row) has all entries 1 and whose other entries are zero, I is the identity matrix of F_n, and J is the matrix of F_n such that $a_{ij}=1$ for $i+j=n+1$ and $a_{ij}=0$ otherwise $(i, j=1, \cdots, n)$. Elements of M_n are called *F-magic squares*. Since the cases $n=1, 2$ are trivial, we restrict our discussion to $n \geqq 3$.

LEMMA 1. *If $n \geqq 3$, then $C_1, \cdots, C_n, R_1, \cdots, R_{n-1}, I, J$ are F-linearly independent unless $n=3$ and $3=0$ in F or $n=4$ and $2=0$ in F.*

Proof. Assume that

$$\alpha_1 C_1 + \cdots + \alpha_n C_n + \beta_1 R_1 + \cdots + \beta_{n-1} R_{n-1} + \gamma I + \delta J = 0. \tag{2}$$

The $(n, 2), \cdots, (n, n-1)$ components in (2) yield $\alpha_2 = \cdots = \alpha_{n-1} = 0$ and the $(1, 2)$ component then gives $\beta_1 = 0$. The $(1, 1)$, $(n, 1)$, and (n, n) components in (2) then give $\alpha_1+\gamma = \alpha_1+\delta = \alpha_n+\gamma = 0$, and it follows that $\alpha_1 = \alpha_n = -\gamma = -\delta$. Since $\alpha_1 = 0$ then implies that $\beta_2 = \cdots = \beta_{n-1} = 0$, we may as well suppose that $\alpha_1 = \alpha_n = 1$, $\gamma = \delta = -1$ and reach a contradiction. If $n > 4$, then the $(2, 3)$ and $(2, 2)$ components in (2) yield $\beta_2 = \beta_2 - 1 = 0$, a contradiction. If $n=4$, the $(2, 1)$ and $(2, 2)$ components in (2) give $\beta_1 = -1$, $\beta_2 - 1 = -2 = 0$, a contradiction. If $n=3$, then the second row of (2) reduces to $(\beta_2+1, \beta_2-2, \beta_2+1)=0$, and we see that $\beta_2 = -1$, $\beta_2 - 2 = -3 = 0$, a contradiction.

We state the following two lemmas without proof. They may be proved by the method used to prove Lemma 1.

LEMMA 2. *Let $n=3$ and $3=0$ in F. Then $C_1, C_2, C_3, R_1, R_2, I$ are F-linearly independent and $J = C_1 + C_3 - I - R_2$.*

LEMMA 3. *Let $n=4$ and $2=0$ in F. Then $C_1, C_2, C_3, C_4, R_1, R_2, R_3, I$ are F-linearly independent and $J = C_1 + C_4 + R_2 + R_3 + I$.*

We are now in a position to prove the following

THEOREM. *If $n \geqq 3$, the dimension of M_n is $n^2 - 2n$ unless $n=4$ and $2=0$ in F when this dimension is n^2-2n+1.*

Proof. Case I: The hypotheses of Lemmas 2 and 3 are false. Define $B_i = C_i$ $(i=1, \cdots, n)$, $B_{j+n} = R_j$ $(j=1, \cdots, n-1)$, $B_{2n} = I$, $B_{2n+1} = J$, $D_k = B_k - B_{k+1}$ $(k=1, \cdots, 2n)$. By Lemma 1, it is clear that $D_1, \cdots, D_{2n}$ are F-linearly independent. It is also clear that the equations (1) are equivalent to

$$(D_k \mid M) = 0 \qquad (k = 1, \cdots, 2n). \tag{3}$$

Regard each D_k, and M as well, as a vector in $V_{n^2}(F)$. Let Δ be the $2n$ by n^2 matrix obtained by arranging each D_k as a row vector by lining up its rows (in order) into a row. Then Δ has rank $2n$ and its null space, $\mathfrak{M}_n$, has dimension n^2-2n.

Case II: $n=3$ and $3=0$ in F. Since $J = C_1 + C_3 - I - R_2$ the equations (1) imply that $(J \mid M) = 0$. By Lemma 2, the equations (1) are equivalent to

$$(C_1 | M) = (C_2 | M) = (C_3 | M) = (R_1 | M) = (R_2 | M) = (I | M) = 0.$$

An argument like that of Case I then gives 9-6 as the dimension of $\mathfrak{M}_3$.

Case III: $n=4$ and $2=0$ in F. Here $J=C_1+C_4+R_2+R_3+I$ and $(J|M)$ is arbitrary in F. With $B_i=C_i$ $(i=1, \cdots, 4)$, $B_{j+4}=R_j$ $(j=1, 2, 3)$, $B_8=I$, $D_k=B_k-B_{k+1}$ $(k=1, \cdots, 7)$, we see by Lemma 3 that the equations (1) are equivalent to

$$(D_k | M) = 0 \qquad (k = 1, \cdots, 7). \tag{4}$$

As in Case I, M_n has dimension $n^2-7=n^2-2n+1$.

Reference

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A NOTE ON ALGEBRAS

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An *n-dimensional associative algebra* A over a field F is an (associative) ring which is simultaneously an n-dimensional vector space over F, and in which the ring multiplication and scalar multiplication are connected by

$$x(cy) = c(xy) = (cx)y \qquad (c \text{ in } F,\ x \text{ and } y \text{ in } A).$$

If $(e_1, \cdots, e_n)$ is a basis for A over F, then the ring-multiplicative structure of A is determined by the n^3 members c_{jki} of F (the "structure constants") defined by

$$e_je_k = \sum_i c_{jki}e_i.$$

The *left regular representation* of A (with respect to this basis) is the scalar-multiplication-preserving homomorphism of A into $M_n(F)$ (the algebra of $n\times n$ matrices over F) in which the image L_j of e_j is given by $(L_j)_{ik}=c_{jki}$; in the analogous *right regular representation* the image R_k of e_k is given by $(R_k)_{ji}=c_{jki}$. A change of basis in A alters these homomorphisms only by a similarity transformation, so that for our purposes there is no loss of generality in working only with the fixed basis $(e_1, \cdots, e_n)$.

The best-known sufficient condition that both regular representations of A be *faithful* (*i.e.*, be isomorphisms into $M_n(F)$) is that A *have a unity*. A weaker sufficient condition, noted in ([1] p. 292, Ex. 5) is that *not every element of A be a zero-divisor*; more precisely, the left (right) regular representation of A is faithful if A has at least one element which is not a right (left) zero-divisor. (An element x of A is a *right zero-divisor* if $yx=0$ for some nonzero y in A; *left zero-divisor* is defined analogously.) This sufficient condition is not necessary (as we show by an example below), and so it is esthetically desirable, though mathematically rather trivial, to state explicitly the "natural" necessary and sufficient condition:

THEOREM. *The left (right) regular representation of A is faithful if and only if A has no nonzero left (right) "annihilators."*

Proof. The image $\sum_j c_j L_j$ of the element $x = \sum_j c_j e_j$ of A under the left regular representation can vanish if and only if

$$\sum_j c_j c_{jki} = 0 \qquad \text{(all, } k, i\text{)},$$

and thus if and only if

$$xe_k = \sum_i \Big(\sum_j c_j c_{jki} \Big) e_i = 0 \qquad \text{(all } k\text{)},$$

and thus if and only if $xy = 0$ for all y in A; *i.e.*, if and only if x is a left "annihilator" of A. The proof of the "right-handed" version is similar.

To round out our discussion, we give an example (found by M. Newman) in which (a) every element of A is both a right zero-divisor and a left zero-divisor, but (b) no nonzero element of A is either a right annihilator or a left annihilator. For such an example, take A to consist of all bordered 3×3 matrices (with entries in F) of the form

$$M_1 = \begin{pmatrix} a_1 & 0 & 0 \\ b_1 & 0 & 0 \\ c_1 & d_1 & f_1 \end{pmatrix}, \qquad M_2 = \begin{pmatrix} a_2 & 0 & 0 \\ b_2 & 0 & 0 \\ c_2 & d_2 & f_2 \end{pmatrix},$$

so that

$$M_1 M_2 = \begin{pmatrix} a_1 a_2 & 0 & 0 \\ b_1 a_2 & 0 & 0 \\ c_1 a_2 + d_1 b_2 + f_1 c_2 & f_1 d_2 & f_1 f_2 \end{pmatrix}.$$

Each element M_1 of A is a left zero-divisor, since we can take $a_2 = d_2 = f_2 = 0$ and then choose nonzero b_2 and c_2 for which $d_1 b_2 + f_1 c_2 = 0$; similarly each element of A is a right zero-divisor. No nonzero element M_1 of A is a left annihilator, for if $M_1 M_2 = 0$ for all M_2 then we can take $a_2 \neq 0$ and $d_2 \neq 0$ to prove $a_1 = b_1 = f_1 = 0$, then take $a_2 \neq 0$ and $b_2 = 0$ to prove $c_1 = 0$, then take $b_2 \neq 0$ to prove $d_1 = 0$; similarly no nonzero element of A is a right annihilator.

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A THEOREM IN THE DECIMAL REPRESENTATION OF RATIONALS

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We present here a theorem concerning the number of times a given digit appears in the repeating sets one gets upon division by a prime. We will assume throughout the paper that m denotes the numerator and n or p denotes the denominator of a fraction and that $(10m, n)=1$, $m<n$. Let s be the number of digits in the smallest repeating set of m/n.

Let us first recall a few facts about decimals:

(1) The quantity s depends only on n, and $s(n)|\phi(n)$, where $\phi(n)$ is the Euler function.

(2) The $\phi(n)$ possible values for m give exactly $\phi(n)/s(n)$ repeating sets such that no two are cyclic permutations of each other. We call these the *distinct repeating sets.*

(3) As m varies, each digit in each distinct repeating set appears exactly once in "first place," that is, immediately after the decimal point.

The proof, which we do not give completely here, depends upon the fact that if $n|(10^t-1)$, then $s|t$, and vice versa; Euler's theorem tells us that $n|(10^{\phi(n)}-1)$. There are $\phi(n)$ repeating sets in all, because there are $\phi(n)$ possible values for m; there are exactly $s(n)$ cyclic permutations for each repeating set, since there are that many digits in each set. Hence there are $\phi(n)/s(n)$ distinct repeating sets.

THEOREM. *Consider the collection of distinct repeating sets of any prime. If $G(d_i)$ is the total number of times the digit d_i appears in these repeating sets, then for all i, j, $|G(d_i)-G(d_j)|\leqq 1$.*

This means simply that as far as possible, every digit appears just as often as any other digit.

Proof. We have already seen that every digit in each distinct repeating set appears exactly once in "first place" as we consider the repeating sets of m/p (p being prime) for all m less than p. And we get this first digit by solving the equation $10m=d_ip+r$, for $0<r<p$. The question now resolves itself into the following: How many m's exist for each d_i so that both the equation and the condition on r are satisfied?

In order for the condition on r to be satisfied, $d_ip<10m<(d_i+1)p$.

Also if $\bar{m}$ is the least such m, and $\bar{m}+k$ is the greatest such m, then $d_ip<10\bar{m}\leqq 10(\bar{m}+k)<(d_i+1)p$. Hence $d_ip+10k<(d_i+1)p$, and $p>10k$.

On the other hand, since $\bar{m}$ is the least such m, and $\bar{m}+k$ is the greatest, $10(\bar{m}+k+1)>(d_i+1)p$, and $10(\bar{m}-1)<d_ip$. Therefore, $10(\bar{m}+k+1)>(d_i+1)p>p+10(\bar{m}-1)$, and $10(k+2)>p$.

Finally, $p/10>k>p/10-2$. Since the total number of possible m's is $k+1$, then for any digit d_i, $p/10+1>G(d_i)>p/10-1$. From this we can see that $G(d_i)$ is one of two possible integers.

COROLLARY. *If we write p as $10q+r$, $(0<r<10)$, then $(11-r)$ digits appear q times, and $(r-1)$ digits appear $q+1$ times.*

Proof. If x digits appear k times, and $(10-x)$ digits appear $(k+1)$ times, then x and k must satisfy the equation $xk+(10-x)(k+1)=p-1$. It is easy to show that the solution to this is unique, and hence $x=11-r$ and $k=q$.

Let us call a digit which appears q times *deficient* and one which appears $(q+1)$ times *excessive.*

If we work with the inequalities $d_i(10q+r)<10m<(d_i+1)(10q+r)$, we see readily that if $r=1$, all digits are deficient;

if $r=3$, there are two excessive digits, 3 and 6;

if $r=7$, four digits are deficient: 0, 3, 6, and 9;

if $r=9$, two digits are deficient: 0 and 9.

For example, take $p=73$. There are nine distinct repeating sets, each with eight digits, as follows:

01369863, 02739726, 04109589, 05479452, 06849315, 08219178, 12328767, 16438356, 24657534.

In accordance with the corollary, the digits 3 and 6 appear eight times each, and every other digit appears seven times.

CLASSROOM NOTES

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ON EXTREMAL OPTICAL PATHS AND THE LAW OF REFLECTION

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Consider the following problem: Given a simple plane curve C having curvature at every point and given two points p_1 and p_2 both on the same side of C, determine that point p of C which minimizes $l=|\mathbf{p}_1\mathbf{p}|+|\mathbf{p}\mathbf{p}_2|$.

When C is a straight line, we have the famous problem of Heron (see [1], pp. 330–1 and [2], pp. 142–4). The solution to Heron's problem is that the lines $\mathbf{pp}_1$ and $\mathbf{pp}_2$ must make equal angles with the given line (see the above-mentioned references for a proof).

The above principle of Heron was generalized by Fermat to a principle of least time for optical ray tracing. In particular, for our general curve C any point p of C such that the path p_1pp_2 obeys the reflection law should minimize l.

Consider, however, the case where C is a circle and p_1 and p_2 are two distinct points on C. The points p which obey the reflection law are the endpoints of that diameter of C which bisects $\mathbf{p}_1\mathbf{p}_2$. It is easily verified that one of these points maximizes l while the other yields at least a relative maximum of l.

ON THE GREGORY INTERPOLATION FORMULA

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A great deal has been said in recent years about the place of inductive reasoning in the teaching of mathematics. George Pólya, in particular, has maintained that if the teaching of mathematics is to reveal to any degree the invention of mathematics then an effort must be made to teach guessing or plausible reasoning. One particularly nice problem, which the author has found stimulating to students and quite suitable for illustrating the inductive approach, is the derivation of the well-known and useful Gregory interpolation formula.

Suppose, for example, one considers the sequence 2, 3, 3, 6, 16, 37, $\cdots$ and asks if a simple rule can be deduced by which successive terms might be formed. The students will readily suggest that the method of taking differences is often helpful in answering such questions and they may be led to guess that taking differences of differences or forming an array of differences may also be helpful. In the present case one would obtain the array

$$\begin{array}{ccccccccccc} 2 & & 3 & & 3 & & 6 & & 16 & & 37 \\ & 1 & & 0 & & 3 & & 10 & & 21 & \\ & & -1 & & 3 & & 7 & & 11 & & \\ & & & 4 & & 4 & & 4 & & & \end{array} \tag{1}$$

which finally ends with a row of constant differences. In this case it is clear that the third row of (1) is simply an arithmetic progression whose terms are generated by the formula

$$h(s) = -1 + 4s, \tag{2}$$

if the first term is assumed to correspond to $s=0$, the second to $s=1$, and so on.

Now the elements of the second row of (1) are easily obtained by adding to 1 the appropriate terms from the third row. That is, the elements of the second row are generated by the formula

$$\begin{aligned} g(t) &= 1 + \sum_{s=0}^{t-1} h(s) \\ &= 1 + \sum_{s=0}^{t-1} (-1 + 4s) \\ &= 1 - t + 4\cdot\frac{t(t-1)}{2}, \end{aligned} \tag{3}$$

where again the first term is assumed to correspond to $t=0$, the second to $t=1$, and so on.

At this point one is tempted to simplify (3) immediately to $g(t)=1-3t+2t^2$ but a more careful look at (1), (2), and (3) makes the correspondence of the

coefficients 1, -1, and 4 seem somewhat more than just coincidental and this correspondence would be obscured if the simplification were made. One might even begin to suspect that a general pattern is beginning to emerge for the form of the generating function for a sequence whose array of differences terminates in a row of constant differences. Of course, ideas of this sort might be tested by guessing the generating function for the given sequence and then deriving the correct formula as we did in (2) and (3) above. Looking again at (1) we conjecture that the generating function for the given sequence ought to be

$$f(n) = \frac{2}{0!} + \frac{n}{1!} - \frac{n(n-1)}{2!} + \frac{4n(n-1)(n-2)}{3!}. \tag{4}$$

There may, of course, be some trouble in guessing the divisor for the last term. Reasonable candidates for this divisor might be 2^2, 3, or 3!, but students familiar with the binomial theorem usually choose 3! and modify the divisors on the first three terms as shown.

Now the nth term of the given sequence is clearly equal to 2 plus the sum of the first $n-1$ terms of the second row of (1). Thus the correct formula for the generating function of the given sequence is given by

$$\begin{aligned}
&2 + \sum_{t=0}^{n-1}\left[1 - t + 4\cdot\frac{t(t-1)}{2}\right] \\
&\quad = 2 + n - \frac{n(n-1)}{2} + 4\sum_{t=0}^{n-1}\frac{t(t-1)}{2} \\
&\quad = 2 + n - \frac{n(n-1)}{2} + 4\sum_{t=0}^{n-1}\left[\frac{(t+1)t(t-1)}{3!} - \frac{t(t-1)(t-2)}{3!}\right] \\
&\quad = \frac{2}{0!} + \frac{n}{1!} - \frac{n(n-1)}{2!} + \frac{4n(n-1)(n-2)}{3!},
\end{aligned} \tag{5}$$

which is precisely the result already guessed in (4). It should also be pointed out that the educated guess in (4) helped us to evaluate the last sum in (5) by suggesting how to convert it into a collapsing sum.

Now let a_0 be the first term of a sequence of terms whose mth row of differences are constant and whose terms correspond to $n=0, 1, 2, \cdots$, and let a_i denote the first term in the ith row of differences for $i=1, \cdots, m$. The students will now readily guess that the generating function for the given sequence should be

$$F(n) = \sum_{i=0}^{m} a_i\binom{n}{i}. \tag{6}$$

Of course, if the terms of the given sequence correspond to $n=k, k+1, \cdots$ for some positive integer k, then $n-k$ should be substituted for n on the right side of (6).

Finally, the interpolation formula can now be proved by mathematical induction if we first prove by induction or by converting to a collapsing sum the identity

$$(7)\qquad \sum_{t=0}^{n-1} \frac{t(t-1)\cdots(t-i)}{(i+1)!} = \frac{n(n-1)\cdots(n-i-1)}{(i+2)!}, \qquad 0 \leqq i \leqq n-2,$$

which is also suggested by the preceding discussion.

A NOTE ON THE FUNCTION $W=A_z+B$

E. S. Elyash and Norman Levine, University of Pittsburgh

A complex function $w=f(z)$ will be said to satisfy condition L if and only if $|z_1-z_3| = |z_1-z_2| + |z_2-z_3|$ always implies that $|f(z_1)-f(z_3)| = |f(z_1)-f(z_2)| + |f(z_2)-f(z_3)|$. It is well known that the function $w=Az+B$, $A\neq 0$, satisfies condition L and is conformal; it is the purpose of this note to give a proof of the converse. To the best of the authors' knowledge this converse does not appear in the above formulation elsewhere.

Theorem. *If $w=f(z)$ is conformal and satisfies condition L, then $w=Az+B$, where A and B are constants and $A\neq 0$.*

Proof. Let z_0 be a fixed point and R a ray from z_0 with $\arg(z-z_0)=\theta$ for all $z\in R$. Since R is connected and $f(z)$ is continuous, the image $f(R)$ is also connected. Furthermore $f(z)$ being conformal, $f(R)$ is nondegenerate and, as an obvious result of condition L, $f(R)$ is a ray or a line-segment. Hence let $\arg(w-w_0)=\phi$ for all $w\neq w_0$ and $w\in f(R)$; thus an argument for $f'(z_0)$ is $\phi-\theta$. By application of the same reasoning to any other point on R, $\arg f'(z) = \arg f'(z_0)$ for all $z\in R$. Indeed, by considering any other ray through z_0 and using the analyticity of $f(z)$ and $f'(z)$ we have $\arg f'(z) = \arg f'(z_0)$ for all z. This permits us to say

$$f'(z) = \rho(x, y)e^{i(\phi-\theta)} \qquad \text{for all } z.$$

The Cauchy-Riemann conditions yield the pair of equations

$$\rho_x \cos(\phi-\theta) = \rho_y \sin(\phi-\theta),$$
$$\rho_y \cos(\phi-\theta) = -\rho_x \sin(\phi-\theta).$$

The unique solution of this system is $\rho_x=\rho_y=0$ and therefore ρ is a constant. Thus $f'(z)=A$ where A is a complex constant different from zero because $f(z)$ is conformal and finally $f(z)=Az+B$.

SIMPLE PROOF OF A FUNDAMENTAL THEOREM OF FIELD THEORY

A. KERTÉSZ, University of Debrecen, Hungary

Let L be an extension of the field K and let A and B be two subsets of L. We say that A algebraically depends on B (over K) if each element of A is algebraic over $K(B)$. A and B are called algebraically equivalent (over K) if A depends on B and B on A. The set A is algebraically independent (over K) if no element x of A depends algebraically on the set $A-x$.

In the theory of fields the following theorem of Steinitz is of fundamental importance.*

THEOREM. *Let A and B be two subsets of L, algebraically independent and equivalent (over K). Then A and B have the same cardinal number.*

It is the aim of this note to give a simple proof of this theorem, which makes use only of the well-known set-theoretical fact that for an arbitrary infinite cardinal $\mathfrak{p}$ one has $\aleph_0\mathfrak{p}=\mathfrak{p}$ (which relation is based on the well-ordering theorem) and of the following corollary of the so-called exchange theorem of Steinitz: If the algebraically independent finite set N of n elements algebraically depends on the set M, then M has at least n elements.†

Let us indeed prove the above theorem. If one of the sets A and B is finite, then by virtue of the mentioned corollary of the exchange theorem A and B have the same cardinal number. Let us suppose, therefore, that A has cardinal number $\mathfrak{m}$, B has cardinal number $\mathfrak{n}$, with both cardinals infinite and $\mathfrak{m}>\mathfrak{n}$. To each element of A we make correspond one (and only one) finite subset of B, on which this element of A algebraically depends. Thus we map the set A into the set S of all finite subsets of B. If the complete inverse image of each element of S occurring in the range of this mapping were finite, then the cardinal number $\mathfrak{m}$ of A must be such that $\mathfrak{m}\leqq\mathfrak{n}$ since the cardinal number of S itself is at most $\aleph_0\mathfrak{n}=\mathfrak{n}$. So, in view of our hypothesis that $\mathfrak{m}>\mathfrak{n}$, the set A must have an infinite subset which algebraically depends on the same finite subset of B. This, however, contradicts the mentioned corollary of the exchange theorem. The reason for this contradiction is clearly the hypothesis that $\mathfrak{m}>\mathfrak{n}$, and in view of the fact that the rôle of A and B is completely symmetric, it follows that $\mathfrak{m}=\mathfrak{n}$, thus completing the proof.

* E. Steinitz, Algebraische Theorie der Körper (edited by R. Baer and H. Hasse), Berlin, 1930, New York, 1950, Section 23.

† *Ibid.*, Section 22, Theorem 9. For a simpler proof see G. Pickert, Einführung in die höhere Algebra, Göttingen 1951; pp. 65–67, 172–173.

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN A. BROWN, University of Delaware, AND
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THE EDUCATION OF MATHEMATICS TEACHERS*

Introduction

A. E. MEDER, JR., Rutgers University

The topic to be discussed by our panelists is an important one. The improvement of instruction in mathematics in the secondary school or of the secondary school curriculum depends in the last analysis on the teacher. He holds the key. Only the teacher can translate recommendations into classroom activity.

But such improvement requires more than good will on the part of the teacher. It requires a better knowledge of subject matter. The improvement of secondary school mathematics does not require improved methodology; it requires enhanced knowledge of mathematics on the part of secondary school teachers.

Moreover, it requires that the teacher continue to learn throughout his professional career. We must have done with the notion that a teacher's education is complete when he receives his Bachelor's degree.

The problem of teacher education is twofold. There is the problem of preservice education: what mathematics should be taught to the undergraduate who is preparing to be a secondary school teacher of mathematics? There is also the problem of in-service education: what mathematics should be taught to the teacher who completed his formal education some time in the past, but who needs to bring his knowledge of mathematics up to date in order to modernize the instruction given in his classroom? It is hoped that our panelists will make remarks on both aspects of the problem of teacher education.

Finally, let me say that this particular panel will almost certainly present proposals that in some measure at least will reflect the recommendations of the Commission on Mathematics of the College Entrance Examination Board, for its members have all had some contact with its work. The basic objective of the program recommended by the Commission for college-capable high school students is that they should be prepared to begin a course in calculus in their freshman year in college. Stress is therefore placed on algebra and on the study of the properties of the elementary functions, with a somewhat lessened emphasis (at least in terms of time allotted to the subject) upon geometry. Skills are regarded as very important, but the development of concepts, of an understand-

* Edited transcriptions of talks given as part of a panel discussion at the annual meeting of the New Jersey Section of the Mathematical Association of America on November 1, 1958. Several others of the panel talks will appear in a later issue.

ing of the nature of deductive reasoning, and of mathematical insight, power and understanding are of even greater importance.

The attainment of such high objectives demands thoroughly prepared teachers. What this preparation should be is the theme of the panel discussion.

Algebra

E. R. LORCH, Columbia University

In a recent discussion with Jean Vilar, the producer of the French National Theater, the question arose as to whether he had to choose special plays to get down to the level of his auditors. In reply he said that he had discovered that some of the most intricate, deepest plays are the ones which are best received by unsophisticated groups. And his counsel was, "What the audience wants is poetry."

Now I think that this is an answer that we could paraphrase for our mathematical work. What the student wants is mathematics, mathematics of the deepest, of the most genuine, of the purest type. Let us give him that, no matter if it now seems a little bit abstruse, or if it should seem to him a little bit difficult. Possibly we should go at it in gentle steps, but no matter what the difficulties are, we should go straight to the core of the subject matter and base our program for the undergraduate on genuine mathematics.

It so happens that mathematics in general has been undergoing a remarkable period of cleaning up. While the nineteenth century was a century in which mathematics was made rigorous—take, for instance, the structure of the real number system—this is the century in which mathematics is being made clean; and possibly algebra is that branch of mathematics in which the cleanliness is most apparent. That being the case, I suggest that we go in for algebra in big doses.

As specific suggestions, I have a list of five main categories. The first concerns the *Algebra of Sets and Theory of Sets in General.* My first subheading is something which everyone would think of at once in connection with algebra of sets, namely, the algebra of complementation, of union and intersection, the theory of duality, and the relation of the calculus of sets with the Boolean algebra of propositions. This material is very standard, and something with which we are all familiar. This is what most people mean, I think, when they speak of introducing set theory into the curriculum.

The next subheading under algebra of sets I consider to be less important. I mention it only for the sake of completeness: the arithmetic of cardinals and ordinals. Since the work of Cantor, this has been considered classical material of great interest; books on the theory of sets have been dedicated virtually to this subject alone. Yet I do not think it particularly important. The relation between the two smallest cardinals, C and aleph-null, should be handled at the college level, but not the general arithmetic.

Going ahead with our subheadings, we should talk about the introduction of certain other set-theoretic operations involving constructions of various types,

starting with given sets. For instance, an obvious set to construct if a set E is given, is the set of all subsets of E. Another very important operation given two sets E and F, is to construct their product, $E\times F$. Indeed, this product is one of the fundamental bricks in the mathematical structure. From the product $E\times F$ one can go on to the definition of relations involving E and F. There are roughly three special types of relations which should be mentioned: the equivalence relation and the related partition theorem that an equivalence relation generates a partition of a certain type in the set in which it operates; functional relations, which lead to the definition of a function and hence are obviously of the utmost importance in all undergraduate work; and finally, order relations: partial ordering, total ordering, and well-ordering. All of these can be illustrated copiously by material which is readily at hand in the general undergraduate curriculum.

So much for the algebra of sets. Heading II is *The Theory of Groups and Rings*. I would start with a discussion of the role of axiomatics in modern mathematics as distinguished from its very limited role of fifty years ago. This modern spirit may be illustrated by considering groups and rings. Of the utmost interest to the high school teacher is first of all the meaning of the symbol $-a$ as applied to a group (in general the meaning of subtraction). Based on this operation are two little things that will have to be proved for the students. One is that $-(-a)$ is equal to a and the other is that $-a$ times $-b$ is equal to ab. These are two little gems in the theory of groups and rings and their proof constitutes an introduction to abstract algebra. The definition of a field should be given, followed by a discussion of the problem of imbedding a domain of integrity (a certain kind of a ring) in a field. For example, starting with the integers one thus obtains the rational numbers. This theorem is a very beautiful result and it shows the student precisely what the rationals are.

Next I put down two other topics: the elements of the theory of ideals, including definitions of ideal, and prime ideal, which can be very simply given, once more with illustrations. In that connection it should be shown how the complex numbers are constructed from the real numbers by certain prime ideal operations. This is a construction which is a little better, I consider, than the classic Cauchy method of pairs, in which multiplication and addition are defined in what seems to be an arbitrary way.

Let me go through the other three headings rapidly. III: *Rings in Analysis*. There are special rings in analysis which require attention. Rings of functions—the functions which one integrates or differentiates form a ring: add them or multiply them, one still has integrable or differentiable functions. And this leads to a notion which might be mentioned sooner or later: that around the corner lurking somewhere is a structure known as a Banach space and even a more complicated one called a Banach algebra. Specifically one could talk about the space or the ring of the continuous functions on the closed interval $[a, b.]$

IV: *Number Theory*. We begin with the theory of ordinary integers. Out of this flows the theory of finite fields, especially the prime fields with p elements. They can be studied in terms of the elementary congruences of number theory.

And possibly something can be done in algebraic number theory to show that factorization of the primes is not necessarily unique, that it is only unique in certain types of situations. Simple examples can be given to illustrate non-uniqueness, remedied once more with the theory of ideals.

This brings me to section V, where I should like to discuss *Linear Transformations* and possibly, for those who would like me to do it in the old-fashioned way (although some of my colleagues would shudder at it), plain ordinary matrices. In that connection beautiful illustrations of the notions of nilpotence, idempotence, and noncommutativity are available.

This, to my mind, constitutes a reasonable program of algebra for undergraduates.

Analysis

A. W. Tucker, Princeton University

It is, of course, clear that the basic part of analysis for the training of a secondary school teacher is calculus. I would like to see courses in calculus improved in various directions, but these directions are not novel. The ideal calculus book in spirit, I feel, is Courant's *Calculus*. Unfortunately, it is not in a form that is easy for an undergraduate to read. It is written more in the style of what we regard as a graduate textbook but the compromise, the wonderful compromise, that it embodies between intuition, applications and careful mathematics is ideal. We have other books in calculus that are very fine in certain directions. There has been a tendency recently to go very heavily in the direction of mathematical rigor. I think that this care on the rigorous side is more appropriate in algebra than in calculus. It is the life blood of the calculus that intuition have full use and that there be applications of all possible sorts—from physics, of course, from engineering, but also the social sciences (marginal concepts from economics, for example). The great value of the calculus is the bringing together of intuition and the tremendous wealth of applications as well as the serious mathematical core that is regarded as the backbone of college mathematics.

The importance for the secondary school teacher of calculus is to shed light on the elementary functions—the polynomial, exponential and logarithmic, and circular functions—and, therefore, the calculus training of the prospective teacher should go into the differential equations that pertain to the exponential and circular functions. Also, since complex numbers should be an essential part of the secondary school mathematics, some introduction to functions of a complex variable would be desirable for a secondary school teacher. So calculus, as you have it in Volume I of Courant's *Differential and Integral Calculus*, should include the differential equations I referred to and an introduction to functions of a complex variable. These I would regard as the desiderata for the analysis training of a prospective secondary school teacher.

If you ask me to go on and mention other possible topics in analysis I could do so but I feel that analysis now does not play quite the dominant role that

it has in the past. Calculus is still probably the most important connection between school mathematics and advanced mathematics, but parallel to the calculus we have algebra, especially, and also probability, so we have to think of mathematical education no longer as a single track but a number of parallel tracks with much switching back and forth between the tracks.

THE BIOLOGICAL SCIENCES CURRICULUM STUDY

ARNOLD B. GROBMAN, University of Colorado

After an initial grant was made to the American Institute of Biological Sciences by the National Science Foundation, the Biological Sciences Curriculum Study was established with Dr. Bentley Glass, Johns Hopkins University, as Chairman of its Steering Committee and with the writer as Director of the Study. Late in January 1959, headquarters for the BSCS were established at the University of Colorado. A newsletter is available upon request.

Several projects were organized at the first meeting of the Steering Committee in early February; these, of course, are but a sample of the investigations that will eventually be undertaken.

Course Content in the Biological Sciences. Probably the central activity will be the work of a special committee that will attempt to answer the general question: What should a person graduating from high school know about the biological sciences? At a later time, studies of university and college curricula will be undertaken.

Innovation in Laboratory Instruction. The BSCS group felt rather strongly that much laboratory instruction in biology is less than inspiring and that a great deal could be done to make laboratory experiences more educationally meaningful. It is hoped that some pilot experiments utilizing some of the laboratory exercises designed by the committee will be attempted in the fall of 1959.

Studies of Successful Biology Teachers. There are a considerable number of high school biology teachers who are generally recognized in their communities as exceptionally effective teachers. The BSCS is interested in assembling information about a representative group of these teachers so that their backgrounds, training, and methods can be analyzed.

Digest of Existing Information. There is a wealth of information available that has been assembled by federal and state agencies, scientific and educational societies, and various other organizations. The BSCS will attempt to evaluate this mass of information and to prepare a digest.

Reviews for Teachers in the Life Sciences. As an aid to teachers now in service, to students preparing for teaching careers who may be unable to elect a full program in the biological sciences, and to laymen with an interest in the general area, the BSCS plans to initiate a pamphlet series. Each pamphlet would be concerned with a specific topic, would be written by an outstanding authority, and would be carefully designed for its appeal to the intended audience.

PROGRAMS IN MATHEMATICS OF ACADEMIES OF SCIENCE, 1959–60

Fifty proposals from 30 regional scientific organizations, usually State Academies of Science, have been approved by the National Science Foundation for participation in the new NSF Academies of Science Program.

A letter was sent by J. R. Mayor to the director of each of these activities requesting information on the place of mathematics in all of these programs. Replies from nine of the 30 directors indicate that their programs will include mathematics as one of the sciences. At least two of the proposals are for mathematics alone.

Brief descriptions of the place of mathematics in these nine programs are given below. According to the outline submitted by the National Science Foundation, it appears that at least four or five of the others, from which no response has been received, will include mathematics in the science education activities.

The *Buffalo Society of Natural Sciences* is planning to include mathematics as a part of its Project Stimulation program this winter, but details are not available. It possibly will take the form of projects for presentation at the Science Congress (science fair) as exhibits or lecture demonstrations. It expects to use visiting lecturers of the university level for this work.

The *Chicago Academy of Sciences* indicated that it would welcome lectures on mathematics (it has had a couple) that show the bearing of mathematics on astronomy, earth movements, biology, and any other similar applications. It would also like to cooperate with the State section of the Mathematical Association of America in any way feasible. It will seek ways in which mathematics can be applied to the several study courses upon which the Academy is embarked.

The *Cooper Union for the Advancement of Science and Art* is sponsoring a Mathematics Speakers Bureau in metropolitan New York. Mathematicians will speak in high schools and participate in discussions involving students and teachers. It is expected that 235 schools will be served.

One of the projects of the *Hawaiian Academy of Science* which was approved by the NSF is that of Teachers' Science Seminar Series which involves sending distinguished scientists out to give lectures designed to give teachers subject-matter information. These illustrated lectures also are open to the public in some cases and the local department of instruction allows partial credit for teachers who participate in a series. So far there have been no lectures in the field of mathematics, but there is a chance that one or two lectures in mathematics might be sponsored during the coming year. This will be a matter for the Teachers' Science Seminar Committee to determine.

The *Indiana Academy of Science* includes mathematics among its twelve divisions. It expects to include all of the divisional areas of science and mathematics in the visiting scientists program if personnel can be found to agree to cooperate. There probably are a number of mathematicians who will be interested since they have been active in the affairs of the Indiana Academy.

The *Louisiana Academy of Sciences* project includes Visiting Lecturers, and

$1500 has been allocated to this phase. It expects to call on local scientists to participate in the program and believes that they will do so because many in educational institutions, professional groups, and industry have volunteered their services in the past. Thus far, there has been no time to compile a list of those who would participate on a state-wide basis, but such a list will be made. It is anticipated that many will not ask for even so much as remuneration for their expenses, but this will be offered. Costs for approximately 100 speakers can be provided with the $1500. Any assistance that comes from scientific organizations such as the Mathematical Association of America will be appreciated and a request will be sent to the MAA for a list of mathematicians in Louisiana who would be willing to take part in the project. Full cooperation with the MAA Committee on Visiting Lecturers to Secondary Schools was offered.

The *Minnesota Academy of Science* has a mathematics section which will be headed during 1959–60 by David R. Lewis, 3167 Lake Johanna Boulevard, St. Paul 12. Mr. Lewis is employed by the Remington Rand-Univac Division of the Sperry Rand Corporation. A committee is being set up to administer the visiting scientist program and to coordinate the programs of the professional societies and other groups. The available talent will be spread among a large number of schools. The committee will include local representatives of the professional societies, which should provide for the necessary exchange of information.

The *Nebraska Academy of Sciences* has a section of the Mathematical Association of America and a section of the National Council of Teachers of Mathematics. Each of these retains its identity but meets with the Academy in its spring meeting. Visiting scientists will include mathematicians who undoubtedly will be members of these groups, and consultations with the mathematicians as individuals will be held. The cooperation and good counsel of the MAA Committee on Visiting Lecturers to Secondary Schools will be welcomed.

The *Oklahoma Academy of Science* project does not involve visiting lectureships, nor does it involve in-service training. The "3-R's" Conference sponsored by the Frontiers of Science Foundation last November attracted considerable interest in science and mathematics education in this state. As a result, some 125 community-level organizations, such as Civic Clubs or Chambers of Commerce, asked for consultative services on how science and mathematics programs may be improved. The Oklahoma project is to furnish these consultative services through the scientists and mathematicians from the various four-year colleges in the state. While the cooperation of the Oklahoma section of the MAA has not been solicited, use will be made of mathematicians as consultants.

There is a Mathematics Section of the *Tennessee Academy of Science*. The Tennessee program for next year is designed to help science and mathematics teachers. It will be organized under the three divisions of Biological Science, Physical Science, and Mathematics. During each quarter of the year—fall, winter, spring—an all-day institute program is planned. Each such program will have three sections according to the above categories.

The *Texas Academy of Science* has a Mathematics Section; or more exactly,

the Texas Section of MAA is affiliated with the Texas Academy of Science and has its winter meeting with the annual meeting of the Academy. Its project is carried out within the Academy with the full cooperation of the Texas Section, MAA. All the members of the steering committee are very active members of MAA. The Texas Academy will be glad to cooperate with the MAA Committee on Visiting Lecturers to Secondary Schools—the lecturers will be mathematicians.

The *Virginia Academy of Science* has a section of mathematics, astronomy, and physics. The mathematicians have been talking about forming a section of their own but this has not yet been done. The committee which will pick the professors to invite as visiting lecturers has on it a mathematician, a biologist, an experimental psychologist, a physicist, a chemist, a geologist, and a biochemist interested in medical science. Every college has been asked to appoint a representative who will work with this committee. These representatives are requested to send names of individuals whom they would like to visit their college as a visiting scientist. Some of them no doubt will ask for a mathematician. The committee would like to consider cooperation with the MAA Committee on Visiting Lecturers to Secondary Schools.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1386. *Proposed by Monte Dernham, San Francisco, California*

In overtaking a freight, a passenger train which is x times as fast takes x times as long to pass it as it takes the two trains to pass when going in opposite directions. Find x.

E 1387. *Proposed by L. T. Van Tassel, San Diego, California*

Find p such that the line $y=p$ divides the area of an arch of the sine curve $y=\sin x$ into parts in a given ratio to one another.

E 1388. *Proposed by H. W. Gould, West Virginia University*

Determine the largest integral value of n such that there exists a solution to the system of inequalities

$$k < x^k < k + 1, \qquad k = 1, \cdots, n.$$

Would it make any difference if $<$ were replaced by $\leqq$?

E 1389. *Proposed by E. M. Scheuer, Space Technology Laboratories, Los Angeles, California*

Evaluate the $n \times n$ determinant A_n whose (i, j)th entry is $a^{|i-j|}$.

E 1390. *Proposed by J. Łoś and J. Mycielski, Wroclaw, Poland*

A *decomposition* of a rectangle (rectangular parallelepiped) R is a dissection of R into smaller rectangles (rectangular parallelepipeds) by lines (planes) parallel to the sides (faces) of R. If a decomposition of R is not a refinement of some other decomposition of R, then the decomposition is said to be *primitive*.

(1) Show that there exist primitive decompositions of a rectangle into any number $n > 1$ of rectangles, except $n = 3, 4, 6$.

(2) Show that there exist primitive decompositions of a rectangular parallelepiped into any number $n > 1$ of rectangular parallelepipeds, except $n = 3, 4$.

SOLUTIONS

A Criterion for a Right Triangle

E 1356 [1959, 233]. *Proposed by F. Leuenberger, Zuoz, Switzerland*

Let I, O, r, R, K denote the incenter, circumcenter, inradius, circumradius, and area of a triangle T. Show that $(IO)^2 + K = r^2 + R^2$ if and only if T is a right triangle.

Solution by Leon Bankoff, Los Angeles, Calif. In any triangle, $K = rs$, where s is the semiperimeter. Now $s = r + 2R$ if and only if T is a right triangle, in which case

$$K = r(r + 2R) = r^2 + R^2 - (R^2 - 2Rr) = r^2 + R^2 - (IO)^2.$$

Also solved by A. N. Aheart, Shlomo Ben-Adam, L. D. Goldstone, R. H. Hou, J. P. Hoyt D. C. B. Marsh, C. S. Ogilvy, N. R. Riesenberg, L. A. Ringenberg, Chih-yi Wang, Dale Woods, Roscoe Woods, and the proposer. Late solution by Kjeld Ejrnaes.

Marsh and Roscoe Woods showed that for any triangle ABC

$$(IO)^2 + K = R^2 + r^2 + 4rR\sqrt{2} \sin (45° - A/2) \sin (45° - B/2) \sin (45° - C/2).$$

An Equation with a Unique Solution

E 1357 [1959, 233]. *Proposed by Albert Wilansky, Lehigh University*

Prove that for every number a the equation $x = -a + \sqrt{2} \sin [(a - x)/\sqrt{2}]$ has a unique solution.

I. *Solution by A. R. Hyde, West Hartford, Conn.* The solution is found by

the intersection of the line $y=(a+x)/\sqrt{2}$ with the curve $y=\sin\,[(a-x)/\sqrt{2}]$. Since the slope of the line is the same as the maximum slope of the sine curve, there must be exactly one intersection.

II. *Solution by D. C. B. Marsh, Colorado School of Mines.* For arbitrary but fixed a, the function

$$f(x) \equiv x + a - \sqrt{2}\sin\,[(a - x)/\sqrt{2}]$$

has slope $1+\cos\,[(a-x)/\sqrt{2}]$, which is nonnegative for all x. Thus $f(x)$ is monotone and, ranging over all real values, takes on the value zero precisely once.

Also solved by E. F. Allen, Shlomo Ben-Adam, D. A. Breault, J. L. Brown, Jr., R. F. Brown and Joel Levy (jointly), C. H. Cunkle, S. J. Einhorn, George Glauberman, Michael Goldberg, L. D. Goldstone, A. G. Grace, Jr., Alfred Gray, R. Holt, Jack Kohne and Richard Rizzo and Robert Whitley (jointly), Viktors Linis, C. S. Ogilvy, F. D. Parker, J. L. Pietenpol, S. D. Pratico, B. E. Rhoades, L. A. Ringenberg, Charles Sampson, W. A. Veech, Dale Woods, and the proposer.

The Roots of a Special Cubic

E 1358 [1959, 234]. *Proposed by N. R. Riesenberg, Brooklyn College*

Show that the roots of the cubic equation $64x^3-192x^2-60x-1=0$ are $\cos^3(2\pi/7)\sec(6\pi/7)$, $\cos^3(4\pi/7)\sec(2\pi/7)$, $\cos^3(6\pi/7)\sec(4\pi/7)$.

Solution by D. C. B. Marsh, Colorado School of Mines. We transform known results into the desired form as follows: $\cos(2k\pi/7)$, $k=0, 1, \cdots, 6$, are the roots of $\operatorname{Re}(w^7-1)=0$, where $w=u+iv$. The latter equation may be rewritten as

$$(u - 1)(8u^3 + 4u^2 - 4u - 1)^2 = 0.$$

Thus $\cos(2k\pi/7)$, $k=1, 2, 3$, are roots of

$$8u^3 + 4u^2 - 4u - 1 = 0,$$

and $\cos^2(2k\pi/7)$, $k=1, 2, 3$, are roots of

$$64t^3 - 80t^2 + 24t - 1 = 0.$$

Setting $t=3x/(4x-1)$, the above equation transforms into

$$64x^3 - 192x^2 - 60x - 1 = 0,$$

whose roots are

$$\begin{aligned} x = t/(4t - 3) &= \cos^2(2k\pi/7)/[4\cos^2(2k\pi/7) - 3] \\ &= \cos^3(2k\pi/7)/\cos 3(2k\pi/7) = \cos^3(2k\pi/7)\sec(6k\pi/7), \end{aligned}$$

$k=1, 2, 3$, as we wished to show.

Also solved by Leon Bankoff, Shlomo Ben-Adam, J. F. Bittner, P. L. Chessin, F. J. Duarte, Michael Goldberg, J. M. Kingston, Albert Nijenhuis, R. E. Shafer, and the proposer. Late solution by Kjeld Ejrnaes.

Bankoff pointed out that the problem is given by Rev. E. M. Radford, *Mathematical Problem Papers*, Cambridge (1923), p. 421.

A Set of Eight Positive Integers

E 1359 [1959, 234]. *Proposed by Leo and William Moser, Universities of Alberta and Saskatchewan*

(1) Given eight positive integers $a_1 < a_2 < \cdots < a_8 \leqq 16$. Prove that there exists a k such that $a_i - a_j = k$ has at least three solutions.

(2) Find a set $a_1, \cdots, a_8$ for which $a_i - a_j = k$ has at most three solutions for any k.

Solution by J. L. Pietenpol, Columbia University. (1) If no three of the seven positive numbers $a_2 - a_1, a_3 - a_2, \cdots, a_8 - a_7$ are to be equal, then their sum must be at least equal to $1+1+2+2+3+3+4=16$. But the sum is $a_8 - a_1 \leqq 15$. We have thus established the stronger result that $a_i - a_{i-1} = k$ has at least three solutions for some k.

(2) There are many examples, such as 1, 2, 3, 4, 7, 9, 12, 16.

Also solved by D. A. Breault, R. F. Brown and Joel Levy (jointly), G. W. Day, N. J. Fine, George Glauberman, Michael Goldberg, J. H. Hodges, A. S. Howard, D. C. B. Marsh, R. A. Melter, D. J. Persico, D. R. Wilder, and the proposers.

k-commutative Elements of a Group

E 1360 [1959, 234]. *Proposed by J. L. Brenner, Stanford Research Institute*

Call two elements a, b of a group "k-commutative" (B. Friedman) if every product of k factors, each factor being a or b, commutes with every other such product. Show that for every two elements a, b of a group, the set of all k for which a, b are k-commutative is an ideal in the set of nonnegative integers.

Solution by J. Hooley, Derby and District College of Technology, Derby, England. With a definition of 0-commutativity analogous to $a^0 = b^0 = e$, the unit of the group, G, every pair a, b in G is 0-commutative.

Let J be the set of nonnegative integers and, for given a, b in G, let I be the subset of J for whose members k we have a, b a k-commutative pair. Let n be a positive integer and let g and g' be products of nk factors, each factor a or b. By the associative law, gg' can be expressed as

$$g_1 g_2 \cdots g_n g_1' g_2' \cdots g_n',$$

where g_j, g_j' $(1 \leqq j \leqq n)$ are products of k factors. Then, using associativity and n^2 applications of k-commutativity (k in I), we see that

$$gg' = g_1' \cdots g_n' g_1 \cdots g_n = g'g,$$

and a, b are (nk)-commutative.

If a, b is a k-commutative and an $(i+k)$-commutative pair, and if h, h' are any products of i factors (a or b), then

$$hh' = a^{-k}a^k hh' a^k a^{-k} = a^{-k}h' a^{2k} h a^{-k},$$

since a, b are $(i+k)$-commutative. Let p be the least member of J such that

$$p + i \equiv 0 \pmod{k}.$$

Then $p \leqq k$ and

$$hh' = a^{-k}h'a^pa^ka^{k-p}ha^{-k} = a^{-k}a^kh'a^{p+k-p}ha^{-k}$$

(using $(p+i)/k$ applications of k-commutativity)

$$= h'a^kha^pa^{-p-k} = h'ha^pa^ka^{-p-k}$$

(again using $(p+i)/k$ applications of k-commutativity) $=h'h$. That is, if k and $k+i$ are in I, then i is in I.

It now follows that I is an ideal in J.

Also solved by James Baugh and T. G. McLaughlin (jointly), D. C. B. Marsh, D. J. Persico, and the proposer.

Baugh and McLaughlin established the stronger result that the ideal in the set J of nonnegative integers is actually a principal ideal in J.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers, The State University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers, The State University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4869. *Proposed by M. S. Klamkin, AVCO Research, Wilmington, Mass.*

A smooth centro-symmetric curve has the property that the centroid of any half-area which is formed by chords through the center is equidistant from the center. Show that the curve is a circle.

4870. *Proposed by D. S. Mitrinovitch, Belgrade, Yugoslavia*

If the variable z traces the circle $|z| = 1$, find and describe the path of the point $w = (z^2 - az)/(az - 1)$, where the parameter a is a complex constant.

4871. *Proposed by Juris Hartmanis, Ohio State University*

Prove that the Boolean algebra of all subsets of the integers has maximal chains whose length is denumerable and has also maximal chains whose length is not denumerable.

4872. *Proposed by Ky Fan, Oak Ridge National Laboratory*

Let A be a matrix (not necessarily square) of rank $r \geqq 1$ and with nonnegative elements. Let A^* denote the transpose of A. Prove that the square matrix AA^* has no eigenvalue different from 0, 1, if and only if, after deleting all identically vanishing rows and columns, the remaining submatrix of A can be brought by a permutation of the rows and a permutation of the columns to the form

$$B = B_1 \dotplus \cdots \dotplus B_r, \tag{1}$$

where, for each i, B_i is a rectangular matrix of rank 1, with all its elements positive and such that the sum of squares of all elements of B_i is 1. Equation (1) means that B is obtained by laying out successively the rectangular blocks $B_1, \cdots, B_r$ with the lower right corner of B_i attached to the upper left corner of B_{i+1}, and with zeros filling in the entire matrix outside these blocks.

4873. *Proposed by W. R. Utz, University of Missouri*

Let S be a connected topological space in which there is a relation $<$ which satisfies: (1) S has a least and a greatest element relative to $<$, (2) $<$ satisfies the trichotomy law, (3) $<$ is continuous. In other words, if $x<y$ in S, then there exist neighborhoods N_x and N_y of x and y, respectively, such that $z \in N_x$ implies $z<w$ for each $w \in N_y$.

Show that any continuous transformation of S into itself has a fixed point and give an example of S which is not an arc.

4874. *Proposed by D. J. Newman, AVCO Research, Wilmington, Mass.*

Let c_n be a given sequence of constants such that $c_n = O(1/n)$, $n = 1, 2, \cdots$. Show that there is a bounded function $f(x)$ with c_n as its positive Fourier coefficients [*i.e.*, such that

$$\left.\frac{1}{2\pi}\int_0^{2\pi} f(x)e^{-inx}dx = c_n\right].$$

SOLUTIONS

Characterization of the Additive Group of Rationals

4822 [1959, 66]. *Proposed by J. deGroot, University of Amsterdam, Netherlands*

Prove that the additive group R of rational numbers is (up to isomorphisms) the only group satisfying the following conditions: (1) R is abelian, (2) R is infinite, and (3) every endomorphism (that is a homomorphic mapping of R into itself) is either an automorphism or a mapping on the null element.

Solution by D. G. Cantor, University of California, Los Angeles. Let G be a group satisfying (1), (2), and (3). By property (3) every nonzero endomorphism of G has an inverse and hence the endomorphisms of G form a division ring.

Let F be the prime field of this division ring. It is easy to verify that G is a vector space over F; and that it is one-dimensional. Otherwise there would be non-trivial endomorphisms of G onto lower dimensional subspaces which would not be automorphisms. Now F is isomorphic to either the integers, mod a prime, or the rationals. In the first case G would have only p elements contrary to (2); thus F is isomorphic to the field of rationals. Now every element of G is of the form $g_0 f$, for g_0 a fixed element of G and f an element of F. Let f' be the image of f in the field of rationals. Then $g_0 f \leftrightarrow f'$ is the desired isomorphism.

Also solved by H. F. Bechtell, Frederick Cunningham, Jr., C. C. Faith, Leopold Flatto, E. R. Gentile, Franklin Haimo, Melvin Henriksen, P. D. Hill, J. M. Horváth, L. E. Ward, Jr., J. V. Whittaker, R. J. Wisner and Charles Osgood, and the proposer.

Editorial Note. Several solvers referred to I. Kaplansky, *Infinite Abelian Groups*, in particular p. 10. Horváth finds the result in a theorem of T. Szele, *Die Abelschen Gruppen ohne eigentliche Endomorphismen*, Acta. Sci. Math., Szeged, vol. 13, pp. 54–56. See also J.-P. Serre, *Sur un théorème de T. Szele*, same volume, pp. 190–191.

By an analysis somewhat similar to the above solution, Gentile proves the more general result: *Let R be a commutative integral domain with identity and let G be an R-module such that every R-endomorphism of G is either an automorphism or a null map; then either G is 0 or G is R-isomorphic to the field of quotients of R/I for some prime ideal I of R.*

Derivative of Powers of a Matrix

4823 [1959, 66]. *Proposed by G. Matthews, St. Dunstan's College, Catford, England*

Let X be a lower-semi-matrix whose elements x_{ij} $(i, j=1, 2, \cdots)$ are independent variables if $j \leqq i$ and zero if $j > i$, and let D be the differential operator matrix defined by $D_{ij} = \partial/\partial x_{ij}$ if $j < i$, $D_{ij} = 0$ if $j > i$ and $D_{ii} = \sum_{k=1}^{i} \partial/\partial x_{kk}$. Prove that $D(X^2) = 2X$, and hence by induction that $D(X^n) = nX^{n-1}$, where n is any positive integer greater than 2.

Solution by Leonard Carlitz, Duke University. We show first that $DX = I$. Indeed

$$DX = \left(\sum_{j \leq k \leq i} D_{ik} x_{kj} \right),$$

$$\sum_{j \leq k \leq i} D_{ik} x_{kj} = \sum_{j \leq k < i} \frac{\partial}{\partial x_{ik}} x_{kj} + \sum_{k=1}^{i} \frac{\partial}{\partial x_{kk}} x_{ij} = 0 + \delta_{ij}.$$

Secondly, since D_{ij} is linear in $\partial/\partial x_{rs}$, it is easily shown that

$$DAB = A \cdot DB + DA \cdot B,$$

where A and B are arbitrary square matrices whose elements are functions of x_{rs}. Therefore, by induction,

$$DX^n = X^{n-1} \cdot DX + DX^{n-1} \cdot X = X^{n-1} I + (n-1) X^{n-2} X = nX^{n-1}$$

for all integral $n \geqq 1$.

Also solved by R. D. Gordon, José M. Isidro, D. C. B. Marsh, B. E. Rhoades, D. H. Trahan, and the proposer.

Conformal Mapping

4824 [1959, 66]. *Proposed by D. J. Newman, AVCO Research and Development, Wilmington, Mass.*

Let $ABCD$ be a rectangle, $AB=1$, $BC=2$. Suppose that it is conformally mapped onto the upper half plane. If $A\to a$, $B\to b$, $C\to c$, $D\to d$, exhibit an elementary relation between a, b, c, d.

Solution by the proposer. We begin with an auxiliary mapping: Let X = midpoint of BC and Y = midpoint of AD. Map triangle ABX onto the semi-circle $|z-1| \leqq 1$, $\operatorname{Im} z \geqq 0$, with $X\to 0$, $B\to 1$, $A\to 2$. By the reflection principle, if we invert through this semi-circle, this mapping can be continued to a mapping of $ABXY$ to the upper half plane with $Y\to\infty$.

Now apply the square root to this mapping and we get a mapping of $ABXY$ onto the first quadrant and so, again by the reflection principle, this can be continued to a mapping of $ABCD$ onto the upper half plane. We find, in this case, $A\to\sqrt{2}$, $B\to 1$, $C\to -1$, $D\to -\sqrt{2}$.

Since the given mapping must be a bilinear function of this one and since this leaves the cross ratio invariant, we have

$$\{abcd\} = \frac{(b-a)(d-c)}{(d-a)(b-c)} = -\frac{3\sqrt{2}-4}{8}.$$

Also solved by E. John Burr making use of the properties of the Schwarz-Christoffel transformation.

A Result of Ramanujan

4826 [1959, 66]. *Proposed by M. S. Klamkin and L. A. Shepp, AVCO Research and Development, Wilmington, Mass.*

If $\phi(x)=x/1^2-x^3/3^2+x^5/5^2-x^7/7^2+\cdots$, express $\phi(1)$ in terms of $\phi(2-\sqrt{3})$, thus obtaining a more rapidly converging expansion.

Note by Emil Grosswald, University of Pennsylvania. In Ramanujan's paper, *On the integral* $\int_0^x t^{-1}\tan^{-1} t\,dt$, he obtains the result

$$2\phi(1) = 3\phi(2-\sqrt{3}) + \tfrac{1}{4}\pi\log(2+\sqrt{3}).$$

(See *J. Indian Math. Soc.*, VII, 1915, pp. 93–96; also *Collected Papers*, pp. 40–43.*) Ramanujan's treatment, with some details filled in, follows.

One observes that

$$\phi(x) = \int_0^x \frac{\arctan t}{t}\,dt. \tag{1}$$

* See also L. Lewin, *Dilogarithms and Associated Functions*, London 1958, p. 39.

Next one shows that, for $0<x<\pi/2$,

$$\text{(2)}\qquad \frac{\sin 2x}{1^2}+\frac{\sin 6x}{3^2}+\frac{\sin 10x}{5^2}+\cdots=\phi(\tan x)-x\log(\tan x)$$

holds. Since both sides of (2) approach zero as $x\to 0$, it is sufficient to show that their derivatives are equal, *i.e.*, using (1), that

$$\text{(3)}\qquad \frac{2\cos 2x}{1}+\frac{2\cos 6x}{3}+\frac{2\cos 10x}{5}+\cdots=-\log(\tan x),\quad 0<x<\pi/2.$$

Since (3) is a known cosine Fourier expansion, (2) is proved. With $x=\pi/12$, $\tan x=2-\sqrt{3}$, (2) becomes:

$$\frac{1}{2}\left\{\frac{1}{1^2}+\frac{2}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}-\frac{2}{9^2}-\frac{1}{11^2}+\frac{1}{13^2}+\frac{2}{15^2}+\cdots\right\}$$

$$=\phi(2-\sqrt{3})-\frac{\pi}{12}\log(2-\sqrt{3}).$$

The left member is $2\phi(1)/3$ as is shown by rewriting it as

$$\frac{1}{2}\left\{\frac{1}{1^2}-\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{7^2}+\cdots\right\}+\frac{3}{2}\left\{\frac{1}{3^2}-\frac{1}{9^2}+\frac{1}{15^2}+\cdots\right\}.$$

The desired result now follows immediately.

Also solved by L. Carlitz, Y. L. Luke, K. K. Norton, C. C. Yalavigi, and the proposers.

Inverse Functions

4827 [1959, 67]. *Proposed by J. Gallego-Diaz, Vanderbilt University*

If the development of the function $y=f(x)$ is given by

$$y=a_1x+a_3x^3+a_5x^5+\cdots,$$

find a function y knowing that if we invert the series we get

$$x=a_1y-a_3y^3+a_5y^5-\cdots.$$

I. *Solution by A. J. Kokar, School of Mines and Industries, Adelaide, Australia.* Noting that $a_1^2=1$ and that it is enough to consider $a_1=1$, we can put the given equations in the form

$$\text{(1)}\qquad \begin{aligned} y&=x(1+A_2x^2+A_4x^4\cdots), & x&=y(1-A_2y^2+A_4y^4-\cdots),\\ y/x&=t=G(x^2), & x/y&=1/t=G(-y^2); \end{aligned}$$

or we can rewrite them using the inverse function as

$$x^2=F(t),\qquad -y^2=F(1/t),$$

whence, upon dividing, we have

$$F(1/t) = -t^2F(t). \tag{2}$$

To solve (2), we may set $F(t)=k^2D(t)E(t)$, k^2 constant, $D(1/t)=t^2D(t)$, $E(1/t)=-E(t)$, and seek satisfactory functions D and E. Simple examples are

$$D(t) = 1/t \quad \text{and} \quad E(t) = (t - 1/t)^{2n+1},$$

$$x^2 = F(t) = k^2(t - 1/t)^{2n+1}/t.$$

Replacing t by y/x we have

$$(xy)^{2n+2} = k^2(y^2 - x^2)^{2n+1}. \tag{3}$$

For $n=0$ this gives

$$y = x(1 - x^2/k^2)^{-1/2} = x\sum_{j=0}^{\infty}\binom{2j}{2}\left(\frac{x}{2k}\right)^{2j},$$

$$x = y(1 + y^2/k^2)^{-1/2} = y\sum_{j=0}^{\infty}(-1)^j\binom{2j}{j}\left(\frac{y}{2k}\right)^{2j}.$$

which fulfills the requirements of the problem.

It is evident that there are infinitely many solutions with different values of n and with different choices of D and E. Furthermore, the function F can be factored in other ways, e.g.,

$$F(t) = kA(t)B(t)C(t),$$

$$A(1/t) = tA(t), \qquad B(1/t) = -tB(t), \qquad C(1/t) = C(t).$$

An entirely general form for the solution seems improbable.

II. *Solution by Leonard Carlitz, Duke University.* A solution of the problem is furnished by the gudermannian $y=\mathrm{gd}x$ (see, for example, Wilson, *Advanced Calculus*, p. 6) defined by $\sinh x=\tan y$ which has the property $\mathrm{gd}ix=i\mathrm{gd}^{-1}x$. We have at once

$$x = \mathrm{gd}^{-1}y = \log(\sec y + \tan y), \qquad dx/dy = \sec y.$$

From the known expansion of $\sec y$ there results

$$x = \sum_{n=0}^{\infty}(-1)^n\frac{E_{2n}}{2n+1}\frac{y^{2n+1}}{(2n)!},$$

$$y = \sum_{n=0}^{\infty}\frac{E_{2n}}{2n+1}\frac{x^{2n+1}}{(2n)!},$$

where E_{2n} are the Euler numbers in the even suffix notation.

Also solved by R. D. Gordon, and the proposer.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

The New Mathematics. By Irving Adler. John Day, New York, 1958. 187 pp. $3.75.

This is a delightful little book. The forward declares that the book is addressed to the average reader. It is the opinion of the reviewer that the average reader would enjoy much of the book but would have some difficulty in understanding everything in the book. Although explanations are as elementary as possible, the average reader would probably find difficulty following the discussions of neighborhoods and vector spaces.

For the teacher of secondary school mathematics and the capable high school student the book should prove very valuable. For the teacher who has studied some modern algebra the book will give a glimpse of some "old friends." The teacher lacking this background will certainly gain a great deal by reading the book. He will have a better understanding of the complex number system and will find some meaning in such terms as "mapping," "sets," and "structure."

The reader is introduced to a variety of mathematical structures: groups, rings, fields, vector spaces, and topological spaces. Most of the concepts are introduced first by the use of familiar examples in the number systems and then by less familiar examples such as residue classes and matrices.

Few proofs are included, so the author moves along rapidly and the book is read easily. Some exercise lists are included which would assist the reader in understanding new ideas.

The book should create a desire on the part of many readers for further study. A bibliography is included.

The author does a successful job of creating a feeling of expectancy in the early chapters. He convinces the reader that he is seeking something and that he does not find it until the last few pages—a number system which would solve all algebraic equations.

In the closing paragraph the author notes that although the names of certain mathematical structures are new, they are closely related to such familiar things as addition and multiplication. He concludes, "Although the world of modern mathematics is a new world in many ways, it has never lost contact with the old world of number and space from which it has grown."

Teachers of secondary school mathematics and laymen with a special interest in mathematics will enjoy this book. It should find a place in high school libraries.

MARIE S. WILCOX
Thomas Carr Howe High School
Indianapolis, Indiana

Vorgriechische Mathematik I. Vorgeschichte und Ägypten. By Kurt Vogel. Mathematische Studienhefte, Hefte 1. Hermann Athen and Georg Wolff, editors. Hermann Schroedel Verlag KG, Hannover, Germany, 1958. 80 pp. 8 D.M. (About $2).

It was the late George Sarton, historian of science, who once asked, "How could we really appreciate the traditions which give our lives their meaning, if we did not continue them. . . . The past is full of beauty, but less so however than the future." Dr. Vogel, evidently, has this for his thesis in his latest work written primarily for gymnasium students in an effort to aid their understanding of both ancient culture and the influence of ancient mathematics upon that of the present day.

The first part of this excellently written and illustrated paper-back volume deals with prehistoric geometry and numbers in Europe, Crete, Mohenjo-Daro, and Sumer. The mathematics of the Egyptians, in which Vogel has been working for many years, is brought up to date with the results of recent Egyptological research. The Egyptian number system, metrology, fundamental operations, arithmetic and algebraic problems, and geometry are explained adequately for the layman without loss of mathematical detail.

Each of the two parts of the book has a set of exercises and literature references; a full index is at the end.

Vogel's interesting work should serve eminently the purpose for which it was intended. With a better understanding of its history, mathematics will emerge as one of the most important cultural interests of our time.

MARTIN LEVEY
Temple University

Introduction to Functional Analysis. By A. E. Taylor. Wiley, New York, 1958. xvi+423 pp. $12.50.

This book attempts to fill a large gap in the textbook literature. Although the book (as is stated in the title) is an introduction, it is not exactly elementary, being at a much higher level than, for example, Kolmogorov and Fomin (Albany, N. Y., 1957). The book is thorough. A degree of completeness is achieved such as is seldom realized in a textbook. There is a vast number of exercises, many of which are important theorems.

After preliminary chapters on vector spaces and the elements of general topology, there is a longer chapter entitled "topological linear spaces." The scope of the discussion is essentially confined to normed spaces so that the title is somewhat misleading. The general theory is continued in chapter 4, which contains basic theorems on metrizable linear spaces such as the operator inverse theorem, the closed graph theorem, and the principle of uniform boundedness. Also included are representations for the duals of special Banach spaces. The next two chapters, which seem to be the core of the book, are concerned with spectral theory, first for Banach spaces in general, and then for bounded oper-

ators in Hilbert space. The last chapter on "integration and linear functionals" gives one the impression of being an afterthought. Here, T is a locally compact Hausdorff space, $C_\infty(T)$ the space of continuous functions, with compact support, on T and $\|x\| = \sup_{t\in T} |x(t)|$. ($C_\infty(T)$ is not complete, in general.) It is proved that the dual, with the strong topology, is isomorphic, as a Banach space, to the Banach space of finite, regular, signed Borel measures on the Borel sets in T. The isomorphism is given by $x'(x) = \int x d\mu$.

The book is to be commended as a worthy intellectual accomplishment. Time will tell its value as a text for graduate courses. It certainly merits a trial.

CASPER GOFFMAN
Purdue University

Topological Analysis. By Gordon Thomas Whyburn. Princeton University Press, Princeton, N. J., 1958. xii+119 pp. $4.00.

Topological analysis consists of those basic theorems of analysis, especially of the functions of a complex variable, which are essentially topological in character, developed and proved entirely by topological and pseudotopological methods. This book expands and gives the details of the program outlined by the author in his lecture "Introductory Topological Analysis" [See *Lectures on Functions of a Complex Variable*, University of Michigan Press, 1955, pp. 1–14.]

Professor Whyburn has been deeply interested in interior or open mappings from the time of their introduction by Stoïlow [See *Principes Topologiques de la Theorie des Fonctions analytiques*, Paris, 1938.] and, in particular, in their intimate connection with analytic functions. Stoïlow early recognized lightness and openness as the two fundamental topological properties of the class of all nonconstant analytic functions. Whyburn, in the intervening years, has made deep and penetrating discoveries in the field of light interior mappings. In particular, he has characterized their action on 2-dimensional manifolds.

The present text develops the surprisingly small amount of background necessary for an understanding of this material, and then studies the circulation index of a mapping about a point. Using this concept the lightness and openness properties of mappings generated by differentiable functions of a complex variable are established. Open mappings are studied in a general topological setting leading up to the complete local analysis of a light open mapping on a 2-dimensional manifold. Global relationships of the Hurwitz Formula type are obtained for the compact manifold case. Rouche's Theorem and other results of this type are proved. One of the outstanding results of the book is the proof of the Hurwitz Theorem on sequences, in its full generality, without using the differentiability of the limit function. The proof, as do most others in the text, uses almost none of the results or techniques of classical analysis. Poles of a function are defined without series expansion.

Analysts who have the patience to study the topological background material (pages 1–40, including a complete point-set proof of the Jordan Curve Theorem), or who have read the same material in Whyburn's book on *Analytic*

Topology [New York, 1942, American Mathematical Society, Colloquium Publications, vol. 28], will find their efforts richly rewarded in the remainder of the book. Professor Whyburn has done mathematics a great service by providing this textbook as a beacon light to guide forward in their research both analysts and topologists who are interested in obtaining further results in both fields.

DICK WICK HALL
Harpur College

An Introduction to Combinatorial Analysis. By John Riordan. Wiley, New York, 1958. x+244 pp. $8.50.

This elegant treatise on the art and science of counting will undoubtedly become a standard text and reference. Chapter 1 treats the usual theory of permutations and combinations, with emphasis on the use of generating functions. The theory of those is developed more fully in Chapter 2, which includes a discussion of the important and oft-recurring Bell polynomials. Chapter 3, on the principle of inclusion and exclusion, is followed by an excellent treatment of permutations in cyclic representation. After a survey of distributions and occupancy (Chapter 5), the author takes up, in Chapter 6, partitions, compositions, and the enumeration of trees and graphs. Here he presents the beautiful and powerful theorem of Pólya. The last two chapters are an extended development of permutations with restricted position, in which field the author has made many contributions.

Some outstanding characteristics of this book are its consistent and systematic use of the generating function as a powerful tool; its wealth of applications, both in the text and in the extensive problem sections; its many useful tables; its clarity of presentation; its accuracy and excellent format. There were only a few instances (*e.g.*, di Bruno's formula) where the development could have been more direct and illuminating. This reviewer would have liked to see at least a statement of Ramanujan's beautiful congruence theorems and of the asymptotic formula for $p(n)$. Indeed, there is a conspicuous lack, throughout, of order of magnitude considerations. But these are only minor flaws in an important work covering an important field.

N. J. FINE
University of Pennsylvania and
Institute for Adanced Study

Difference Methods for Initial-Value Problems. By Robert D. Richtmyer. Interscience, New York, 1958. xii+238 pp. $7.25.

When formulated mathematically, many of the most important problems in theoretical physics lead to a partial differential or integro-differential equation whose solution, subject to appropriate subsidiary conditions, specifies the state of the physical system under study. Although some of the simpler problems (such as the vibrating string and heat flow equations of the elementary textbooks) have explicit solutions in terms of Fourier series and integrals, most

realistically formulated problems allow only approximate numerical solutions. Under such circumstances, the continuous space and time variables are made discrete and systems of finite-difference equations are solved in a step-wise fashion to furnish approximations to the solution of the corresponding continuous problem.

Studying the book under review is an excellent way to learn of the present state of knowledge concerning such finite-difference methods. Since the beginnings of the theory in the classic paper by Courant, Friedrichs, and Lewy [*Über die partiellen Differenzengleichungen der mathematischen Physik*, Math. Annalen, vol. 100, 1928, pp. 32–74] much progress has been made, especially at the wartime atomic energy laboratories and at New York University's Institute of Mathematical Sciences. Nevertheless, high-speed computers have been and are still being used to solve highly complex problems in the absence of rigorous proof that the machine methods do what they are supposed to do. Convergence proofs and error estimates are not available for many important problems. These get solved numerically by methods based more on empirical tests and intuition than on mathematical demonstration.

Theory and practice are both treated in this book. Convergence of the difference equation solution to the true solution of the continuous problem, rates of convergence and stability of various difference equation systems, and methods of solving implicit difference systems are discussed in Part I. Here the main theorem is the equivalence between stability and convergence proved by Peter Lax. The general theory, expressed in the language of linear operators, is then specialized to pure initial-value problems with constant coefficients, and the von Neumann condition for stability is derived and studied in great detail. Part II (approximately two-thirds of the book) is devoted to a discussion of difference-equation methods that have proved useful for initial-value problems arising in heat flow, neutron diffusion, radiation transfer, sound waves, elastic vibrations, and fluid dynamics.

SAMUEL GOLDBERG
Oberlin College

Mathematics in Fun and in Earnest. By Nathan Altshiller Court. Dial Press, New York, 1958. 250 pp. $4.75.

Altshiller Court's book is an illustration of his own credo: mathematics in earnest should be fun; mathematics in fun may be earnest. The book has no index, but a five-page table of contents furnishes reasonable orientation. The enthusiasm of the author carries the reader along. Everything is grist to Court's mill: philosophy of mathematics, logic and set-theory, paradoxes, basic concepts, games and puzzles and other topics. The result is a buffet where you sample one dish after another for aroma and flavour. References at the end of each chapter offer opportunity for further investigation. In many cases this

seems indispensable, most markedly perhaps in connection with the following passage (p. 65):

"Hilbert labored for many years trying to produce a proof that the axioms of his 'Grundlagen der Geometrie' satisfied this requirement [of being logically consistent with each other]. But all his persistent zeal and his enormous intellectual resources proved unequal to the task, though he could find some personal consolation in the proposition, proved by K. Goedel in 1931, that the Grundlagen could not yield a proof of its own consistency. Georges Bouligand (*Le déclin des absolus mathématico-logiques*, Paris, 1949, p. 16) formulated the argument as follows: 'To find within a body of doctrine G a proof that G is consistent is impossible, for to accept the validity of such a proof is to concede to a part of G a special privilege: an abusive procedure, if the coherence of G as a whole is in doubt.' *Simple and obvious, David Hilbert to the contrary notwithstanding.*" (Italics mine).

Since our author's main scientific interest has been consistently in the domain of geometry, as is well known to readers of this MONTHLY, it is not surprising that his condensed discussions of this field, for example on Euclidean and on projective geometry, and on the role of the infinite in these disciplines, should be rewarding in their clarity.

Among mathematical puzzles the graphical solution of the classical "pouring problems" is worthy of notice (credited to Perelman). One runs across intriguing statements such as "[the mathematical] proofs are not an integral part of the mathematical doctrine." In "Mathematical Asides" we find a quantitative discussion of the fact that men and beasts lost or blindfolded tend to move in circles, based on the assumption of slight differences in the lengths of legs. "The Perplexities of a Potato Pusher," pp. 171–181, may seem a little out of place, both in form and content, and two problems in elementary geometry may not have gained by the forced humorous treatment.

On p. 239 we read that Vinogradov proved in the thirties of our own century that any odd number (instead of: every sufficiently large odd number) is the sum of three primes.

Very careless proofreading is hard to excuse. Of many misprints and inaccuracies, I mention only misspelled author names: Halstead for Halsted (consistently); Lengendre for Legendre (p. 161); Frenel for Fresnel (p. 178); Danzig for Dantzig (p. 134); Appollonius for Apoll. (p. 102); Non-Arguesian for Non-Desargu. (p. 179).

All in all, we must be grateful to our author for expressing without inhibition his thoughts on a wide range of mathematical and philosophical topics. Even in disagreeing with him, one appreciates his adventurous spirit. The words of Goethe come to one's mind: Wer Vieles bringt wird Manchem etwas bringen.

AUBREY J. KEMPNER (Emeritus)
University of Colorado

Fundamentals of Mathematics, Rev. Ed. By M. Richardson. Macmillan, New York, 1958. xviii+507 pp. $6.50.

This book, as did the original edition (for a review of the original see the Aug.-Sept. 1941 MONTHLY), "attempts to provide a suitable terminal course for students of the arts and social sciences, stressing the fundamental concepts and applications of mathematics rather than its formal techniques." In this revision some sections have been rewritten (especially the section on probability), and sections have been added on: electronic computers, information theory, the algebra of propositions and truth tables, the application of Boolean algebra to electrical networks, political structures, inequalities, linear programming, the theory of games of strategy, and individual and social preferences.

If one looks over these additions and the chapter titles it may be observed that the topics come quite close to those in the experimental texts for the freshman Universal Mathematics course recommended by the Committee on the Undergraduate Program of the Association. Thus, those teachers looking for a useable text which covers approximately the material suggested by the C.U.P. might well consider this one.

Since Richardson's 500-page book also "covers" topics such as topology, non-Euclidean geometry, and transfinite cardinals, which are not suggested by the C.U.P., it is fairly clear that much of the coverage must lack depth. The lack of depth may be gauged from the following items. Many properties of the natural numbers are deduced in the first part of the book but induction is not mentioned until a chapter near the end. (This actually may be wise pedagogically.) Complex numbers are introduced but the student is not told what it means to add or multiply two of them. A rational fraction is defined to be a symbol. (A footnote explains that this is not above logical reproach.) The chapter on trigonometry does not discuss addition formulas or identities. "Random sample" is not defined (in fact, it is confused with "representative sample") and much of the chapter on statistics seems to be much "thinner" than it should be for a terminal math course for social scientists. There are too few problems in the statistics chapter.

Several of the newly added sections are excellent introductions to some of the new applications of mathematics and the book should succeed in giving students a good "feeling" for the nature of mathematics and some of its applications. Whether a teacher will find it suitable as a text or not will depend on whether he considers the attainment of this feeling adequate compensation for loss of a deeper treatment of (necessarily fewer) topics.

All in all, the book is a valuable attempt at a compromise between talking about mathematics and doing mathematics.

FRANK L. WOLF
Carleton College

An Introduction to Mathematics. By Alfred North Whitehead. Oxford University Press, New York, 1958. A Galaxy Book. v+191 pp. $1.50.

This small volume, which appeared in England in 1911, was not published in this country until 1948, a year after its author's death. At that time new drawings were prepared, but the text remained unchanged, and here appears in "paper-back" form.

Written "not to teach mathematics, but to enable students from the beginning of their course to know what science is about" by freeing it from a mass of technical details, the book deals with the basic ideas of those topics of classical mathematics that were of interest to the budding scholar of 1911: the nature of mathematics, variables, symbolism, conic sections, geometry, trigonometry and others. In fact the budding scholar of today, struggling with the notions of continuity and limits, might be directed to the sections on functions, the differential calculus, and infinite series.

But it is its treatment of the development of scientific ideas through the centuries, showing why they languished or came to fruition when they did, as well as its lucidity of thought and expression, that gives the book its chief claim to distinction. And for the reader who does not know Whitehead, his warmth, his simplicity, his quiet humor, his utter lack of pretension, it will provide a rewarding introduction to a man as well as to a philosopher and scientist.

BESS E. ALLEN
Wayne State University

Lectures on Ordinary Differential Equations. By Witold Hurewicz. Technology Press (M.I.T.) and Wiley, New York, 1958. xvii+122 pp. $5.00.

This monograph constitutes a rigorous and lively introduction to the theory of ordinary differential equations. The approach is essentially geometric and extremely appealing to the intuition. The book was prepared from mimeographed notes left by Professor Hurewicz under the editorship of Professor Levinson. The reviewer feels that the editor did an excellent job of completing these notes into book form without losing any of Professor Hurewicz's style. Chapter I studies the first order system and starts with direction fields, goes into approximate solutions, existence and uniqueness theorems and extensions of these theorems. Chapter II is concerned with systems of first order equations, Lipschitz conditions, properties of solutions, and the reduction of higher ordered systems. In Chapter III matrix notation is used and the author makes a very careful presentation of fundamental systems and their properties, the Wronskian, Green's function, eigenvalues and eigenfunctions. Chapter IV introduces autonomous systems and, for the linear systems, the singularities are classified in terms of the local behavior of the characteristics. Certain nonlinear systems are reduced to this situation under carefully stated conditions. The last chapter (V) deals with solutions of autonomous systems in the large. The principal result

here is the Poincaré-Bendixson Theorem. Poincaré's index is used to characterize orbital stability of limit cycles and simple singularities. In the opinion of the reviewer, the careful and lucid way in which this book is written puts it well within the grasp of the senior mathematics student.

PASQUALE PORCELLI
Mathematics Research Center (U. S. Army)
The University of Wisconsin

BRIEF MENTION

An Introduction to the Geometry of N Dimensions. By D. M. Y. Sommerville. Dover, New York, 1958. xvii+196 pp. $1.50.

This reprint first appeared in 1929.

The Supervisor of Mathematics, His Role in the Improvement of Mathematics Instruction. National Council of Teachers of Mathematics, Washington 6, D. C., January, 1959. 10 pp. 15 cents each; 10 or more copies, 10 cents each.

A very timely publication, since more and more school systems are now providing the services of a separate supervisor of mathematics. Some of these posts are currently being filled by persons with little or no graduate training *in mathematics;* a deplorable situation.

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

PERSONAL ITEMS

Professor E. B. Mode, Wellesley, Massachusetts, represented the Association at the inauguration of President A. S. Knowles of Northeastern University on September 8, 1959.

University of Colorado: Professor B. W. Jones will be on leave at the University of Puerto Rico until June 1960; Professor L. J. Mordell, Cambridge University, England, will be a Visiting Professor until June 1960.

University of Georgia: Professor M. K. Fort, Jr., has been appointed Head of the Mathematics Department; Professor A. C. Cohen, Jr. has been appointed Director of the newly established Institute of Statistics; Assistant Professor R. P. Hunter will be on leave for one year to Oxford, England; Assistant Professor B. J. Ball, University of Virginia, has been appointed Associate Professor; Assistant Professor J. G. Horne, Jr., University of Kentucky, has been appointed Assistant Professor.

Massachusetts Institute of Technology: Professors Warren Ambrose, C. C. Lin, I. M. Singer, and Assistant Professor M. L. Minsky will be on leave during 1959–60; Associate Professor G. B. Whitham, New York University, has been appointed Professor; Dr. Sigurdur Helgason, University of Chicago, has been appointed Assistant Professor and will be on leave at Columbia University during the academic year 1959–60; Assistant Professor D. M. Kan, Hebrew University, Jerusalem, Israel, and Dr. G. C. Rota, Har-

vard University, have been appointed Assistant Professors; Dr. J. G. Clunie, Imperial College, University of London, will be a Visiting Assistant Professor; Dr. R. J. Gribben, Sloan Foreign Postdoctoral Fellow during 1958–59 will be Lecturer; Drs. Peter Dombrowski, Bonn University, Germany, G. H. Fullerton, Queen's University, Belfast, Ireland, Harry Furstenberg, Princeton University, Walter Koppelman and P. C. Shields, Yale University, W. G. Strang and E. O. Thorp, University of California, Los Angeles, have been appointed C. L. E. Moore Instructors; Drs. G. F. Feeman, Muhlenberg College, Patrick Gallagher, Princeton University, and J. D. Thomas, Los Alamos Scientific Laboratories, have been appointed Instructors; Drs. R. F. Chisnell, University of Manchester, England, Genjiro Fujisaki, University of Tokyo, Japan, Yosutaka Sibuya, University of California, Los Angeles, and Ankira Sakurai, New York University, have been appointed Research Associates; Dr. Takayoshi Mitsui, University of Tokyo, Japan, has been awarded a Sloan Foreign Postdoctoral Fellowship in the School for Advanced Study at M.I.T.

University of Wisconsin: Professor P. C. Hammer, Director of the Numerical Analysis Laboratory, has been given a research grant by the University Research Committee for the year 1959–60, and will spend the year in Europe. In his absence Professor E. A. Robinson will be Acting Director.

Mr. A. C. Ahlin, Massachusetts Institute of Technology, has accepted a position as Mathematician with the Boeing Airplane Company, Renton, Washington.

Mr. R. C. Anderson, Sr., Alabama College, has accepted a position as Mathematician with the Army Ballisic Missile Agency, Redstone Arsenal, Huntsville, Alabama.

Dr. K. I. Appel, University of Michigan, has accepted a position with the Institute for Defense Analyses, Princeton, New Jersey.

Assistant Professor R. R. Archer, Massachusetts Institute of Technology, has been appointed Assistant Professor at the University of Massachusetts.

Assistant Professor W. G. Bade, University of California, Berkeley, has been promoted to Associate Professor.

Mr. C. R. Berndtson, AVCO Manufacturing Corporation, Stratford, Connecticut, has been appointed Mathematics Teacher at Maine Central Institute, Pittsfield, Maine.

Mr. J. A. Berton, University of Illinois, has been appointed Instructor at Indiana State Teachers College.

Associate Professor A. A. Blank, University of Tennessee, has been appointed Associate Professor at the Institute of Mathematical Sciences, New York University.

Professor F. E. Bowling, Lincoln Memorial University, has been appointed Professor and Head of the Department of Mathematics at Tennessee Wesleyan College.

Assistant Professor J. R. Boyd, Lamar State College, has been appointed Assistant Professor at Arlington State College.

Dr. D. G. Brennan, Massachusetts Institute of Technology, has been appointed Assistant Group Leader at M.I.T., Lexington Laboratory.

Mr. J. T. Brogan, Mississippi State University, has accepted a position as Research Engineer with Autonetics, Downey, California.

Brother L. R. Foy was appointed President of Marian College, Poughkeepsie, New York.

Mr. Thomas Courtney, Texas Technological College, has accepted a position as Junior Engineer with the Philco Corporation, Philadelphia, Pennsylvania.

Dr. C. H. Cunkle, Cornell Aeronautical Laboratories, Inc., Buffalo, New York, has been appointed Associate Professor at Utah State University.

Assistant Professor Frederic Cunningham, Jr., Wesleyan University, has been appointed Associate Professor at Bryn Mawr College.

Dr. G. L. Davey, Edwards Air Force Base, California, has been appointed a Member of the technical staff at Hughes Aircraft Company, Culver City, California.

Mr. E. R. Deal, University of Michigan, has been appointed Assistant Professor at Colorado State University.

Mr. Walter Ehrenpreis, University of Connecticut, has been appointed Assistant Professor at Trenton State College.

Dr. H. T. Engstrom, National Security Agency, Washington, D. C., has been appointed Vice President of Remington Rand, New York.

Professor Frances E. Falvey, Millikin University, has been appointed Dean of Milwaukee-Downer College.

Mr. C. K. Fendall, Boeing Airplane Company, Seattle, Washington, has accepted a position as Staff Member, Mathematical Services Department, Aeronutronic, Newport Beach, California.

Miss Barbara A. Flores, Baylor University, has accepted a position as Chemical Technician with the Dow Chemical Company, Freeport, Texas.

Mr. George Forester, Ross Laboratories, Columbus, Ohio, has been appointed Science Supervisor for the Catlin Gabel School, Portland, Oregon.

Assistant Professor D. J. R. Foulis, Lehigh University, has been appointed Assistant Professor at Wayne State University.

Dr. Stanley Frank, University of Florida, has been appointed Teacher and Acting Head of the Mathematics Department at St. Johns River Junior College, Palatka, Florida.

Mr. R. B. Gardner has been appointed Lecturer at Columbia University.

Dr. J. M. Gary, California Institute of Technology, has been appointed Assistant Professor at Harvey Mudd College.

Dr. R. E. Gaskell, Boeing Airplane Company, Seattle, Washington, has been appointed Professor at Oregon State College.

Associate Professor Samuel Goldberg, on leave from Oberlin College, has been appointed Visiting Associate Professor to the Graduate School of Business Administration, Harvard University.

Associate Professor F. M. C. Goodspeed, University of British Columbia, had been appointed Associate Professor at Laval University, Quebec.

Associate Professor Bernard Greenspan has been appointed Professor on his return to Drew University.

Dr. D. S. Greenstein, University of Michigan, has been promoted to Assistant Professor.

Professor Marshall Hall, Jr., Ohio State University, has been appointed Professor at the California Institute of Technology.

Dr. Carl Hammer, Sylvania Electrical Products, Inc., Needham, Massachusetts, has accepted a position as Administrator, Technical Projects Coordination, Radio Corporation of America, New York.

Professor J. R. Hanna, on leave from the University of Wichita, is Design Engineer II at the Martin Company, Denver, Colorado.

Mr. G. G. Harrington, Southern Methodist University, has accepted a position as Associate Research Engineer with the Boeing Airplane Company, Seattle, Washington.

Mr. W. G. Hazlett, Chance Vought Aircraft, Inc., Dallas, Texas, has accepted a position as Program Instructor for the System Development Corporation, Santa Monica, California.

Assistant Professor Henry Hiz, on leave from Pennsylvania State University, has been appointed Visiting Professor at the University of Pennsylvania.

Mr. R. H. Hoskins, John Hancock Mutual Life Insurance Company, Boston, Massachusetts, has been promoted to Associate Group Actuary.

Mr. A. R. Hyde, Kingswood School, West Hartford, Connecticut, has accepted a position as an Editor with the Ronald Press Company.

Mr. R. Q. Jennett, Convair, Fort Worth, Texas, has accepted a position as a Design Specialist with the Martin Company, Orlando, Florida.

Mr. B. A. Jensen, Dana College, has been appointed Assistant Professor at Nebraska Wesleyan University.

Mr. W. J. Jonsson, Defence Research Board, Operations Research Group, St. Hubert, Quebec, has been appointed a Lecturer at the University of Manitoba, Fort Garry, Manitoba.

Mr. N. M. Kendall, Kansas University, has accepted a position as a Mathematician with the Bendix Computer Division, Los Angeles, California.

Professor E. C. Kiefer, Millikin University, has retired and been given the title Professor Emeritus. He will be at Western Illinois University during 1958–59 as a Visiting Professor.

Mr. R. W. King, Vanderbilt University, has been appointed Assistant Professor at Mississippi College.

Mr. W. R. Knight, University of Toronto, has been appointed Assistant Professor at the University of New Brunswick, Fredericton.

Professor C. F. Kossack, Purdue University, has accepted the position of Manager of the newly formed statistics and operations research department of International Business Machines Corporation, Lamb Estate Research Center, Town of Cortlandt, New York.

Assistant Professor R. G. Kuller, Wayne State University, has been appointed Assistant Professor at Dartmouth College.

Mr. G. L. Lane, University of Kansas, has been appointed a Staff Member of the Sandia Corporation, Albuquerque, New Mexico.

Mr. B. L. Lercher, Pennsylvania State University, has been appointed an Instructor at the University of Rochester.

Mr. Shen Lin, Ohio State University, has been appointed Assistant Professor at Ohio University.

Associate Professor F. L. Lynch, Jr., Seton Hall University, has been appointed Chairman of the Mathematics and Physics Departments.

Mrs. Helen M. Marston, Educational Testing Service, Princeton, New Jersey, has been appointed Instructor at Douglass College of Rutgers, The State University.

Mr. C. L. McCarty, Jr., Georgia Institute of Technology, has been appointed Research Mathematician at the University of North Carolina Research Computation Center.

Assistant Professor J. H. McKay, Seattle University, has been appointed Associate Professor at Michigan State University Oakland, Rochester, Michigan.

Dr. R. D. McWilliams, Princeton University, has been appointed Assistant Professor at Florida State University.

Mr. H. L. Nelson, University of Kansas, has accepted a position as Mathematical Analyst with the Autonetics Division of North American Aviation, Los Angeles, California.

Mr. R. S. Nicholls, Miles Laboratories, Inc., Elkhart, Indiana, has been promoted to Director, Research Coordination.

Assistant Professor J. A. Nickel, Willamette University, has been appointed Associate Professor and Chairman of the Mathematics Department at Oklahoma City University.

Mr. G. A. Paxson, College of the Holy Names, has accepted a position as Mathematician with the Mathematical Services Group of the California Research Corporation, Richmond, California.

Professor W. T. Reid, Northwestern University, has been appointed Professor and Head of the Department of Mathematics at the State University of Iowa.

Dr. Azriel Rosenfeld, Ford Instrument Company, Long Island City, New York, has accepted a position as Staff Consultant with the Lewyt Manufacturing Corporation, Long Island City, New York.

Dr. C. V. L. Smith, Aberdeen Proving Ground, Maryland, has accepted a position as Chief, Data Systems Division, NASA Goddard Space Flight Center, Washington, D. C.

Mr. W. G. Spohn, Jr., Bowling Green State University, has been appointed Mathematician with the Applied Physics Laboratory, Johns Hopkins University, Silver Spring, Maryland.

Mr. D. C. Stevens, Ohio State University, has been appointed an Instructor at Alabama College.

Associate Professor W. L. Strother, University of Miami, has been appointed Professor and Chairman of the Mathematics Department, Miami University, Oxford, Ohio.

Professor G. L. Thompson, Ohio Wesleyan University, has been appointed Associate Professor in the Graduate School of Industrial Administration, Carnegie Institute of Technology.

Assistant Professor E. P. Tovani, University of Chicago, has accepted a position as Senior Research Engineer with Convair Astronautics, San Diego, California.

Mr. George Van Zwalenberg, University of California, Berkeley, has been appointed Instructor at Bowling Green State University.

Dr. Warren Weaver has retired from his position as Vice President for the Natural and Medical Sciences of The Rockefeller Foundation and has become Vice President of the Alfred P. Sloan Foundation.

Dr. Harold Weitzner, University of California, Berkeley, has been appointed Associate Research Scientist, Institute of Mathematical Sciences, New York University.

Dr. W. F. Whitmore, Special Projects Office, United States Navy, has accepted a position as Consulting Scientist on the staff of the Chief Scientist, Lockheed Missiles and Space Division, Sunnyvale, California.

Professor Hazel S. Wilson, Doane College, has been appointed Professor at Jacksonville University.

Mr. C. R. Woodrow, Oklahoma State University, has been appointed an Instructor at Austin College.

Professor F. L. Wren, George Peabody College, has been appointed Professor at San Fernando Valley State College.

Mr. C. E. Yingst, Lockheed Aircraft Corporation, Burbank, California, has accepted a position as Computer Programmer at the Thompson-Ramo-Wooldridge Products Company, Los Angeles, California.

Professor Emeritus R. E. Gaines, University of Richmond, died on June 19, 1959. He was a charter member of the Association.

Assistant Professor Lena E. Reynolds, Chapman College, died July 13, 1959. She was a member of the Association for thirty-six years.

NATIONAL ACADEMY OF SCIENCES—NATIONAL RESEARCH COUNCIL DIVISION OF MATHEMATICS

FELLOWSHIP AND RESEARCH OPPORTUNITIES

The Division of Mathematics calls attention to the fact that several foundations and offices offer a number of fellowships as well as financial support for basic research in mathematics during the year 1960–61. A partial list, with comments, is given below; other sources of support are given in the bulletin, "A Selected List of Major Fellowship Opportunities and Publications for Educational Support," available from the Fellowship Office, National Academy of Sciences—National Research Council, 2101 Constitution Avenue, Washington 25, D. C.

1. *National Science Foundation.* The National Science Foundation sponsors various fellowship programs in the sciences, including mathematics. Awards, available only to citizens of the United States, are made solely on the basis of ability.

A. *Predoctoral Fellowships*—offered annually in three programs.

(1) *Graduate* fellowships are awarded at the First Year, Intermediate, and Terminal Year levels of study. Applications for 1960–61 will be available in October 1959 from the National Academy of Sciences—National Research Council, 2101 Constitution Avenue, N.W., Washington 25, D. C., until the closing date, January 1, 1960. Award date—March 15, 1960.

(2) *Cooperative Graduate* fellowships are tenable at approximately 150 participating institutions. Application materials are available at, and are submitted through, the selected institution. For information and list of participating institutions, write The Fellowships Section, Division of Scientific Personnel and Education, National Science Foundation, Washington 25, D. C. Closing date for receipt of applications—November 6, 1959. Award date—approximately April 1, 1960.

(3) *Summer Fellowships for Graduate Teaching Assistants* make it possible for graduate teaching assistants to continue full-time academic study and/or research at approximately 150 participating institutions. Application materials are available at, and are submitted through, the teaching assistants' own institutions. For information and list of participating institutions write to the National Science Foundation. Closing date—December 11, 1959. Award date—approximately April 1, 1960.

B. *Postdoctoral Fellowships*

(4) *Postdoctoral* fellowships for recent recipients of the doctoral degree who desire additional advanced training preparatory to specialized scientific work. Application material available from the National Academy of Sciences—National Research Council. Two award periods: (1) Closing date for first competition—September 1, 1959. Award date—October 15, 1959. (2) Second award period from October 1959 to December 22, 1959. Award date—March 15, 1960.

(5) *Senior Postdoctoral* fellowships are awarded to scientists who have demonstrated ability and special aptitude for productive scholarship in the sciences and who have held the doctoral degree for a minimum of 5 years at time of application. Application material may be obtained from the National Science Foundation after May 15, 1959 for submission prior to October 5, 1959. Award date—December 7, 1959.

C. *Faculty Fellowships*

(6) *Science Faculty* fellowships are awarded to college teachers of science (including mathematics) who plan to continue teaching and wish to increase their competence as teachers. Eligibility requirements include a baccalaureate degree and three (3) years of full time experience at the collegiate level. Application material may be obtained from the National Science Foundation after May 15, 1959 for submission prior to October 5, 1959. Award date—December 7, 1959.

(7) *Summer Fellowships for Secondary School Teachers of Science and Mathematics* provide opportunities for teachers of high ability to pursue individually planned programs at the graduate level. Tenures from one to three summers are available. Information and application material will be available in October 1959 from Secondary School Fellowships, American Association for the Advancement of Science, 1515 Massachusetts Ave., N.W., Washington 5, D. C. Closing date early January 1960. Award date—March 15, 1960.

D. *Research Grants.* The National Science Foundation also supports basic research in the mathematical sciences by means of grants. Proposals for such support are ac-

cepted at any time. They should be submitted about six months before the applicant wishes to receive notification of the Foundation's decision. Instructions for the preparation of proposals, contained in a booklet entitled, "Grants for Scientific Research," may be obtained upon request from the Program Director for Mathematical Sciences, National Science Foudation.

2. *Office of Naval Research.* The Office of Naval Research, through contracts with universities and other organizations, supports basic research in broadly selected fields of mathematics. Proposals should be directed to the Mathematics Branch, Office of Naval Research, Washington 25, D. C. In addition, postdoctoral research associateships in pure mathematics are established under contracts with the ONR at selected universities. For details and application forms write to the above address.

3. *Air Force Office of Scientific Research.* The Air Force Office of Scientific Research supports research in mathematics directly through contracts with colleges, universities, foundations and industrial laboratories. Interested research mathematicians are encouraged to submit proposals, through their organizations, for research in mathematical fields in which they specialize. Proposals should be mailed to the Commander, Air Force Office of Scientific Research, Attn.: Director of Mathematical Sciences, Washington 25, D. C.

4. *Office of Ordnance Research, U. S. Army.* Among the functions of the Office of Ordnance Research is the support of basic research in mathematics. Proposals for projects are ordinarily made by individual scientists or groups of scientists in a form which leads to a contract between the Office of Ordnance Research and a university or research laboratory. For further information write to Commanding Officer, Office of Ordnance Research, Box CM, Duke Station, Durham, North Carolina.

5. *Fulbright Awards—Public Law 584 (79th Congress).* Approximately 400 awards are offered annually for university lecturing and postdoctoral research in all academic fields in Argentina, Australia, Brazil, Burma, Chile, Colombia, Ecuador, India, New Zealand, Pakistan, Paraguay, Peru, the Philippines and Thailand (the next competition for the preceding countries, for the 1961–62 academic year, closes April 25, 1960); Austria, Belgium-Luxembourg, Republic of China, Denmark, Finland, France, Germany, Greece, Iceland, Iran, Ireland, Israel, Italy, Japan, the Netherlands, Norway, Spain, Turkey, and the United Kingdom including colonial dependencies (competition for the latter countries, for the 1960–61 academic year, closes October 1, 1959). 1960–61 lectureships in mathematics are offered in the Republic of China, Finland, Ireland, Japan, Singapore, and the United Kingdom. Awards are payable in foreign currency and usually include travel for the grantee, but not for members of his family, and a maintenance allowance, which may be adjusted in relation to the number of accompanying dependents up to four. Requests for information should be addressed to the Committee on International Exchange of Persons, Conference Board of Associated Research Councils, 2101 Constitution Avenue, Washington 25, D. C.

6. *National Bureau of Standards. Naval Research Laboratory. Air Research and Development Command. Naval Ordnance Laboratory. Navy Electronics Laboratory.* These highly desirable appointments for postdoctoral study and research are similar to postdoctoral fellowships. Postdoctoral resident research associateships are available in a variety of sciences including mathematics and are tenable at the Washington, D. C. and Boulder, Colorado, laboratories of the National Bureau of Standards; at the Naval Research Laboratory in Washington, D. C.; at selected development and research centers of the Air Research and Development Command; at the Naval Ordnance Laboratory in Silver Spring, Maryland; and at the Naval Electronics Laboratory in San Diego, California.

7. *Atomic Energy Commission.* The Division of Research of the Atomic Energy Commission through contracts with universities and other organizations supports research in the fields of numerical analysis, digital computer design, programming research, and related topics. Proposals should be submitted to the Division of Research, Atomic Energy Commission, Washington 25, D. C.

Brookhaven National Laboratory. Brookhaven National Laboratory, operated by Associated Universities, Inc. under contract with the Atomic Energy Commission offers postdoctoral research appointments in the fields of numerical analysis, digital computing, mathematical physics, differential equations, probability and statistics, and various specialized branches including reactor theory, hydrodynamics, and orbit theory. Applications should be directed to M. E. Rose, Head, Applied Mathematics Division, Brookhaven National Laboratory, Upton, Long Island, New York.

MEETING OF MATHEMATICS SECTION—AAAS

The American Association for the Advancement of Science will meet December 26–31 in Chicago. Section A (Mathematics) will have three meetings. On December 26 at 4 P.M. in the Promenade Room of the Morrison Hotel, R. H. Bing will give his retiring vice-presidential address "Topology of Euclidean Three-space."

On December 27 at 9 A.M. in the Promenade Room of the Morrison Hotel there will be four invited papers on "The New Look in Mathematical Education":

G. Baley Price, the University of Kansas and the California Institute of Technology, "Report on the Work of the Committee on the Undergraduate Program."

Henry Swain, Winnetka, Illinois, "The Ninth Grade Course of the School Mathematics Study Group."

Morris Kline, New York University, "New Curriculum or New Pedagogy?"

William L. Duren, Jr., University of Virginia, "Early Calculus in the United States."

On December 28 at 9 A.M. in Parlor F of the Morrison Hotel there will be a symposium on "Trends in the Applications of Mathematics" cosponsored by Section A and the Society for Industrial and Applied Mathematics:

R. F. Drenick, Bell Telephone Laboratories, "Probability and Stochastic Processes."

Philip Wolfe, Rand Corporation, "Mathematical Programming."

Frank Wagner, North American Airplane Company, "Computers and Computer Languages."

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Treasurer, announces that the following 144 persons have been elected to membership by the Board of Governors on applications duly certified.

ALBERT W. ALEXANDER, A.B.(Atlantic Christian) Teacher, Contentnea High School, Kinston, North Carolina.

HOWARD M. AMEN, M.E.(South Dakota) Instr., South Dakota State College.

THOMAS A. ATCHISON, B.S.(Texas) Grad. Student, University of Texas.

MICHAEL W. ATKINSON, B.S.(Ball S.T.C.) Grad. Student, Ball State Teachers College.

MRS. KEMP H. BALDWIN, M.A.(East Carolina) Teacher, J. H. Rose High School, Greenville, North Carolina.

MARY E. BEAUMONT, M.S.(Wisconsin) Instr., Ripon College.

ALBERT F. BENINATI, M.A.(N.Y.S. T.C., Albany) Math. Supervisor, Nyack High School, New York.

CHARLES A. BERGREN, M.A.(Columbia) Engineer, Airborne Instruments Lab.

GLORIA A. BERNIER, B.S.(St. Lawrence) Research Mathematician, St. Regis Paper Co.

DEBORAH W. BEVERIDGE, M.A.T.(Radcliffe) Teacher, Milton Academy, Massachusetts.

GERALD G. BILODEAU, Ph.D.(Harvard) Advanced Research Engineer, Sylvania Electronic Systems.

HAMILTON S. BLUM, Prof. diploma (Columbia) Chairman of Dept., East Meadow High School, New York.
LEONIE D. BOEHMER, B.S. (Coll. of St. Joseph) Calculating Machine Operator, Fox Vliet Drug Co.
PETER M. BRADY, JR., Student, Stevens Institute of Technology.
LOUISE BRAKE, M.A. (East Carolina) Teacher, Snow Hill High School, North Carolina.
OSCAR W. BRANNON, A.B. (Atlantic Christian) Teacher, Windsor High School, North Carolina.
A. MARTIN BUONCRISTIANI, Student, University of Santa Clara.
H. BLAIR BURNER, JR., M.S. (North Dakota) Instr., University of North Dakota.
JACK F. BUSSIO, A.A. (Sacramento J.C.) Student, University of California, Davis.
DAVID G. CANTOR, B.S. (California Tech.) NSF Fellow, University of California, Los Angeles.
MARION E. CLARK, M.S. (Michigan) Asso. Professor, King College.
EILEEN C. COX, A.M. (Columbia) Acting Chairman of Dept., New Dorp High School, Staten Island.
CHARLES R. DAVIS, B.S. (East Carolina) Teacher, Morehead City School, North Carolina.
FELIX T. DAVIS, III, Student, University of Kansas.
JOHN B. DAVIS, JR., M.A. (East Carolina Coll.) Asso. Professor, Wilmington College.
LOUIS E. DENOYA, Student, Oklahoma State University.
EARL W. DENTON, M.A. (East Carolina) Teacher, Grifton High School, North Carolina.
MAURICE C. DEVORE, B.S. (Nevada) James Marshall High School, W. Sacramento, California.
EDWARD D. EBERT, M.S. (Iowa) Asst. Professor, University of Toledo.
LOUIS M. EDWARDS, M.Ed. (Florida) Teacher, Edgewater High School, Orlando, Florida.
MARY ELIZABETH EKSTRAND, B.A. (California, Davis) Mathematician, Naval Ordnance Test Station.
SAMUEL FEDER, M.S. (New York) Mathematician, Bulova Research & Development Labs.
ENRICO T. FEDERIGHI, Ph.D. (Indiana) Sr. Mathematician, Applied Physics Lab.
KIMBROUGH S. GARRETT, Student, University of Texas.
ELDRIDGE W. GARTEN, B.A. (Shelton Coll.) Engineering Change Analyst, Republic Aviation Corp.
RALPH H. GOEBEL, M.S. (S.U. of Iowa) Asso. Actuary, Northwestern National Life Insurance Co.
GLENN W. GOODRICH, Student, University of Tulsa.
EDMOND B. GORMAN, B. S. in E.E. (Healds Eng.) Design Engineer, Federal Pacific Electric Co.
GEORGE C. GRAFF, B.S. (Brooklyn Coll.) Grad. Asst., University of Illinois.
JACK S. GRAY, B.S. (East Carolina) Asst. Principal, Gar-Field High School, Woodbridge, Virginia.
MRS. BETTY S. GUILFORD, B.S. (Appalachian S.T.C.) Teacher, Chocowinity High School, North Carolina.
ROBERT A. HALL, M.S. (Washington) Instr., University of Minnesota, Duluth.
EVA C. HANNEMAN, A.M. (Cornell) Teacher, Waverly Senior High School, New York.
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COL. N. L. HEAD, Ed.M. (Boston) Teacher, Lexington High School, Massachusetts.
JOSEPH A. HEBERT, Student, Bradford Durfee College of Technology.
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EUGENE M. HUGHES, M.S. (Kansas S.U.) Head of Dept., Nebrsaka State Teachers College.
STANLEY S. ISAACS, Student, Antioch College.
ARNOLD A. JOHNSON, Ph.D. (Case Inst. of Tech.) Project Engineer, Case Institute of Technology.
MRS. EMMA D. JOHNSON, M.A. (Ohio State) Asst. Professor, Ohio Wesleyan University.
HARLEY R. JORDAN, B.S. (Denver) Physicist, Navy Department.
JOANNE M. KAMINSKY, B.S. (Harpur) Mathematician, International Business Machines.
HAROLD M. KAPLAN, M.A. (Princeton) Asst. Professor, United States Naval Academy.
MAURICE KAPLOWITZ, M.A. (New York) Teacher, Boys High School, Brooklyn, New York.
HENRY C. KASHIAN, M.A. (Boston) Sr. Engineer, Sylvania Electric Products.
GORDON E. KELLER, Student, Houghton College.
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JOHN R. MESSINGSCHLAGER, Student, University of Detroit.
URSULA MRAZEK, Student, University of Alabama.
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JACK O. PURDUE, Ph.D. (Oklahoma) Chairman, Div. of Natural Sciences, Oklahoma Baptist University.
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OVED SHISHA, Ph.D. (Hebrew) Research Asso., Harvard University.
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SISTER RITA JEAN, C.S.J., M.A. (Minnesota) The College of St. Catherine.
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HARRY L. STEIN, Ph.D. (Minnesota) Supervisor of Grad. Studies and Math. Lecturer, University of British Columbia.
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RALPH E. STRAUCH, A.B. (California, Los Angeles) Ensign, United States Navy.
GRACE STRECKER, M.A. (St. Louis) Teacher, Normandy High School, St. Louis County, Missouri.
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HOWARD H. W. UN, Student, Beloit College.
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DAVID L. WOOD, B.S. (Miami) Teacher, Coral Gables Senior High School, Florida.
LT. JIMMIE D. WOODS, B.S. (U.S. Coast Guard Acad.) U.S. Coast Guard Base, San Juan, Puerto Rico.
RUBIN P. WOOTEN, Student, Mississippi State University.
RONALD L. WRIGHT, Student, Santa Ana College.
JUDITH ZAGRODNICK, B.S. (Pittsburgh) Sr. Technical Asst., Bell Telephone Labs.

THE FORTIETH SUMMER MEETING OF THE ASSOCIATION

The fortieth summer meeting of the Mathematical Association of America was held at the University of Utah, Salt Lake City, Utah, from Monday, August 31 through Thursday, September 3, 1959 in conjunction with summer meetings of the American Mathematical Society, the Association for Symbolic Logic, the Society for Industrial and Applied Mathematics, Pi Mu Epsilon, and Mu Alpha Theta. There were registered 985 persons, including 423 members of the Association.

Sessions of the MAA were held on Monday morning and afternoon, on Tuesday and Wednesday mornings and on Thursday afternoon. All sessions were held in Spencer Hall Auditorium of the University of Utah. Presiding officers were President C. B. Allendoerfer, Vice-president Harley Flanders, and Professors C. B. Tompkins, A. Nijenhuis, R. F. Rinehart and F. E. Hohn. The eighth series of Earle Raymond Hendrick Lectures were delivered by Professor William Feller of Princeton University. The Program Committee for the meeting consisted of J. V. Wehausen, Chairman; G. E. Forsythe, F. E. Hohn, A. Nijenhuis, and R. F. Rinehart.

FIRST SESSION OF THE ASSOCIATION

The Earle Raymond Hedrick Lectures: "Topics Connected with Ordinary Differential Operators," Lecture I, by Professor William Feller, Princeton University.

"Training of High School Mathematics Teachers," by Professor J. L. Kelley, University of California, Berkeley. Panel: C. B. Allendoerfer, E. G. Begle, Harley Flanders, L. C. Lay.

"The Role of Films and Television in Teaching Mathematics," by Professor H. M. MacNeille, Washington University.

SECOND SESSION OF THE ASSOCIATION

Hedrick Lecture II, by Professor Feller.

"High Speed Computing in a University," Chairman, Professor C. B. Tompkins, University of California, Los Angeles.

"Recent Developments in Computer Languages," by Professor Bernard A. Galler, University of Michigan.

"Organizing a University Computation Center," by Professor J. G. Herriot, Stanford University.

"The Acquisition and the Role of the Digital Computer in Research," by Dean Henry Eyring, University of Utah.

THIRD SESSION OF THE ASSOCIATION

Hedrick Lecture III, by Professor Feller.

Business Meeting of the Association.

"The Role of Geometry for the Mathematics Student," by Professor Herbert Busemann, University of Southern California.

"Geometry for Teachers in the Undergraduate Curriculum," by Professor E. E. Moise, University of Michigan.

FOURTH SESSION OF THE ASSOCIATION

"Stimulations to Mathematics from Operations Research," Chairman, Professor R. F. Rinehart, Duke University.

"Mathematical Programming," by Dr. Robert Kalaba, Rand Corporation.

"Stochastic Processes; Queuing Theory," by Dr. Donald P. Gaver, Jr., Westinghouse Electric Corporation (by title).

"Calculus of Variations," by Professor J. M. Danskin, Rutgers University.

FIFTH SESSION OF THE ASSOCIATION

Retiring Presidential Address: "Analytic Functions of a Bicomplex Variable," by Professor G. B. Price, University of Kansas.

"Theory of Digital Circuits," Chairman, Professor F. E. Hohn, University of Illinois.

"Some Applications of Logic to the Theory of Sequential Circuits," by Professor F. B. Fitch, Yale University.

"The Logico-Mathematical Theory of Synchronous Circuits," by Dr. C. C. Elgot, Willow Run Laboratories, University of Michigan.

"The Mathematical Theory of Asynchronous Circuits," by Professor D. E. Muller, University of Illinois.

SPECIAL SESSIONS OF THE ASSOCIATION

The set of four films on "The Theory of Limits" starring E. J. McShane and produced by the MAA's Committee on Production of Films was shown in Kingsbury Hall Auditorium on Wednesday evening. The first two films were also shown in Spencer Hall on Thursday afternoon.

On Thursday evening an open conference on High School Contests was held in the Union Building, with about 25 persons present. President Allendoerfer presided. There was general agreement that the contest should be continued along its present lines with some suggested modifications in the nature of the questions.

MEETING OF THE BOARD OF GOVERNORS

The Board of Governors of the Association met on Monday evening in the Union Building of the University of Utah with twenty-one members present. Among the more important items of business transacted were the following:

Professor R. D. Wagner of the University of Wisconsin was elected to the Board of Governors to fill the balance of the unexpired term of the late Professor H. P. Evans.

The Board approved the following schedule of future meetings: Hotel Conrad Hilton, Chicago, Illinois, January 28–30, 1960; Michigan State University, August 29–September 1, 1960; Hotel Willard, Washington, D. C., January, 1961; Oklahoma State University, August, 1961; Kansas City, Missouri, January, 1962; University of British Columbia, August, 1962; University of Colorado, August, 1963.

The Board voted to invite Professor Ivan Niven of the University of Oregon, to deliver the ninth series of Earle Raymond Hedrick Lectures at the 1960 Summer Meeting.

Publication was authorized of Carus Monograph No. 13 entitled "A Primer of Real Functions" by R. P. Boas.

Action was taken to increase the size of the MONTHLY so that the normal issue hereafter will consist of 112 pages rather than 96 pages. This will provide for more rapid publication of accepted papers.

The Board voted to make an appropriation for the support of the Mathematics Magazine in order to provide for its continued publication.

BUSINESS MEETING OF THE ASSOCIATION

A business meeting of the Association was held on Thursday morning with President Allendoerfer presiding. The Secretary-Treasurer reported that the membership of the Association was 8633 on August 17, an increase of 11% since the corresponding date last year. Over 10,000 copies of the MONTHLY are being printed for each issue.

Professor Henry L. Alder, the new Secretary of the Association was then presented to the meeting. He expressed the appreciation of the members of the Association for the services of Professor Gehman as Secretary-Treasurer.

Professor Rothwell Stephens reported for the Committee on Visiting Lecturers and Dr. John R. Mayor reported for the Committee on Secondary School Lecturers.

MEETING OF SECTION OFFICERS

A meeting of representatives of the Sections of the Association was held on Tuesday evening in the Union Building. Forty-eight persons were present representing 25 of the 27 Sections of the Association.

Reports were made on special activities of the various sections, such as speakers bureaus, traveling lecturers programs, and committees on high school-college relations. There was also a general discussion of programs of section meetings.

MEETINGS OF OTHER ORGANIZATIONS

The American Mathematical Society held its sessions from Tuesday afternoon through Friday. The colloquium speaker was Professor J. L. Doob and invited addresses were given by Professors P. A. Smith and E. A. Coddington.

The Association for Symbolic Logic met on Friday. SIAM met from Tuesday evening through Friday morning.

Pi Mu Epsilon held a luncheon meeting on Tuesday. Mu Alpha Theta, the national high school and junior college mathematics club, held a luncheon meeting on Wednesday.

ARRANGEMENTS, ENTERTAINMENT, AND RECREATION

The Committee on Arrangements for the meeting consisted of: W. J. Coles, Chairman; J. H. Barrett, Lida K. Barrett, F. C. Biesele, J. H. Curtiss, H. M. Gehman, I. O. Horsfall, N. C. Hunsaker, E. E. Kohlbecker, R. S. Pierce, D. W. Robinson, C. R. Wylie.

Registration headquarters was located in the Union Building. Dormitory and cafeteria accommodations were provided by the University of Utah. The text book exhibit and the Mathematical Sciences Employment Register were located in the Union

Building. The Register was augmented by a Placement Service consisting of listings of individuals available for positions.

An informal reception for the women attending the meeting was given by the wives of the Department of Mathematics, University of Utah, on Tuesday afternoon in Carlson Hall Lounge. A picnic at Brighton at the head of Big Cottonwood Canyon was held on Wednesday afternoon. The usual SIAM social evening was held on Wednesday evening. A musical program was presented on Thursday evening at Temple Square especially for persons attending the meetings.

A resolution of thanks prepared by Professor H. M. MacNeille and adopted by the participating organizations expressed warm appreciation to our host, the University of Utah, for all that it has contributed to make our stay here both pleasant and memorable. Thanks was also expressed to the University staff and in particular to the members of the local Committee on Arrangements for so ably coordinating the many phases of the arrangements which have contributed so much to our comfort and enjoyment.

HARRY M. GEHMAN, *Secretary-Treasurer*

CALENDAR OF FUTURE MEETINGS

Forty-third Annual Meeting, Conrad Hilton Hotel, Chicago, Illinois, January 28–30, 1960.

Forty-first Summer Meeting, Michigan State University, East Lansing, Michigan, August 29-September 1, 1960.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Grove City College, Grove City, Pennsylvania, April 30, 1960.

ILLINOIS, Illinois Wesleyan University, Bloomington, May 13–14, 1960.

INDIANA

IOWA, State University of Iowa, Iowa City, April 22, 1960.

KANSAS, Kansas State College of Pittsburg, April 30, 1960.

KENTUCKY, University of Kentucky, Lexington, April, 1960.

LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 19–20, 1960.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, American University, Washington, D. C., December 5, 1959.

METROPOLITAN NEW YORK

MICHIGAN, University of Michigan, Ann Arbor, March 26, 1960.

MINNESOTA

MISSOURI, Central Missouri State College, Warrensburg, April 30, 1960.

NEBRASKA, University of Nebraska, Lincoln, April 23, 1960.

NEW JERSEY, Princeton University, November 7, 1959.

NORTHEASTERN, Boston College, Chestnut Hill, Massachusetts, November 28, 1959.

NORTHERN CALIFORNIA, University of California, Berkeley, January 16, 1960.

OHIO, Kent State University, May 7, 1960.

OKLAHOMA

PACIFIC NORTHWEST, State University of Montana, Missoula, June 17, 1960.

PHILADELPHIA, University of Delaware, Newark, November 28, 1959.

ROCKY MOUNTAIN, United States Air Force Academy, Colorado Springs, May 6–7, 1960.

SOUTHEASTERN, University of South Carolina, Columbia, April 1–2, 1960.

SOUTHERN CALIFORNIA, Los Angeles State College, March 12, 1960.

SOUTHWESTERN, Air Force Missile Development Center, Holloman Air Force Base, New Mexico, April, 1960.

TEXAS, San Antonio College, April, 1960.

UPPER NEW YORK STATE, University of Rochester, May 7, 1960.

WISCONSIN, Mount Mary College, Milwaukee, May 7, 1960.

GEORGE BANTA COMPANY, INC., MENASHA, WISCONSIN, U.S.A.

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LOGIC

MISCELLANY

PROBABILITY, STATISTICS, GAME THEORY

TOPOLOGY

MATHEMATICAL NOTES

Edited by ROY DUBISCH, Fresno State College

CLASSROOM NOTES

Edited by C. O. OAKLEY, Haverford College

MATHEMATICAL EDUCATION NOTES

Edited by JOHN A. BROWN, University of Delaware, and JOHN R. MAYOR, AAAS and University of Maryland

PROBLEMS AND SOLUTIONS

Edited by HOWARD EVES, University of Maine and E. P. STARKE, Rutgers, The State University

AUTHORS

Numbers refer to pages, **boldface** type indicating a problem solved and solution published; *italics*, a problem solved but the complete solution not published; ordinary type, a problem proposed.

SOLUTIONS

Numbers in **boldface** type refer to problems, those in lightface, to pages.

RECENT PUBLICATIONS

Edited by RICHARD V. ANDREE, University of Oklahoma

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ON A FAMILY OF GENERALIZED CONCHOIDS*

R. M. WINGER, University of Washington

Curves with the polar equation $\rho = a \cos p\theta/q + k$ were studied by Moritz† under the name of cyclic-harmonic, using mainly polar coordinates. Following Moritz, Stratton‡ discussed the "polar tangent curves" $\rho = a \tan p\theta/q + k$. The author§ wrote a paper on each of these curves, employing a compact parametric representation. In this paper, which is the third of a trilogy, we consider the remaining case

(1) $\rho = a \sec p\theta/q + k$, a, k real, p, q relatively prime integers.

For our purpose it is convenient to introduce circular or absolute coordinates

$$x \equiv X + iY, \qquad \bar{x} \equiv X - iY, \qquad i^2 = -1,$$

X, Y rectangular, which in turn can be expressed in polar coordinates in the usual way. Then we have

$$x = X + iY = \rho(\cos\theta + i \sin\theta) = (a \sec p\theta/q + k)(\cos\theta + i\sin\theta). \tag{2}$$

Now replace θ by $q\phi$ and let $t \equiv \cos\phi + i \sin\phi$ be the parameter. By De Moivre's theorem

$$\cos p\phi + i \sin p\phi = t^p, \qquad \cos p\phi - i \sin p\phi = t^{-p},$$

whence $2 \cos p\phi = t^p + t^{-p}$. Thus after reduction we get for x and similarly for $\bar{x}$

$$x = \frac{t^q(kt^{2p} + 2at^p + k)}{t^{2p} + 1}, \qquad \bar{x} = \frac{kt^{2p} + 2at^p + k}{t^q(t^{2p} + 1)}, \tag{3}$$

which are parametric equations of the curve in circular coordinates. In this representation t is a complex number of absolute value 1, and real points (*i.e.*, those which can be plotted) have conjugate complex coordinates.#

Equations (3) may be made homogeneous in the parameter or in the coordinates or both by the substitutions $t = t_1/t_2$, and $x = x_1/x_3$, $\bar{x} = x_2/x_3$. We get thus

$$\begin{aligned} x_1 &= t_1^{2q}(kt_1^{2p} + 2at_1^p t_2^p + kt_2^{2p}), \\ x_2 &= t_2^{2q}(kt_1^{2p} + 2at_1^p t_2^p + kt_2^{2p}), \\ x_3 &= t_1^q t_2^q (t_1^{2p} + t_2^{2p}). \end{aligned} \tag{4}$$

The triangle of reference then consists of the circular rays, x_1, x_2 and the line

* Read under the title "A generalization of Cotes' spirals," at the sixty-first summer meeting of the American Mathematical Society, Seattle, 1956.

† Univ. Washington Publ. Math., vol. 1, 1923, pp. 1–58.

‡ This MONTHLY, vol. 43, 1936, p. 398.

§ Univ. Washington Publ. Math., vol. 3, No. 1, 1948, and No. 2, 1952.

R. M. Winger, An Introduction to Projective Geometry, Boston, 1923, p. 326.

at infinity, $x_3=0$, with vertices O, the origin, and I, J, the circular points.

The line equations are, removing the factor $2(p+q)(t_1t_2)^{q-1}$,

$$(5)\qquad \begin{aligned} u_1 &= -kqt^{4p} - 2a(p+q)t^{3p} - 2kqt^{2p} + 2a(p-q)t^p - kq, \\ u_2 &= t^{2q}[-kqt^{4p} + 2a(p-q)t^{3p} - 2kqt^{2p} - 2a(p+q)t^p - kq], \\ u_3 &= 2qt^q(kt^{2p} + 2at^p + k)^2, \end{aligned}$$

where we have returned to the nonhomogeneous parameter.

The simple conchoid, $p=q=1$ is a rational circular quartic, having a line of symmetry, a node at the origin, a tacnode at infinity whose tangent is the only real asymptote. We shall find analogues for all these properties for the more general conchoids.

The existence of polynomial parametric equations proves that the general conchoid is rational. The form of (4) and (5) shows that the order in general is $2p+2q$ and the class $4p+2q$. We see also that there is a multiple point O at the origin whose parameters are given by the common factor in x_1 and x_2. Further there is a q-fold point with coincident parameters and $2q$-fold point contact tangent at each of the circular points, absorbing $q-1$ cusps, since $(t_1t_2)^{q-1}$ factors out of the line equations. The other intersections with the line at infinity are double points since $t=\pm i^{1/p}$, where $i^{1/p}$ is any one of the pth roots of i, name the same point.

We ask now whether these are tacnodes. The condition is that the two tangents of each node coincide. Setting $t=i^{1/p}$ in (5), we find as the coordinates of the tangent line of one branch, after removing the factor $4ai$, $(p,\ -pt^{2q},\ 2iaqt^q)$. Next the coordinates of the other branch $t=-i^{1/p}$ are, removing the factor $(-1)^p4ai$, $(p,\ -p^{2q},\ (-1)^p2iaqt^q)$, which differ from the former only in the third coordinate. We now observe that the two sets of coordinates will be the same if and only if p and q are both odd or both even. But they cannot both be even under the original hypotheses, hence we conclude

*The p nodes on the line at infinity whose parameters are $t^{2p}+1=0$ are tacnodes if and only if p and q are both odd.**

Further we note that these nodes are cut out by the p lines $x_1^p+x_2^p=0$ when q is odd and by $x_1^p-x_2^p=0$ when q is even. For, replacing x_1, x_2 by their values from (4) we obtain respectively

$$(kt^{2p} + 2at^p + k)^p(t^{2pq} + 1) \quad \text{and} \quad (kt^{2p} + 2at^p + k)^p(t^{2pq} - 1),$$

the first of which is divisible by $t^{2p}+1$ when q is odd, while the second contains the factor when q is even. Obviously, only one of the nodes can lie on each of the lines.

The flex form, after removing the factor $4q(p+q)(2p+2q-1)^2(t_1t_2)^{3q-3}$ is, returning to the nonhomogeneous form,

* If they were cusps which also have coincident tangents, the parameters would be factors of the line equation.

$$(6)\quad \begin{aligned}&-k^2q^2(t^{6p}+1)-2ak(p^2+2q^2)(t^{5p}+t^p)\\&\qquad+[4a^2(p^2-q^2)-3k^2q^2](t^{4p}+t^{2p})+4ak(3p^2-2q^2)t^{3p}.\end{aligned}$$

The discarded factor indicates that $q-1$ flexes are absorbed at each of the circular points, in addition to the $2q-2$ needed for the cusps at each point. Other cases of coincident flexes will be noted later.

Symmetry. The curve in the general case is invariant under dihedral groups, binary and ternary, of order $2p$. Generators are: of the binary group,

$$(7)\qquad t'=\epsilon t \quad\text{and}\quad t'=1/t, \qquad \epsilon=e^{2\pi i/p};$$

and of the ternary group,

$$(8)\qquad x'=\epsilon x,\quad \bar{x}'=\epsilon^{-1}\bar{x},\quad\text{and}\quad x'=\bar{x},\quad \bar{x}'=x.$$

The binary group contains p involutions, $t'=\epsilon^k/t$, $k=1,\cdots,p$ and, when p is even, an extra one $t'=-t$. Likewise the ternary group contains p reflexions

$$(9)\qquad x'=\epsilon^k\bar{x},\quad \bar{x}'=\epsilon^{-k}x$$

and, when p is even, an additional one

$$(10)\qquad x'=-x,\quad \bar{x}'=-\bar{x},$$

whose center is the origin and whose axis is the line at infinity. The axes of (9) together are $x^p-\bar{x}^p=0$, thus forming p equispaced lines on the origin, and the centers lie on the line at infinity. When p is odd the centers lie on the perpendicular set $x^p+\bar{x}^p=0$. When p is even the centers lie on the axes.

We consider now two subcases of the general case.

p odd. The axes of reflexion are now conjugate as are also the centers. The axis $x-\bar{x}=0$ cuts out the points $(kt^{2p}+2at^p+k)(t^{2q}-1)=0$, which include ± 1, the fixed points of the involution $t'=1/t$. These are contacts of tangents from the corresponding center. The other intersections aside from the multiple point, given by $(t^{2q}-1)/(t^2-1)$, form $q-1$ double points. Similarly for the other axes of symmetry, accounting thus for $p(q-1)$ double points. When q is also odd, as shown above, the nodes on the line at infinity are tacnodes and are cut out by the lines $x^p+\bar{x}^p=0$, in other words they are the centers of reflexion. But when q is even these nodes are ordinary and lie on the axes and are not centers of reflexion. In either case the nodal tangents are asymptotes, real because parallel to the lines $x^p+\bar{x}^p=0$ or $x^p-\bar{x}^p=0$, which are all real in this representation. Or

The curve has p or $2p$ real asymptotes according as q is odd or even, one perpendicular to each axis of symmetry when q is odd, but parallel in pairs when q is even.

Each tacnodal tangent is a sort of double asymptote, two branches of the curve approaching it in either direction.*

* *Cf.* the simple conchoid, $p=q=1$.

p even, q necessarily odd. The curve now admits the extra reflexion (10) and the origin is a center of symmetry. While the parameters of the multiple point O form a general set under the binary group, they are paired in the involution $t' = -t$ so that the tangents at t and $-t$ coincide, hence

The multiple point at the origin contains p tacnodes, absorbing p additional nodes, the p tangents forming a special conjugate set of lines.

There are $2p$ real asymptotes, parallel in pairs, viz., the pairs of tangents of the p nodes on the line at infinity.

Special cases

$a = k$. Equations (3) reduce to

$$x = \frac{kt^q(t^p + 1)^2}{t^{2p} + 1}, \qquad \bar{x} = \frac{k(t^p + 1)^2}{t^q(t^{2p} + 1)} \tag{11}$$

The multiple point O now includes p cusps, parameters $t^p + 1 = 0$.

For x, $\bar{x}$ both contain the factor $(t^p + 1)^2$ while the class is reduced by p, $t^p + 1$ factoring out of the line equations.

From symmetry properties we infer that the axes must be cusp tangents. Now the axes are conjugate when p is odd but divide into two sets of $p/2$ lines when p is even. We verify readily:

If p is odd the cusp tangents are the axes of symmetry. But if p is even the cusp tangents coincide in pairs forming double cusp tangents which comprise half the axes, viz., $x^{p/2} + \bar{x}^{p/2} = 0$.

The asymptotic properties remain the same as in the earlier cases when $a \neq k$.

There are cusps at $t^p - 1$ when $a = -k$, the curve however being projectively equivalent to the case $a = k$. Since each cusp absorbs two flexes, the cusp parameters must factor doubly out of the flex form. We verify that $(t^p \pm 1)^2$ is a factor of the flex form when $a = \pm k$, in other words $k^2 - a^2$ is a factor of the discriminant of the flex form. We can get another square factor of the flex form when undulations occur. We saw above that $t = 1$ lies on the axis $x - \bar{x} = 0$ and is a contact of tangent from the corresponding center. The tangent at this point is $x + \bar{x} = 2(a + k)$ which cuts the curve doubly at $t = 1$. If we ask that the tangent cut out the point $t = 1$ triply, then from symmetry it must cut it out quadruply. The condition for triple intersection is $k/a = (p^2 - q^2)/q^2$. The property must be shared by each point conjugate to 1. We conclude that

There are undulations at $t^p - 1 = 0$ when $k/a = (p^2 - q^2)/q^2$.

Likewise we find that there are undulations at $t^p + 1 = 0$ when $k/a = -(p^2 - q^2)/q^2$ and that $(t^p \mp 1)^2$ is a factor of the flex form when $k/a = \pm(p^2 - q^2)/q^2$, where upper signs are to be taken together. Thus another factor of the discriminant of the flex form is $q^4k^2 - (p^2 - q^2)^2a^2$.

$q=1$. The circular rays x, $\bar{x}$ now have simple contact at the circular points. We can readily account for all the double points of which the quota is $2p^2+p$. For the $2p$-fold point O is normally equivalent to $2p^2-p$ simple nodes. If p is odd the p double points at infinity are tacnodes, each equivalent to two simple nodes. These with the nodes in O make up the full complement. If p is even the nodes at infinity are ordinary but there are then p tacnodes in O so that again all nodes are accounted for. The equation of the p tacnodal tangents in O when p is even is $kx^p+2a(x\bar{x})^{p/2}+k\bar{x}^p=0$.

$k=0$. These curves are degenerate, the order reducing to $2p$ or $2q$ according as $p>q$ or $p<q$. If however p and q are both odd, the parameters occur only in even powers. Then, replacing t_1^2, t_2^2 by τ_1, τ_2, we see that the order is reduced to p and q, respectively. They are inverse to the rose curves $\rho'=1/\rho$, including as special cases many famous curves, and have been extensively studied under the names Cotes spirals, epi or Ähren curves.* We note a few properties.

$p>q$. The homogeneous point equations, after removing the factor $(t_1t_2)^q$, reduce to

$$x_1 = 2at_1^{p+q}t_2^{p-q}, \qquad x_2 = 2at_1^{p-q}t_2^{p+q}, \qquad x_3 = t_1^{2p} + t_2^{2p} \tag{12}$$

and the line equations, factoring out $(t_1t_2)^{p-q-1}$, to

$$\begin{aligned} u_1 &= t_2^{2q}[(p+q)t_1^{2p} - (p-q)t_2^{2p}], \\ u_2 &= t_1^{2q}[-(p-q)t_1^{2p} + (p+q)t_2^{2p}], \\ u_3 &= -\,4aq(t_1t_2)^{p+q}. \end{aligned} \tag{13}$$

The flex form, neglecting a numerical factor, becomes

$$(t_1t_2)^{2p-3}(t_1^{2p} + t_2^{2p}). \tag{14}$$

The (isolated) multiple point at the origin is a $(2p-2q)$-fold point with just two branches, formed by two $(p-q)$-fold points each with coincident parameters, falling together. The point absorbs $2p-2q-2$ cusps, since the class is reduced by this number, and $2p+4q-2$ additional flexes.

We observe that the section by the x_3-axis is a factor of the flex form and we infer

The p nodes at infinity are biflecnodes.

The curve admits a group of order $4p$, having $2p$ axes of symmetry, and thus enjoys the maximum symmetry for an algebraic curve of order $2p$.§ The biflecnodes are centers of reflexion, lying on the axes $x^p+\bar{x}^p=0$ when q is odd

* For references, see G. Loria, Spezielle Algebraische und Transzendente Ebene Kurven, Leipzig, 1902, p. 367.

§ R. M. Winger, Some applications of groups to geometry, this MONTHLY, vol. 37, 1930, p. 4, where through a slip the maximum number of symmetry axes is given as $2n$ instead of n.

but on the complementary set $x^p - \bar{x}^p = 0$ when q is even. There are $2p$ real asymptotes, the biflecnodal tangents, parallel in pairs to the axes on which the nodes lie.

$p < q$. The point equations now are

$$x_1 = 2at_1^{2q}, \quad x_2 = 2at_2^{2q}, \quad x_3 = (t_1t_2)^{q-p}(t_1^{2p} + t_2^{2p}). \tag{15}$$

The line equations have the same form as (13) but are really not the same because of the different hypotheses regarding p and q.

The flex form, except for a numerical factor, is

$$(t_1t_2)^{3q-p-3}(t_1^{2p} + t_2^{2p}), \tag{16}$$

so that the nodes at infinity are again biflecnodes.

There is a $(q-p)$-fold point at I and at J with coincident parameters and a single tangent with $2p$-point intersection, each absorbing $q-p-1$ cusps and $q+p-1$ additional flexes.

Again the group is of order $4p$, containing $2p$ reflexions, but the symmetry is less than maximum since $p<q$. The biflecnodes and the $2p$ real asymptotes are disposed as in the previous case.

We append some graphs for lower values of p and q.

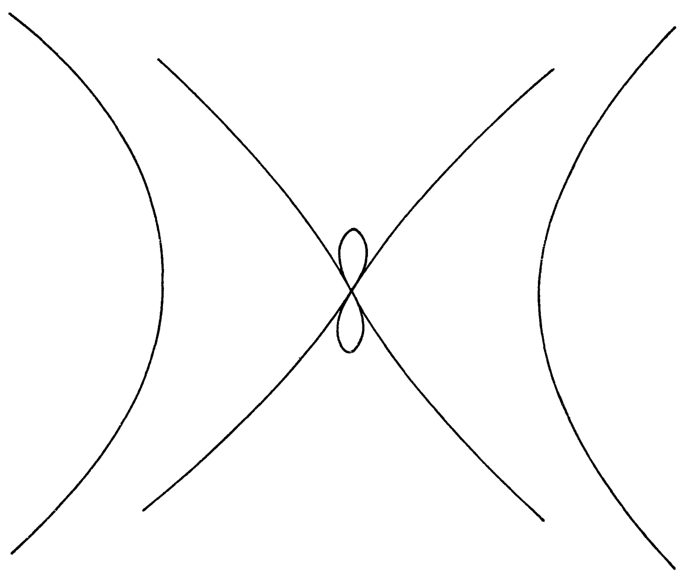

FIG. 1: $p=2$, $q=1$, $k=2a$.

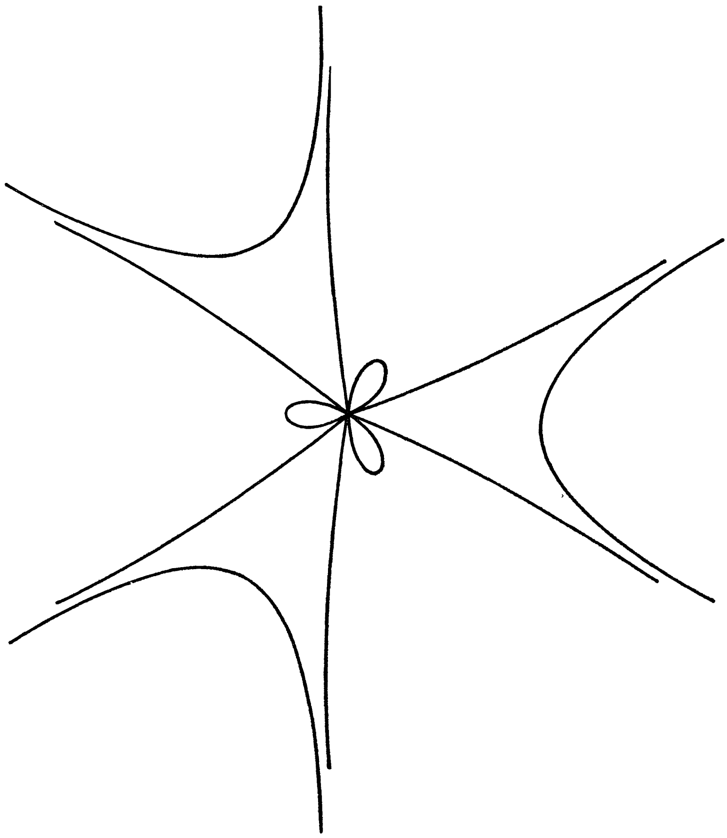

FIG. 2: $p=3$, $q=1$, $k=2a$.

LEONHARD EULER'S INTEGRAL: A HISTORICAL PROFILE OF THE GAMMA FUNCTION

IN MEMORIAM: MILTON ABRAMOWITZ

PHILIP J. DAVIS, National Bureau of Standards, Washington, D. C.

Many people think that mathematical ideas are static. They think that the ideas originated at some time in the historical past and remain unchanged for all future times. There are good reasons for such a feeling. After all, the formula for the area of a circle was πr^2 in Euclid's day and at the present time is still πr^2. But to one who knows mathematics from the inside, the subject has rather the feeling of a living thing. It grows daily by the accretion of new information, it changes daily by regarding itself and the world from new vantage points, it maintains a regulatory balance by consigning to the oblivion of irrelevancy a

fraction of its past accomplishments.

The purpose of this essay is to illustrate this process of growth. We select one mathematical object, the gamma function, and show how it grew in concept and in content from the time of Euler to the recent mathematical treatise of Bourbaki, and how, in this growth, it partook of the general development of mathematics over the past two and a quarter centuries. Of the so-called "higher mathematical functions," the gamma function is undoubtedly the most fundamental. It is simple enough for juniors in college to meet but deep enough to have called forth contributions from the finest mathematicians. And it is sufficiently compact to allow its profile to be sketched within the space of a brief essay.

The year 1729 saw the birth of the gamma function in a correspondence between a Swiss mathematician in St. Petersburg and a German mathematician in Moscow. The former: Leonhard Euler (1707–1783), then 22 years of age, but to become a prodigious mathematician, the greatest of the 18th century. The latter: Christian Goldbach (1690–1764), a savant, a man of many talents and in correspondence with the leading thinkers of the day. As a mathematician he was something of a dilettante, yet he was a man who bequeathed to the future a problem in the theory of numbers so easy to state and so difficult to prove that even to this day it remains on the mathematical horizon as a challenge.

The birth of the gamma function was due to the merging of several mathematical streams. The first was that of interpolation theory, a very practical subject largely the product of English mathematicians of the 17th century but which all mathematicians enjoyed dipping into from time to time. The second stream was that of the integral calculus and of the systematic building up of the formulas of indefinite integration, a process which had been going on steadily for many years. A certain ostensibly simple problem of interpolation arose and was bandied about unsuccessfully by Goldbach and by Daniel Bernoulli (1700–1784) and even earlier by James Stirling (1692–1770). The problem was posed to Euler. Euler announced his solution to Goldbach in two letters which were to be the beginning of an extensive correspondence which lasted the duration of Goldbach's life. The first letter dated October 13, 1729 dealt with the interpolation problem, while the second dated January 8, 1730 dealt with integration and tied the two together. Euler wrote Goldbach the merest outline, but within a year he published all the details in an article *De progressionibus transcendentibus seu quarum termini generales algebraice dari nequeunt*. This article can now be found reprinted in Volume I_{14} of Euler's *Opera Omnia*.

Since the interpolation problem is the easier one, let us begin with it. One of the simplest sequences of integers which leads to an interesting theory is 1, $1+2$, $1+2+3$, $1+2+3+4$, $\cdots$. These are the triangular numbers, so called because they represent the number of objects which can be placed in a triangular array of various sizes. Call the nth one T_n. There is a formula for T_n which is learned in school algebra: $T_n=\frac{1}{2}n(n+1)$.

What, precisely, does this formula accomplish? In the first place, it simplifies

computation by reducing a large number of additions to three fixed operations: one of addition, one of multiplication, and one of division. Thus, instead of adding the first hundred integers to obtain T_{100}, we can compute $T_{100}=\frac{1}{2}(100)(100+1)=5050$. Secondly, even though it doesn't make literal sense to ask for, say, the sum of the first $5\frac{1}{2}$ integers, the formula for T_n produces an answer to this. For whatever it is worth, the formula yields $T_{5\frac{1}{2}}=\frac{1}{2}(5\frac{1}{2})(5\frac{1}{2}+1)=17\frac{7}{8}$. In this way, the formula extends the scope of the original problem to values of the variable other than those for which it was originally defined and solves the problem of interpolating between the known elementary values.

This type of question, one which asks for an extension of meaning, cropped up frequently in the 17th and 18th centuries. Consider, for instance, the algebra of exponents. The quantity a^m is defined initially as the product of m successive a's. This definition has meaning when m is a positive integer, but what would $a^{5\frac{1}{2}}$ be? The product of $5\frac{1}{2}$ successive a's? The mysterious definitions $a^0=1$, $a^{m/n}=\sqrt[n]{a^m}$, $a^{-m}=1/a^m$ which solve this enigma and which are employed so fruitfully in algebra were written down explicitly for the first time by Newton in 1676. They are justified by a utility which derives from the fact that the definition leads to continuous exponential functions and that the law of exponents $a^m\cdot a^n=a^{m+n}$ becomes meaningful for all exponents whether positive integers or not.

Other problems of this type proved harder. Thus, Leibnitz introduced the notation d^n for the nth iterate of the operation of differentiation. Moreover, he identified d^{-1} with $\int$ and d^{-n} with the iterated integral. Then he tried to breathe some sense into the symbol d^n when n is any real value whatever. What, indeed, is the $5\frac{1}{2}$th derivative of a function? This question had to wait almost two centuries for a satisfactory answer.

THE FACTORIALS

n:	1	2	3	4	5	6	7	8	$\cdots$
$n!$:	1	2	6	24	120	720	5040	40,320	$\cdots$

FIG. 1

INTELLIGENCE TEST

Question: What number should be inserted in the lower line half way between the upper 5 and 6?

Euler's Answer: $287.8852\cdots$. Hadamard's Answer: $280.3002\cdots$.

But to return to our sequence of triangular numbers. If we change the plus signs to multiplication signs we obtain a new sequence: $1, 1\cdot 2, 1\cdot 2\cdot 3, \cdots$. This is the sequence of factorials. The factorials are usually abbreviated $1!, 2!, 3!, \cdots$ and the first five are 1, 2, 6, 24, 120. They grow in size very rapidly. The number 100! if written out in full would have 158 digits. By contrast, $T_{100}=5050$ has a

mere four digits. Factorials are omnipresent in mathematics; one can hardly open a page of mathematical analysis without finding it strewn with them. This being the case, is it possible to obtain an easy formula for computing the factorials? And is it possible to interpolate between the factorials? What should $5\frac{1}{2}!$ be? (See Fig. 1.) This is the interpolation problem which led to the gamma function, the interpolation problem of Stirling, of Bernoulli, and of Goldbach. As we know, these two problems are related, for when one has a formula there is the possibility of inserting intermediate values into it. And now comes the surprising thing. There is no, in fact there can be, no formula for the factorials which is of the simple type found for T_n. This is implicit in the very title Euler chose for his article. Translate the Latin and we have *On transcendental progressions whose general term cannot be expressed algebraically.* The solution to factorial interpolation lay deeper than "mere algebra." Infinite processes were required.

In order to appreciate a little better the problem confronting Euler it is useful to skip ahead a bit and formulate it in an up-to-date fashion: find a reasonably simple function which at the integers $1, 2, 3, \cdots$ takes on the factorial values $1, 2, 6, \cdots$. Now today, a function is a relationship between two sets of numbers wherein to a number of one set is assigned a number of the second set. What is stressed is the relationship and not the nature of the rules which serve to determine the relationship. To help students visualize the function concept in its full generality, mathematics instructors are accustomed to draw a curve full of twists and discontinuities. The more of these the more general the function is supposed to be. Given, then, the points $(1,1)$, $(2, 2)$, $(3, 6)$, $(4, 24)$, $\cdots$ and adopting the point of view wherein "function" is what we have just said, the problem of interpolation is one of finding a curve which passes through the given points. This is ridiculously easy to solve. It can be done in an unlimited number of ways. Merely take a pencil and draw some curve—any curve will do—which passes through the points. Such a curve automatically defines a function which solves the interpolation problem. In this way, too free an attitude as to what constitutes a function solves the problem trivially and would enrich mathematics but little. Euler's task was different. In the early 18th century, a function was more or less synonymous with a formula, and by a formula was meant an expression which could be derived from elementary manipulations with addition, subtraction, multiplication, division, powers, roots, exponentials, logarithms, differentiation, integration, infinite series, *i.e.*, one which came from the ordinary processes of mathematical analysis. Such a formula was called an *expressio analytica*, an analytical expression. Euler's task was to find, if he could, an analytical expression arising naturally from the corpus of mathematics which would yield factorials when a positive integer was inserted, but which would still be meaningful for other values of the variable.

It is difficult to chronicle the exact course of scientific discovery. This is particularly true in mathematics where one traditionally omits from articles and books all accounts of false starts, of the initial years of bungling, and where one may develop one's topic forward or backward or sideways in order to heighten

the dramatic effect. As one distinguished mathematician put it, a mathematical result must appear straight from the heavens as a *deus ex machina* for students to verify and accept but not to comprehend. Apparently, Euler, experimenting with infinite products of numbers, chanced to notice that if n is a positive integer,

$$\left[\left(\frac{2}{1}\right)^n \frac{1}{n+1}\right]\left[\left(\frac{3}{2}\right)^n \frac{2}{n+2}\right]\left[\left(\frac{4}{3}\right)^n \frac{3}{n+3}\right]\cdots = n!. \tag{1}$$

Leaving aside all delicate questions as to the convergence of the infinite product, the reader can verify this equation by cancelling out all the common factors which appear in the top and bottom of the left-hand side. Moreover, the left-hand side is defined (at least formally) for all kinds of n other than negative integers. Euler noticed also that when the value $n=\frac{1}{2}$ is inserted, the left-hand side yields (after a bit of manipulation) the famous infinite product of the Englishman John Wallis (1616–1703):

$$\left(\frac{2\cdot 2}{1\cdot 3}\right)\left(\frac{4\cdot 4}{3\cdot 5}\right)\left(\frac{6\cdot 6}{5\cdot 7}\right)\left(\frac{8\cdot 8}{7\cdot 9}\right)\cdots = \pi/2. \tag{2}$$

With this discovery Euler could have stopped. His problem was solved. Indeed, the whole theory of the gamma function can be based on the infinite product (1) which today is written more conventionally as

$$\lim_{m\to\infty} \frac{m!(m+1)^n}{(n+1)(n+2)\cdots(n+m)}. \tag{3}$$

However, he went on. He observed that his product displayed the following curious phenomenon: for some values of n, namely integers, it yielded integers, whereas for another value, namely $n=\frac{1}{2}$, it yielded an expression involving π. Now π meant circles and their quadrature, and quadratures meant integrals, and he was familiar with integrals which exhibited the same phenomenon. It therefore occurred to him to look for a transformation which would allow him to express his product as an integral.

He took up the integral $\int_0^1 x^e(1-x)^n dx$. Special cases of it had already been discussed by Wallis, by Newton, and by Stirling. It was a troublesome integral to handle, for the indefinite integral is not always an elementary function of x. Assuming that n is an integer, but that e is an arbitrary value, Euler expanded $(1-x)^n$ by the binomial theorem, and without difficulty found that

$$\int_0^1 x^e(1-x)^n dx = \frac{1\cdot 2\cdots n}{(e+1)(e+2)\cdots(e+n+1)}. \tag{4}$$

Euler's idea was now to isolate the $1\cdot 2\cdots n$ from the denominator so that he would have an expression for $n!$ as an integral. He proceeds in this way. (Here we follow Euler's own formulation and nomenclature, marking with an * those

formulas which occur in the original paper. Euler wrote a plain $\int$ for $\int_0^1$.) He substituted f/g for e and found

$$(5)\qquad \int_0^1 x^{f/g}(1-x)^n dx = \frac{g^{n+1}}{f+(n+1)g}\cdot\frac{1\cdot 2\cdot\cdots\cdot n}{(f+g)(f+2\cdot g)\cdots(f+n\cdot g)}.$$

And so,

$$(6)^*\qquad \frac{1\cdot 2\cdots n}{(f+g)(f+2\cdot g)\cdots(f+n\cdot g)} = \frac{f+(n+1)g}{g^{n+1}}\int x^{f/g}dx(1-x)^n.$$

He observed that he could isolate the $1\cdot 2\cdots n$ if he set $f=1$ and $g=0$ in the left-hand member, but that if he did so, he would obtain on the right an indeterminate form which he writes quaintly as

$$(7)^*\qquad \int\frac{x^{1/0}dx(1-x)^n}{0^{n+1}}.$$

He now proceeded to find the value of the expression (7)*. He first made the substitution $x^{g/(f+g)}$ in place of x. This gave him

$$(8)^*\qquad \frac{g}{f+g}\,x^{-f/(g+f)}dx$$

in place of dx and hence, the right-hand member of (6)* becomes

$$(9)^*\qquad \frac{f+(n+1)g}{g^{n+1}}\int\frac{g}{f+g}\,dx(1-x^{g/(f+g)})^n.$$

Once again, Euler made a trial setting of $f=1$, $g=0$ having presumably reduced this integral first to

$$(10)\qquad \frac{f+(n+1)g}{(f+g)^{n+1}}\int_0^1\left(\frac{1-x^{g/(f+g)}}{g/(f+g)}\right)^n dx,$$

and this yielded the indeterminate

$$(11)^*\qquad \int dx\frac{(1-x^0)^n}{0^n}.$$

He now considered the related expression $(1-x^z)/z$, for vanishing z. He differentiated the numerator and denominator, as he says, by a known (l'Hospital's) rule and obtained

$$(12)^*\qquad \frac{-x^z dz lx}{dz}\qquad (lx=\log x),$$

which for $z=0$ produced $-lx$. Thus,

$$(13)^*\qquad (1-x^0)/0 = -lx$$

and

$$(1 - x^0)^n/0^n = (-lx)^n. \tag{14*}$$

He therefore concluded that

$$n! = \int_0^1 (-\log x)^n dx. \tag{15}$$

This gave him what he wanted, an expression for $n!$ as an integral wherein values other than positive integers may be substituted. The reader is encouraged to formulate his own criticism of Euler's derivation.

Students in advanced calculus generally meet Euler's integral first in the form

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt, \qquad e = 2.71828 \cdots. \tag{16}$$

This modification of the integral (15) as well as the Greek Γ is due to Adrien Marie Legendre (1752–1833). Legendre calls the integral (4) with which Euler started his derivation the first Eulerian integral and (15) the second Eulerian integral. The first Eulerian integral is currently known as the Beta function and is now conventionally written

$$B(m, n) = \int_0^1 x^{m-1}(1 - x)^{n-1}dx. \tag{17}$$

With the tools available in advanced calculus, it is readily established (how easily the great achievements of the past seem to be comprehended and duplicated!) that the integral possesses meaning when $x>0$ and thus yields a certain function $\Gamma(x)$ defined for these values. Moreover,

$$\Gamma(n + 1) = n! \tag{18}$$

whenever n is a positive integer.* It is further established that for all $x>0$

$$x\Gamma(x) = \Gamma(x + 1). \tag{19}$$

This is the so-called recurrence relation for the gamma function and in the years following Euler it plays, as we shall see, an increasingly important role in its theory. These facts, plus perhaps the relationship between Euler's two types of integrals

$$B(m, n) = \Gamma(m)\Gamma(n)/\Gamma(m + n) \tag{20}$$

and the all important Stirling formula

* Legendre's notation shifts the argument. Gauss introduced a notation $\pi(x)$ free of this defect. Legendre's notation won out, but continues to plague many people. The notations Γ, π, and ! can all be found today.

$$\Gamma(x) \sim e^{-x}x^{x-1/2}\sqrt{(2\pi)}, \tag{21}$$

which gives us a relatively simple approximate expression for $\Gamma(x)$ when x is large, are about all that advanced calculus students learn of the gamma function. Chronologically speaking, this puts them at about the year 1750. The play has hardly begun.

Just as the simple desire to extend factorials to values in between the integers led to the discovery of the gamma function, the desire to extend it to negative values and to complex values led to its further development and to a more profound interpretation. Naive questioning, uninhibited play with symbols may have been at the very bottom of it. What is the value of $(-5\frac{1}{2})!$? What is the value of $\sqrt{(-1)}!$? In the early years of the 19th century, the action broadened and moved into the complex plane (the set of all numbers of the form $x+iy$, where $i=\sqrt{(-1)}$) and there it became part of the general development of the theory of functions of a complex variable that was to form one of the major chapters in mathematics. The move to the complex plane was initiated by Karl Friedrich Gauss (1777–1855), who began with Euler's product as his starting point. Many famous names are now involved and not just one stage of action but many stages. It would take too long to record and describe each forward step taken. We shall have to be content with a broader picture.

Three important facts were now known: Euler's integral, Euler's product, and the functional or recurrence relationship $x\Gamma(x)=\Gamma(x+1)$, $x>0$. This last is the generalization of the obvious arithmetic fact that for positive integers, $(n+1)n!=(n+1)!$ It is a particularly useful relationship inasmuch as it enables us by applying it over and over again to reduce the problem of evaluating a factorial of an arbitrary real number whole or otherwise to the problem of evaluating the factorial of an appropriate number lying between 0 and 1. Thus, if we write $n=4\frac{1}{2}$ in the above formula we obtain $(4\frac{1}{2}+1)!=5\frac{1}{2}(4\frac{1}{2})!$ If we could only find out what $(4\frac{1}{2})!$ is, then we would know that $(5\frac{1}{2})!$ is. This process of reduction to lower numbers can be kept up and yields

$$(5\tfrac{1}{2})! = (3/2)(5/2)(7/2)(9/2)(11/2)(1/2)! \tag{22}$$

and since we have $(\frac{1}{2})!=\frac{1}{2}\sqrt{\pi}$ from (1) and (2), we can now compute our answer. Such a device is obviously very important for anyone who must do calculations with the gamma function. Other information is forthcoming from the recurrence relationship. Though the formula $(n+1)n!=(n+1)!$ as a condensation of the arithmetic identity $(n+1)\cdot 1\cdot 2 \cdots n=1\cdot 2 \cdots n\cdot(n+1)$ makes sense only for $n=1, 2$, etc., blind insertions of other values produce interesting things. Thus, inserting $n=0$, we obtain $0!=1$. Inserting successively $n=-5\frac{1}{2}$, $n=-4\frac{1}{2}, \cdots$ and reducing upwards, we discover

$$(-5\tfrac{1}{2})! = (2/1)(-2/1)(-2/3)(-2/5)(-2/7)(-2/9)(1/2)! \tag{23}$$

Since we already know what $(\frac{1}{2})!$ is, we can compute $(-5\frac{1}{2})!$ In this way the recurrence relationship enables us to compute the values of factorials of negative

numbers.

Turning now to Euler's integral, it can be shown that for values of the variable less than 0, the usual theorems of analysis do not suffice to assign a meaning to the integral, for it is divergent. On the other hand, it is meaningful and yields a value if one substitutes for x any complex number of the form $a+bi$ where $a>0$. With such substitutions the integral therefore yields a complex-valued function which is defined for all complex numbers in the right-half of the complex plane and which coincides with the ordinary gamma function for real values. Euler's product is even stronger. With the exception of $0, -1, -2, \cdots$ any complex number whatever can be inserted for the variable and the infinite product will converge, yielding a value. And so it appears that we have at our disposal a number of methods, conceptually and operationally different for extending the domain of definition of the gamma function. Do these different methods yield the same result? They do. But why?

The answer is to be found in the notion of an analytic function. This is the focal point of the theory of functions of a complex variable and an outgrowth of the older notion of an analytical expression. As we have hinted, earlier mathematics was vague about this notion, meaning by it a function which arose in a natural way in mathematical analysis. When later it was discovered by J. B. J. Fourier (1768–1830) that functions of wide generality and functions with unpleasant characteristics could be produced by the infinite superposition of ordinary sines and cosines, it became clear that the criterion of "arising in a natural way" would have to be dropped. The discovery simultaneously forced a broadening of the idea of a function and a narrowing of what was meant by an analytic function.

Analytic functions are not so arbitrary in their behavior. On the contrary, they possess strong internal ties. Defined very precisely as functions which possess a complex derivative or equivalently as functions which possess power series expansions $a_0+a_1(z-z_0)+a_2(z-z_0)^2+\cdots$ they exhibit the remarkable phenomenon of "action at a distance." This means that the behavior of an analytic function over any interval no matter how small is sufficient to determine completely its behavior everywhere else; its potential range of definition and its values are theoretically obtainable from this information. Analytic functions, moreover, obey the principle of the permanence of functional relationships; if an analytic function satisfies in some portions of its region of definition a certain functional relationship, then it must do so wherever it is defined. Conversely, such a relationship may be employed to extend its definition to unknown regions. Our understanding of the process of analytic continuation, as this phenomenon is known, is based upon the work of Bernhard Riemann (1826–1866) and Karl Weierstrass (1815–1897). The complex-valued function which results from the substitution of complex numbers into Euler's integral is an analytic function. The function which emerges from Euler's product is an analytic function. The recurrence relationship for the gamma function if satisfied in some region must be satisfied in any other region to which the function

can be "continued" analytically and indeed may be employed to effect such extensions. All portions of the complex plane, with the exception of the values $0, -1, -2, \cdots$ are accessible to the complex gamma function which has become the unique, analytic extension to complex values of Euler's integral (see Fig. 3).

THE GAMMA FUNCTION

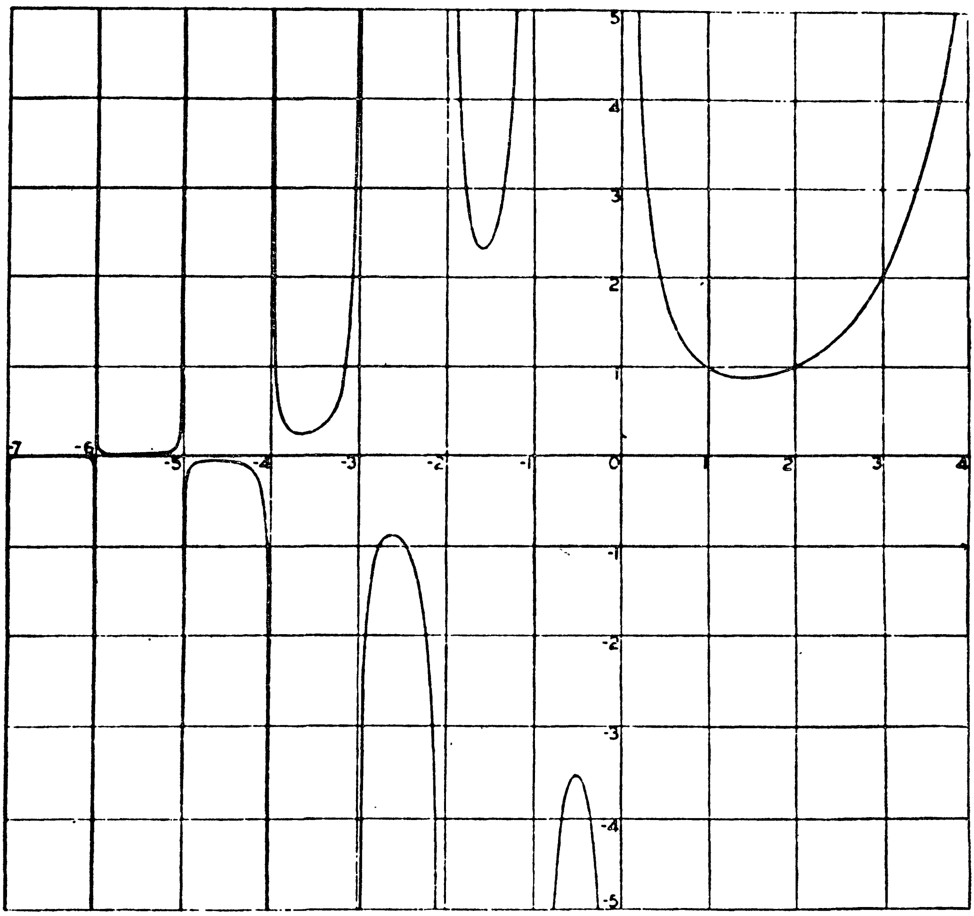

FIG. 2*

To understand why there should be excluded points observe that $\Gamma(x) = \Gamma(x+1)/x$, and as x approaches 0, we obtain $\Gamma(0) = 1/0$. This is $+\infty$ or $-\infty$ depending whether 0 is approached through positive or negative values. The

* From: H. T. Davis, Tables of the Higher Mathematical Functions, vol. I, Bloomington, Indiana, 1933.

functional equation (19) then, induces this behavior over and over again at each of the negative integers. The (real) gamma function is comprised of an infinite number of disconnected portions opening up and down alternately. The portions corresponding to negative values are each squeezed into an infinite strip one unit in width, but the major portion which corresponds to positive x and which contains the factorials is of infinite width (see Fig. 2). Thus, there are excluded points for the gamma function at which it exhibits from the ordinary (real variable) point of view a somewhat unpleasant and capricious behavior.

THE ABSOLUTE VALUE OF THE COMPLEX GAMMA FUNCTION, EXHIBITING THE POLES AT THE NEGATIVE INTEGERS

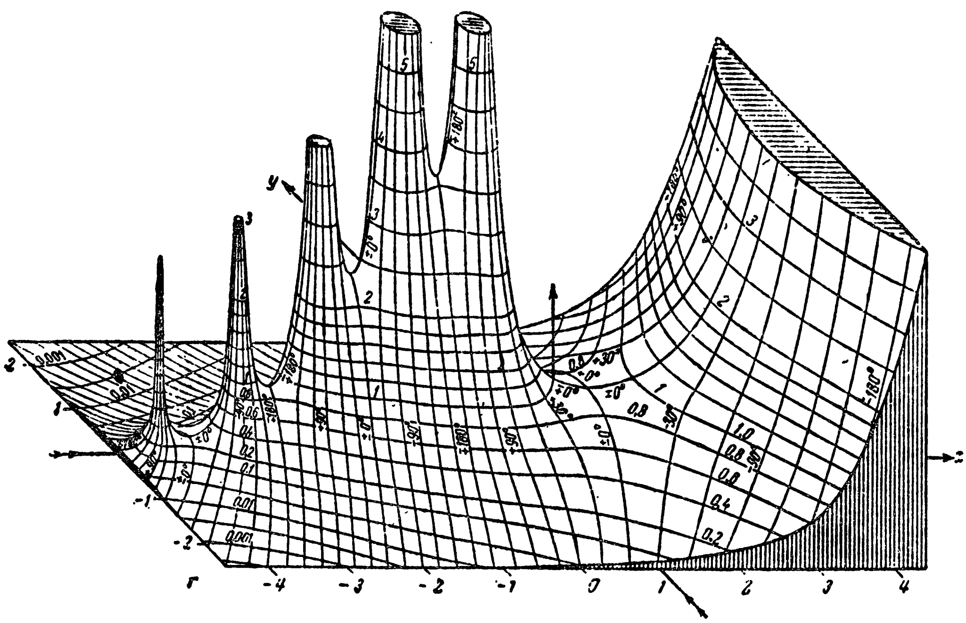

FIG. 3*

But from the complex point of view, these points of singular behavior (singular in the sense of Sherlock Holmes) merit special study and become an important part of the story. In pictures of the complex gamma function they show up as an infinite row of "stalagmites," each of infinite height (the ones in the figure are truncated out of necessity) which become more and more needlelike as they go out to infinity (see Fig. 3). They are known as poles. Poles are points where the function has an infinite behavior of especially simple type, a behavior which is akin to that of such simple functions as the hyperbola $y=1/x$ at $x=0$ or of $y=\tan x$ at $x=\pi/2$. The theory of analytic functions is especially interested

* From: E. Jahnke and F. Emde, Tafeln höherer Funktionen, 4th ed., Leipzig, 1948.

in singular behavior, and devotes much space to the study of the singularities. Analytic functions possess many types of singularity but those with only poles are known as meromorphic. There are also functions which are lucky enough to possess no singularities for finite arguments. Such functions form an elite and are known as entire functions. They are akin to polynomials while the meromorphic functions are akin to the ratio of polynomials. The gamma function is meromorphic. Its reciprocal, $1/\Gamma(x)$, has on the contrary no excluded points. There is no trouble anywhere. At the points $0, -1, -2, \cdots$ it merely becomes zero. And the zero value which occurs an infinity of times, is strongly reminiscent of the sine.

In the wake of the extension to the complex many remarkable identities emerge, and though some of them can and were obtained without reference to complex variables, they acquire a far deeper and richer meaning when regarded from the extended point of view. There is the reflection formula of Euler

$$\Gamma(z)\Gamma(1-z) = \pi/\sin \pi z. \tag{24}$$

It is readily shown, using the recurrence relation of the gamma function, that the product $\Gamma(z)\Gamma(1-z)$ is a periodic function of period 2; but despite the fact that $\sin \pi z$ is one of the simplest periodic functions, who could have anticipated the relationship (24)? What, after all, does trigonometry have to do with the sequence 1, 2, 6, 24 which started the whole discussion? Here is a fine example of the delicate patterns which make the mathematics of the period so magical. From the complex point of view, a partial reason for the identity lies in the similarity between zeros of the sine and the poles of the gamma function.

There is the duplication formula

$$\Gamma(2z) = (2\pi)^{-1/2}2^{2z-1/2}\Gamma(z)\Gamma(z+\tfrac{1}{2}) \tag{25}$$

discovered by Legendre and extended by Gauss in his researches on the hypergeometric function to the multiplication formula

$$\Gamma(nz) = (2\pi)^{1/2(1-n)}n^{nz-1/2}\Gamma(z)\Gamma\left(z+\frac{1}{n}\right)\Gamma\left(z+\frac{2}{n}\right)\cdot\cdots\cdot\Gamma\left(\frac{z+n-1}{n}\right). \tag{26}$$

There are pretty formulas for the derivatives of the gamma function such as

$$d^2 \log \Gamma(z)/dz^2 = \frac{1}{z^2} + \frac{1}{(z+1)^2} + \frac{1}{(z+2)^2} + \cdots. \tag{27}$$

This is an example of a type of infinite series out of which G. Mittag-Leffler (1846–1927) later created his theory of partial fraction developments of meromorphic functions. There is the intimate relationship between the gamma function and the zeta function which has been of fundamental importance in studying the distribution of the prime numbers,

$$\zeta(z) = \zeta(1 - z)\Gamma(1 - z)2^z\pi^{z-1}\sin\tfrac{1}{2}\pi z, \tag{28}$$

where

$$\zeta(z) = 1 + \frac{1}{2^z} + \frac{1}{3^z} + \cdots. \tag{29}$$

This formula has some interesting history related to it. It was first proved by Riemann in 1859 and was conventionally attributed to him. Yet in 1894 it was discovered that a modified version of the identity appears in some work of Euler which had been done in 1749. Euler did not claim to have proved the formula. However, he "verified" it for integers, for $\frac{1}{2}$, and for 3/2. The verification for $\frac{1}{2}$ is by direct substitution, but for all the other values, Euler works with divergent infinite series. This was more than 100 years in advance of a firm theory of such series, but with unerring intuition, he proceeded to sum them by what is now called the method of Abel summation. The case 3/2 is even more interesting. There, invoking both divergent series and numerical evaluation, he came out with numerical agreement to 5 decimal places! All this work convinced him of the truth of his identity. Rigorous modern proofs do not require the theory of divergent series, but the notions of analytic continuation are crucial.

In view of the essential unity of the gamma function over the whole complex plane it is theoretically and aesthetically important to have a formula which works for all complex numbers. One such formula was supplied in 1848 by F. W. Newman:

$$1/\Gamma(z) = ze^{\gamma z}\{(1+z)e^{-z}\}\{(1+z/2)e^{-z/2}\}\cdots, \quad \text{where } \gamma = .57721\,56649\cdots. \tag{30}$$

This formula is essentially a factorization of $1/\Gamma(z)$ and is much the same as a factorization of polynomials. It exhibits clearly where the function vanishes. Setting each factor equal to zero we find that $1/\Gamma(z)$ is zero for $z=0$, $z=-1$, $z=-2, \cdots$. In the hands of Weierstrass, it became the starting point of his particular discussion of the gamma function. Weierstrass was interested in how functions other than polynomials may be factored. A number of isolated factorizations were then known. Newman's formula (30) and the older factorization of the sine

$$\sin \pi z = \pi z(1 - z^2)\left(1 - \frac{z^2}{4}\right)\left(1 - \frac{z^2}{9}\right)\cdots \tag{31}$$

are among them. The factorization of polynomials is largely an algebraic matter but the extension to functions such as the sine which have an infinity of roots required the systematic building up of a theory of infinite products. In 1876 Weierstrass succeeded in producing an extensive theory of factorizations which included as special cases these well-known infinite products, as well as certain doubly periodic functions.

In addition to showing the roots of $1/\Gamma(z)$, formula (30) does much more.

It shows immediately that the reciprocal of the gamma function is a much less difficult function to deal with than the gamma function itself. It is an entire function, that is, one of those distinguished functions which possesses no singularities whatever for finite arguments. Weierstrass was so struck by the advantages to be gained by starting with $1/\Gamma(z)$ that he introduced a special notation for it. He called $1/\Gamma(u+1)$ the *factorielle* of u and wrote $Fc(u)$.

The theory of functions of a complex variable unifies a hotch-potch of curves and a patchwork of methods. Within this theory, with its highly developed studies of infinite series of various types, was brought to fruition Stirling's unsuccessful attempts at solving the interpolation problem for the factorials. Stirling had done considerable work with infinite series of the form

$$A+Bz+Cz(z-1)+Dz(z-1)(z-2)+\cdots.$$

This series is particularly useful for fitting polynomials to values given at the integers $z=0, 1, 2, \cdots$. The method of finding the coefficients $A, B, C, \cdots$ was well known. But when an infinite amount of fitting is required, much more than simple formal work is needed, for we are then dealing with a bona fide infinite series whose convergence must be investigated. Starting from the series $1, 2, 6, 24, \cdots$, Stirling found interpolating polynomials via the above series. The resultant infinite series is divergent. The factorials grow too rapidly in size. Stirling realized this and put out the suggestion that if perhaps one started with the logarithms of the factorials instead of the factorials themselves the size might be cut down sufficiently for one to do something. There the matter rested until 1900 when Charles Hermite (1822–1901) wrote down the Stirling series for $\log \Gamma(1+z)$:

$$\log \Gamma(1+z) = \frac{z(z-1)}{1\cdot 2}\log 2 + \frac{z(z-1)(z-2)}{1\cdot 2\cdot 3}(\log 3 - 2\log 2) + \cdots \tag{32}$$

and showed that this identity is valid whenever z is a complex number of the form $a+ib$ with $a>0$. The identity itself could have been written down by Stirling, but the proof would have been another matter. An even simpler starting point is the function $\psi(z)=(d/dz)\log\Gamma(z)$, now known as the digamma or psi function. This leads to the Stirling series

$$\begin{aligned}&\frac{d}{dz}\log\Gamma(z)\\ &\qquad = -\gamma + (z-1) - \frac{(z-1)(z-2)}{2\cdot 2!} + \frac{(z-1)(z-2)(z-3)}{3\cdot 3!}\cdots,\end{aligned}\tag{33}$$

which in 1847 was proved convergent for $a>0$ by M. A. Stern, a teacher of Riemann. All these matters are today special cases of the extensive theory of the convergence of interpolation series.

Functions are the building blocks of mathematical analysis. In the 18th and 19th centuries mathematicians devoted much time and loving care to develop-

ing the properties and interrelationships between special functions. Powers, roots, algebraic functions, trigonometric functions, exponential functions, logarithmic functions, the gamma function, the beta function, the hypergeometric function, the elliptic functions, the theta function, the Bessel function, the Matheiu function, the Weber function, Struve function, the Airy function, Lamé functions, literally hundreds of special functions were singled out for scrutiny and their main features were drawn. This is an art which is not much cultivated these days. Times have changed and emphasis has shifted. Mathematicians on the whole prefer more abstract fare. Large classes of functions are studied instead of individual ones. Sociology has replaced biography. The field of special functions, as it is now known, is left largely to a small but ardent group of enthusiasts plus those whose work in physics or engineering confronts them directly with the necessity of dealing with such matters.

The early 1950's saw the publication of some very extensive computations of the gamma function in the complex plane. Led off in 1950 by a six-place table computed in England, it was followed in Russia by the publication of a very extensive six-place table. This in turn was followed in 1954 by the publication by the National Bureau of Standards in Washington of a twelve-place table. Other publications of the complex gamma function and related functions have appeared in this country, in England, and in Japan. In the past, the major computations of the gamma function had been confined to real values. Two fine tables, one by Gauss in 1813 and one by Legendre in 1825, seemed to answer the mathematical needs of a century. Modern technology had also caught up with the gamma function. The tables of the 1800's were computed laboriously by hand, and the recent ones by electronic digital computers.

But what touched off this spate of computational activity? Until the initial labors of H. T. Davis of Indiana University in the early 1930's, the complex values of the gamma function had hardly been touched. It was one of those curious turns of events wherein the complex gamma function appeared in the solution of various theoretical problems of atomic and nuclear theory. For instance, the radial wave functions for positive energy states in a Coulomb field leads to a differential equation whose solution involves the complex gamma function. The complex gamma function enters into formulas for the scattering of charged particles, for the nuclear forces between protons, in Fermi's approximate formula for the probability of β-radiation, and in many other places. The importance of these problems to physicists has had the side effect of computational mathematics finally catching up with two and a quarter centuries of theoretical development.

As analysis grew, both creating special functions and delineating wide classes of functions, various classifications were used in order to organize them for purposes of convenient study. The earlier mathematicians organized functions from without, operationally, asking what operations of arithmetic or calculus had to be performed in order to achieve them. Today, there is a much greater tendency to look at functions from within, organically, considering their construction as

achieved and asking what geometrical characteristics they possess. In the earlier classification we have at the lowest and most accessible level, powers, roots, and all that could be concocted from them by ordinary algebraic manipulation. These came to be known as algebraic functions. The calculus, with its characteristic operation of taking limits, introduced logarithms and exponentials, the latter encompassing, as Euler showed, the sines and cosines of trigonometry which had been available from earlier periods of discovery. There is an impassable wall between the algebraic functions and the new limit-derived ones. This wall consists in the fact that try as one might to construct, say, a trigonometric function out of the finite material of algebra, one cannot succeed. In more technical language, the algebraic functions are closed with respect to the processes of algebra, and the trigonometric functions are forever beyond its pale. (By way of a simple analogy: the even integers are closed with respect to the operations of addition, subtraction, and multiplication; you cannot produce an odd integer from the set of even integers using these tools.) This led to the concept of transcendental functions. These are functions which are not algebraic. The transcendental functions count among their members, the trigonometric functions, the logarithms, the exponentials, the elliptic functions, in short, practically all the special functions which had been singled out for special study. But such an indescriminate dumping produced too large a class to handle. The transcendentals had to be split further for convenience. A major tool of analysis is the differential equation, expressing the relationship between a function and its rate of growth. It was found that some functions, say the trigonometric functions, although they are transcendental and do not therefore satisfy an algebraic equation, nonetheless satisfy a differential equation whose coefficients are algebraic. The solutions of algebraic differential equations are an extensive though not all-encompassing class of transcendental functions. They count among their members a good many of the special functions which arise in mathematical physics.

Where does the gamma function fit into this? It is not an algebraic function. This was recognized early. It is a transcendental function. But for a long while it was an open question whether the gamma function satisfied an algebraic differential equation. The question was settled negatively in 1887 by O. Hölder (1859–1937). It does not. It is of a higher order of transcendency. It is a so-called transcendentally transcendent function, unreachable by solving algebraic equations, and equally unreachable by solving algebraic differential equations. The subject has interested many people through the years and in 1925 Alexander Ostrowski, now Professor Emeritus of the University of Basel, Switzerland, gave an alternate proof of Hölder's theorem.

Problems of classification are extremely difficult to handle. Consider, for instance, the following: Can the equation x^7+8x+1 be solved with radicals? Is π transcendental? Can $\int dx/\sqrt{(x^3+1)}$ be found in terms of specified elementary functions? Can the differential equation $dy/dx=(1/x)+(1/y)$ be resolved with quadratures? The general problems of which these are representatives are

even today far from solved and this despite famous theories such as Galois Theory, Lie theory, theory of Abelian integrals which have derived from such simple questions. Each individual problem may be a one-shot affair to be solved by individual methods involving incredible ingenuity.

HADAMARD'S FACTORIAL FUNCTION

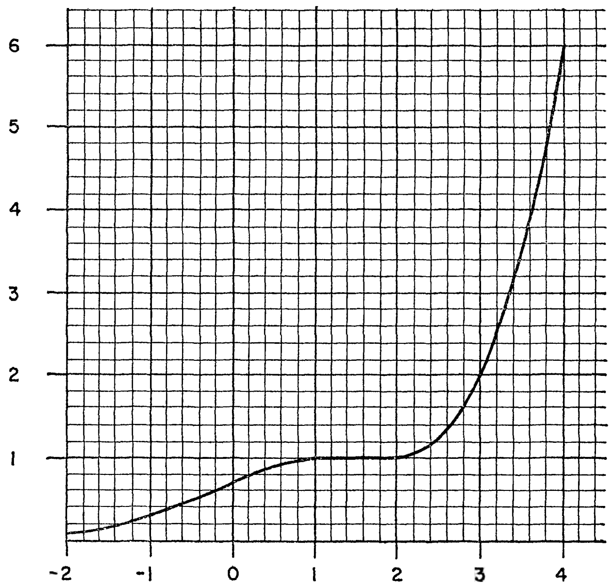

FIG. 4

There are infinitely many functions which produce factorials. The function

$$F(x) = (1/\Gamma(1-x))\,(d/dx)\log\{\Gamma((1-x)/2)/\Gamma(1-x/2)\}$$

is an entire analytic function which coincides with the gamma function at the positive integers. It satisfies the functional equation $F(x+1)=xF(x)+(1/\Gamma(1-x))$.

We return once again to our interpolation problem. We have shown how, strictly speaking, there are an unlimited number of solutions to this problem. To drive this point home, we might mention a curious solution given in 1894 by Jacques Hadamard (1865–). Hadamard found a relatively simple formula involving the gamma function which also produces factorial values at the positive integers. (See Figs. 1 and 4.) But Hadamard's function

$$(34)\qquad y = \frac{1}{\Gamma(1-x)}\frac{d}{dx}\log\left[\Gamma\left(\frac{1-x}{2}\right)\Big/\Gamma\left(1-\frac{x}{2}\right)\right],$$

in strong contrast to the gamma function itself, possesses no singularities anywhere in the finite complex plane. It is an entire analytic solution to the interpolation problem and hence, from the function theoretic point of view, is a simpler solution. In view of all this ambiguity, why then should Euler's solution

be considered the solution par excellence?

From the point of view of integrals, the answer is clear. Euler's integral appears everywhere and is inextricably bound to a host of special functions. Its frequency and simplicity make it fundamental. When the chips are down, it is the very form of the integral and of its modifications which lend it utility and importance. For the interpolatory point of view, we can make no such claim. We must take a deeper look at the gamma function and show that of all the solutions of the interpolation problem, it, in some sense, is the simplest. This is partially a matter of mathematical aesthetics.

A PSEUDOGAMMA FUNCTION

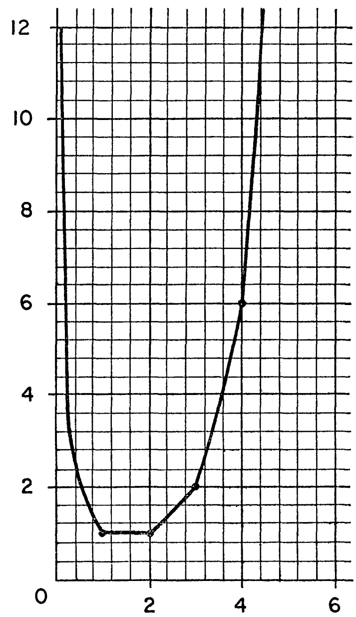

FIG. 5

The function illustrated produces factorials, satisfies the functional equation of the gamma function, and is convex.

We have already observed that Euler's integral satisfies the fundamental recurrence equation, $x\Gamma(x)=\Gamma(x+1)$, and that this equation enables us to compute all the real values of the gamma function from knowledge merely of its values in the interval from 0 to 1. Since the solution to the interpolation problem is not determined uniquely, it makes sense to add to the problem more conditions and to inquire whether the augmented problem then possesses a unique solution. If it does, we will hope that the solution coincides with Euler's. The recurrence relationship is a natural condition to add. If we do so, we find that the gamma function is again not the only function which satisfies this recurrence

relation and produces factorials. One may easily construct a "pseudo" gamma function $\Gamma_S(x)$ by defining it between, say, 1 and 2 in any way at all (subject only to $\Gamma_S(1)=1$, $\Gamma_S(2)=1$), and allowing the recurrence relationship to extend its values everywhere else.

If, for instance, we let $\Gamma_S(x)$ be 1 everywhere between 1 and 2, the recurrence relation leads us to the function (see Fig. 5).

$$\begin{aligned} \Gamma_S(x) &= 1/x & 0 < x \leqq 1; \\ \Gamma_S(x) &= 1, & 1 \leqq x \leqq 2; \\ \Gamma_S(x) &= x - 1, & 2 \leqq x \leqq 3; \\ \Gamma_S(x) &= (x-1)(x-2), & 3 \leqq x \leqq 4; \cdots . \end{aligned} \tag{35}$$

We might end up with a fairly weird result, depending upon what we start with. Even if we require the final result to be an analytic function, there are ways of doing it. For instance, take any function which is both analytic and periodic with period 1. Call it $p(x)$. Make sure that $p(1)=1$. The function $1+\sin 2\pi x$ will do for $p(x)$. Now multiply the ordinary gamma function $\Gamma(x)$ by $p(x)$ and the result $\Gamma(x)p(x)$ will be a "pseudo" gamma function which is analytic, satisfies the recurrence relation, and produces factorials! Thus, we still do not have enough conditions. We must augment the problem again. But what to add?

By the middle of the 19th century it was recognized that Euler's gamma function was the only continuous function which satisfied simultaneously the recurrence relationship, the reflection formula and the multiplication formula. Weierstrass later showed that the gamma function was the only continuous solution of the recurrence relationship for which $\{\Gamma(x+n)\}/\{(n-1)n^x\}\to 1$ for all x. These conditions added to the interpolation problem will serve to produce a unique solution and one which coincides with Euler's. But they appear too heavy and too much like Monday morning quarterbacking. That is to say, the added conditions are hardly "natural" for they are tied in with the deeper analytical properties of the gamma function. The search went on.

Aesthetic conditions were not to be found in the older, analytic considerations, but in a newer, inner, organic approach to function theory which was developing at the turn of the century. Backed up by Cantor's set theory and an emerging theory of topology, the new function theory looked not so much at equations and identities as at the fundamental geometrical properties. The desired condition was found in notions of convexity. A curve is convex if the following is true of it: take any two points on the curve and join them by a straight line; then the portion of the curve between the points lies below the line. A convex curve does not wiggle; it cannot look like a camel's back. At the turn of the century, convexity was in the mathematical air. It was found to be intrinsic to many diverse phenomena. Over the period of a generation, it was sought out, it was generalized, it was abstracted, it was investigated for its own sake, it was applied. Called to attention by the work of H. Brunn in 1887

and of H. Minkowski in 1903 on convex bodies and given an independent interest in 1906 by the work of J. L. W. V. Jensen, the idea of convexity spread and established itself in mean value theory, in potential theory, in topology, and most recently in game theory and linear programming. At the turn of the century then, an application of convexity to the gamma function would have been natural and in order.

The individual curves which make up the gamma function are all convex. A glance at Figure 2 shows this to be true. If, as in the previous paragraph, a pseudogamma function satisfying the recurrence formula were produced by introducing the ripple $1+\sin 2\pi x$ as a factor, it would no longer be true. It must have occurred to many mathematicians to find out whether the gamma function is the only function which yields the factorial values, satisfies the recurrence relation, and is convex downward for $x>0$. Unfortunately, this is not true. Figure 5 shows a pseudogamma function which possesses just these properties. It remained until 1922 to discover a correct formulation. But it was not at too far a distance. The gamma function is not only convex, it is also logarithmically convex. That is to say, the graph of $\log \Gamma(x)$ is also convex down for $x>0$. This fact is implicit in formula (27). Logarithmic convexity is a stronger condition than ordinary convexity for logarithmic convexity implies, but is not implied by, ordinary convexity. Now Harald Bohr and J. Mollerup were able to show the surprising fact that the gamma function is the only function which satisfies the recurrence relationship and is logarithmically convex. The original proof was simplified several years later by Emil Artin, now professor at Princeton University, and the theorem together with Artin's method of proof now constitute the Bohr-Mollerup-Artin theorem. Its precise wording is this:

The Euler gamma function is the only function defined for $x>0$ which is positive, is 1 at $x=1$, satisfies the functional equation $x\Gamma(x)=\Gamma(x+1)$, and is logarithmically convex.

This theorem is at once so striking and so satisfying that the contemporary synod of abstractionists who write mathematical canon under the pen name of N. Bourbaki has adopted it as the starting point for its exposition of the gamma function. The proof: one page; the discovery: 193 years.

There is much that we know about the gamma function. Since Euler's day more than 400 major papers relating to it have been written. But a few things remain that we do not know and that we would like to know. Perhaps the hardest of the unsolved problems deal with questions of rationality and transcendentality. Consider, for instance, the number $\gamma=.57721\cdots$ which appears in formula (30). This is the Euler-Mascheroni constant. Many different expressions can be given for it. Thus,

$$\gamma = -\,d\Gamma(x)/dx\,\big|_{x=1}, \tag{36}$$

$$\gamma = \lim_{n\to\infty}\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right) - \log n. \tag{37}$$

Though the numerical value of γ is known to hundreds of decimal places, it is not known at the time of writing whether γ is or is not a rational number. Another problem of this sort deals with the values of the gamma function itself. Though, curiously enough, the product $\Gamma(1/4)/\sqrt[4]{\pi}$ can be proved to be transcendental, it is not known whether $\Gamma(1/4)$ is even rational.

George Gamow, the distinguished physicist, quotes Laplace as saying that when the known areas of a subject expand, so also do its frontiers. Laplace evidently had in mind the picture of a circle expanding in an infinite plane. Gamow disputes this for physics and has in mind the picture of a circle expanding on a spherical surface. As the circle expands, its boundary first expands, but later contracts. This writer agrees with Gamow as far as mathematics is concerned. Yet the record is this: each generation has found something of interest to say about the gamma function. Perhaps the next generation will also.

The writer wishes to thank Professor C. Truesdell for his helpful comments and criticism and Dr. H. E. Salzer for a number of valuable references.

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LINEAR DIFFERENTIAL OR DIFFERENCE EQUATIONS WITH CONSTANT COEFFICIENTS

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1. Introduction.* Solutions of a system of linear differential or difference equations with *real* constant coefficients a_{ij}, such as

$$(1)\qquad dx_i/dt = \sum_{j=1}^{n} a_{ij}x_j \quad\text{and}\quad x_i(t+h) = \sum_{j=1}^{n} a_{ij}x_j(t),$$

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where h is a positive constant and $i=1, \cdots, n$, are frequently desired. In discussing such systems it is usually convenient to rewrite them in the vector notation

$$d\overline{X}(t)/dt = A\overline{X}(t) \quad \text{and} \quad \overline{X}(t+h) = A\overline{X}(t), \tag{2}$$

where the vector

$$\overline{X}(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

and the element in the ith row and jth column of the n by n square constant matrix A is the coefficient a_{ij}. Here a bar over a capital letter is used to emphasize that a vector is under consideration.

A classical procedure for solving (2) is to first consider the corresponding matrix equations

$$dX/dt = AX \quad \text{and} \quad X(t+h) = AX(t), \tag{3}$$

where $X=X(t)$ is an n by n square matrix $[x_{ij}(t)]$ and then to make an appropriate nonsingular linear transformation

$$X(t) = PY(t), \tag{4}$$

which will reduce the equations of (3) to new equations of the form

$$dY/dt = BY \quad \text{and} \quad Y(t+h) = BY(t), \tag{5}$$

where $B=P^{-1}AP$ and B has the classical Jordan canonical form.

Once the constant matrices P and B have been computed, an explicit formula for a fundamental solution $Y=Y_f(t)$ can be written out in full detail. In terms of this fundamental solution the general solutions of the respective systems (2) can be written in the form

$$\overline{X} = PY_f(t) + \overline{K}_1 \quad \text{and} \quad \overline{X}(t) = PY_f(t) + \overline{K}_2(t), \tag{6}$$

where $\overline{K}_1$ is an arbitrary n-component constant vector and $\overline{K}_2(t)$ is an n-component vector, each component of which is an arbitrary function of t of period h.

In general, matrix P will contain elements which are complex constants. This means transformation (4), leading to the Jordan canonical form, can not in general be carried out in the real domain and this makes it awkward to deduce properties of the real solutions of the systems (1). Furthermore, in applications it is often desirable to have answer (6) in a form free from complex numbers. It is therefore the primary objective of this paper to introduce a new nonsingular *real* transformation which will lead to a *real* canonical form and a solution of relatively simple structure free from possible complex numbers.

2. Proposed real canonical forms. Turnbull and Aitken [1] have shown that any *real* system (3) can be reduced by a suitable *real* nonsingular transforma-

tion (4) to a new system (5), where matrix B has a special diagonal structure

$$B = \begin{bmatrix} B_1 & 0 & \cdots & & 0 \\ 0 & \ddots & & & \vdots \\ \vdots & & B_p & & \vdots \\ \vdots & & & C_1 & \vdots \\ & & & \ddots & 0 \\ 0 & \cdots & & \cdot & C_q \end{bmatrix} = [B_1, \cdots, B_p; C_1, \cdots, C_q]. \tag{7}$$

To describe in detail the structure of a particular square matrix B_i, $(i=1, \cdots, p)$, appearing as a block on the main diagonal in (7), let λ_i be a real root of the characteristic equation $|A-\lambda I|=0$ and let $(\lambda-\lambda_i)^{m_i}$ be an elementary divisor of the λ-matrix $[A-\lambda I]$; then corresponding to this elementary divisor there is a square matrix on the diagonal in (7) of the form

$$B_i = \begin{bmatrix} \lambda_i & 0 & \cdots & & 0 \\ 1 & \ddots & & & \vdots \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & 1 & \lambda_i \end{bmatrix} \qquad (i = 1, \cdots, p),$$

with m_i rows and columns.

On the other hand, if $\lambda_j=\mu_j+i\nu_j$ is a characteristic root, where μ_j and ν_j are real and $\nu_j\neq 0$, then $\bar{\lambda}_j=\mu_j-i\nu_j$ is also a characteristic root and, if $(\lambda-\lambda_j)^{m_j}$ is an elementary divisor, then $(\lambda-\bar{\lambda}_j)^{m_j}$ is also an elementary divisor. Furthermore, corresponding to this pair of elementary divisors in the complex domain, there is a real square matrix on the diagonal of (7) of the form

$$C_j = \begin{bmatrix} D_j & E & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ \vdots & & & \ddots & E \\ 0 & \cdots & & 0 & D_j \end{bmatrix} \qquad (j = 1, \cdots, q), \tag{8}$$

with $2m_j$ rows and columns when the elements are written out in detail, where

$$D_j = \begin{bmatrix} 0 & 1 \\ -(\mu_j^2+\nu_j^2) & 2\mu_j \end{bmatrix} \quad (\nu_j \neq 0), \qquad \text{and} \qquad E = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Although (7) is a real canonical form, for purposes of computing solutions of system (1) it is very inconvenient. In view of this fact the author then decided

to see if it were not better in (8) to replace block D_j by a new matrix

$$J_j = \begin{bmatrix} \mu_j & -\nu_j \\ \nu_j & \mu_j \end{bmatrix}$$

and then proceed to compute the corresponding solutions of (1); see [2]. Shortly thereafter, it was discovered that Niemitski and Stepanov [2] in their Russian text had already proposed this new canonical form, although the corrresponding solutions given in the text are plainly in error. Furthermore, the correct solution turned out to be rather difficult to obtain and very lengthy to write out in full detail.

3. The recommended canonical form. A later reading of a paper by Coddington and Levinson [4] suggested to the author that a far more convenient real canonical form would be obtained if in (7) each matrix C_j, corresponding to a pair of conjugate elementary divisors, were replaced by the matrix

$$G_j = \begin{bmatrix} J_j & 0 & 0 & \cdots & 0 \\ L & \cdot & & & \vdots \\ 0 & \cdot & \cdot & & \vdots \\ \vdots & & \cdot & \cdot & 0 \\ 0 & \cdots & 0 & L & J_j \end{bmatrix},$$

where G_j has $2m_j$ rows and columns when the elements are written out in detail and

$$L = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The matrices C_j and G_j are, of course, equivalent in the sense of similarity under a real nonsingular transformation.

Once the new recommended canonical form

$$B = [B_1, \cdots, B_p; G_1, \cdots, G_q] \tag{9}$$

is adopted, systems (5) can be decomposed at once into a finite number of matrix equations in general of lower order; namely

$$(10)\quad dW_i/dt = B_iW_i, \qquad (11)\quad W_i(t+h) = B_iW_i(t), \qquad (i = 1, \cdots, p),$$

and

$$(12)\quad dZ_j/dt = G_jZ_j, \qquad (13)\quad Z_j(t+h) = G_jZ_j(t), \qquad (j = 1, \cdots, q),$$

where the solution for (5) takes the form

$$Y(t) = [W_1(t), \cdots, W_p(t); Z_1(t), \cdots, Z_q(t)]. \tag{14}$$

The advantage of this decomposition is that fundamental solutions for (10)–(13) can now be written out explicitly and in fairly compact form as follows: For (10) a fundamental solution is

$$W_i = e^{\lambda_i t}\begin{bmatrix} 1 & 0 & & \cdots & & 0 \\ t/1! & 1 & & & & \vdots \\ t^2/2! & t/1! & & & & \vdots \\ \vdots & & & & & \\ & & & \cdot & \cdot & 0 \\ t^{m_i-1}/(m_i-1)! & \cdots & & t^2/2! & t/1! & 1 \end{bmatrix}. \tag{15}$$

For (11), if $\lambda_i \neq 0$, a fundamental solution takes the form

$$W_i(t) = \lambda_i^{t/h}[\lambda_i^{m_i-1}, \lambda_i^{m_i-2}, \cdots, \lambda_i, 1]$$

$$\times\begin{bmatrix} 1 & 0 & & \cdots & & 0 \\ \binom{t/h}{1} & 1 & & & & \vdots \\ \binom{t/h}{2} & \binom{t/h}{1} & & & & \\ \vdots & & & & & \\ & & & \cdot & \cdot & 0 \\ \binom{t/h}{m_i-1} & & \cdots & \binom{t/h}{2} & \binom{t/h}{1} & 1 \end{bmatrix}, \tag{16}$$

where $\binom{t}{m} = t(t-1)\cdots(t-m+1)/m!$, $m \geqq 1$; $\binom{t}{0} = 1$. In the event that $\lambda_i = 0$, then

$$W_i(t) \equiv 0. \tag{17}$$

For (12) we have the fundamental solution

$$Z_j = e^{\mu_j t}\begin{bmatrix} M_j & 0 & & \cdots & & 0 \\ tM_j/1! & M_j & & & & \\ t^2M_j/2! & tM_j/1! & & & & \vdots \\ \vdots & & & & & \\ & & & \cdot & \cdot & 0 \\ t^{m_j-1}M_j/(m_j-1)! & & \cdots & t^2M_j/2! & tM_j/1! & M_j \end{bmatrix}, \tag{18}$$

where M_j is the 2 by 2 matrix

$$M_j = \begin{bmatrix} \cos \nu_j t & -\sin \nu_j t \\ \sin \nu_j t & \cos \nu_j t \end{bmatrix}$$

and the 0's in (18) denote 2 by 2 zero matrices.

Likewise for (13) one has the fundamental solution

$$Z_j(t) = \rho_j^{t/h}[J_j^{m-1}, J_j^{m-2}, \cdots, J_j, L]Q_j(t), \tag{19}$$

where

$$Q_j(t) = \begin{bmatrix} N_j(t) & 0 & & \cdots & & 0 \\ \binom{t/h}{1} N_j(t) & \cdot & & & & \cdot \\ \binom{t/h}{2} N_j(t) & \cdot & & & & \cdot \\ \vdots & & & & & \cdot \\ \cdot & & & \cdot & \cdot & 0 \\ \binom{t/h}{m_j - 1} N_j(t) & \cdots & & \binom{t/h}{2} N_j(t) & \binom{t/h}{1} N_j(t) & N_j(t) \end{bmatrix},$$

$$\rho_j = (\mu_j^2 + \nu_j^2)^{1/2}, \qquad \mu_j + i\nu_j = \rho_j e^{i\theta_j}, \qquad 0 \leqq \theta_j < 2\pi, \qquad \text{and}$$

$$N_j(t) = \begin{bmatrix} \cos(\theta_j t/h) & -\sin(\theta_j t/h) \\ \sin(\theta_j t/h) & \cos(\theta_j t/h) \end{bmatrix} \qquad (j = 1, \cdots, q).$$

It is not difficult to verify by direct substitution in (10)–(13) that these solutions (15)–(19) are correct. Furthermore, by substituting these real fundamental solutions (15)–(19) respectively in (14), one obtains a real fundamental solution $Y = Y_f(t)$, where the corresponding P in (6) is also real.

Although we now have outlined how the solutions of (1) can be found in a convenient real form, the computations required in a particular case to find the real transformation matrix P and the corresponding recommended canonical form (9) might be quite formidable. Nevertheless it is worth knowing in what direction the computational effort should be focused.

4. The trajectory pattern in a special case. Without pursuing the matter further here, as an example of an application of the real canonical form (9), we present the following

THEOREM. *If the real parts of all the characteristic roots for the differential equation* (1) *are negative, then the entire x-phase space is filled with similar, nested, nonintersecting, n-dimensional ellipsoids centered on the origin such that once any*

trajectory satisfying (1) *contacts any one of the ellipsoids it immediately enters the ellipsoid never to come out again as t increases.*

Proof. [2].

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ON THE LOCATION OF THE CENTROID OF CERTAIN SOLIDS

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The first remark that led eventually to the results of this paper came in a class discussion. The members of the class were high school teachers, participants in a National Science Foundation Institute held at Stanford University in 1957–58. The first-named author was one of the participants, the second-named author the instructor, and the remark was made by another participant, Mr. R. D. Fellman.

1. Definitions and results. A *trianguloid* is a polygon connected with a circle. The center of the circle is a special vertex of the trianguloid, called its *apex*. Those sides of the trianguloid of which the apex is not an endpoint are tangent to the circle and form jointly the *base* of the trianguloid.

Let O be the apex of the trianguloid, C_A the centroid (center of gravity) of the area of the trianguloid, C_B the centroid of its base. Both mass distributions involved are understood to be uniform; the distribution centered in C_A has constant superficial density, that centered in C_B constant linear density.

THEOREM I. *The points O, C_A and C_B lie on the same straight line, C_A between O and C_B, and $OC_A = 2C_AC_B$.*

A *pyramidoid* (sorry, not yet in the dictionary) is a polyhedron connected with a sphere. The center of the sphere is a special vertex of the pyramidoid, called its *apex*. Those faces of the pyramidoid of which the apex is not a vertex are tangent to the sphere and form jointly the *base* of the pyramidoid.

Let O be the apex of the pyramidoid, C_V the centroid of its volume, C_B the centroid of its base. Both mass distributions involved are understood to be uniform.

THEOREM II. *The points O, C_V and C_B lie on the same straight line, C_V between O and C_B, and $OC_V = 3C_VC_B$.*

When the base consists of just one side, the trianguloid is simply a triangle; in this particular case the fact asserted by Theorem I is well known. When the base consists of just one face, the pyramidoid is simply a pyramid; in that still more particular case when the pyramid is a tetrahedron, the fact asserted by Theorem II is well known. A polygon circumscribed about a circle can be regarded as a trianguloid, the base of which is the perimeter of the polygon. Similarly, a polyhedron circumscribed about a sphere can be regarded as a pyramidoid, the base of which is the surface of the polyhedron (the faces of the pyramidoid not belonging to the base degenerate by coalescing in one line which joins the center of the sphere to a vertex). What Theorems I and II state regarding circumscribed polygons and polyhedra, respectively, does not seem to be well known, and perhaps it is not known at all. Even the particular fact that the centroid C_B of the perimeter of a triangle can be so simply constructed from the well-known centroid C_A of its area and the center O of its inscribed circle, as Theorem I asserts, does not seem to be known.

Other particular and limiting cases of the two theorems will be briefly mentioned in Section 4. The proof of Theorem II will be given in Section 3 after preparatory remarks in Section 2. The proof of Theorem I is so similar to that of Theorem II that it need not be given.

2. Lemmas. Theorems I and II become intuitive if we view some familiar facts in the proper light. Let us start with a definition. Let us call two material systems *equivalent* (as seen from the standpoint of the geometry of masses) if they have the same mass and the same centroid. There are important elementary cases in which a continuous mass distribution is equivalent to a discrete mass distribution in the sense just defined.

LEMMA I. *A total mass M, distributed with constant linear density along a segment of a straight line, is equivalent to the system of two material points, each with mass $M/2$, located at the two endpoints of the segment.*

LEMMA II. *A total mass M, distributed with constant superficial density over the area of a triangle, is equivalent to the system of three material points, each with mass $M/3$, located at the three vertices of the triangle.*

LEMMA III. *A total mass M, distributed with constant density over the volume of a tetrahedron, is equivalent to the system of four material points, each with mass $M/4$, located at the four vertices of the tetrahedron.*

The language of these lemmas may be new, but the facts that they express are most familiar. For instance, to prove Lemma III, it is enough to show that the centroids of both material systems considered, the continuous and the discrete, coincide. And to prove this it is enough to show that any median plane of the tetrahedron (a plane passing through an edge and the midpoint of the

opposite edge) contains both centroids.*

LEMMA IV. *Two material systems are equivalent if they can be divided into corresponding equivalent (nonoverlapping) parts.*

Let us restate this with appropriate symbols. We express by the equation $D=D_1+D_2+\cdots+D_n$ that the mass distribution D consists of the nonoverlapping parts $D_1, D_2, \cdots, D_n$. We express by the equation $D\sim D'$ that the mass distributions D and D' are equivalent to each other in the sense defined above. With this notation we restate Lemma IV as follows:

LEMMA IV′. *If* $D=D_1+\cdots+D_n$, $D'=D_1'+\cdots+D_n'$ *and* $D_1\sim D_1', \cdots, D_n\sim D_n'$ *then* $D\sim D'$.

(Still shorter: *Equivalences between nonoverlapping mass distributions can be added.*)

To prove Lemma IV, we assume a well-known fact: If the part D_1 of the system D is replaced by one particle (material point) located at the centroid of D_1 and having a mass equal to the total mass of D_1, D is changed into another distribution with the *same centroid.*† By $2n$ consecutive changes of this nature we can change both D and D', the two mass distributions satisfying the hypothesis of Lemma IV′, into the *same* system of n material points, which renders the conclusion of Lemma IV′ evident.

3. Proof of Theorem II. Dissect each face that belongs to the base of the pyramidoid into triangles and call any triangle so obtained a *component triangle.* Call a tetrahedron of which a component triangle is the base and O, the apex of the pyramidoid, is the opposite vertex, a *component tetrahedron.* The base of the pyramidoid is dissected into component triangles and, correspondingly, the volume of the pyramidoid is dissected into component tetrahedra.

The centroid of the mass that uniformly fills the volume of the pyramidoid is C_V. We redistribute the mass within each component tetrahedron as follows: we first concentrate one quarter of the total mass contained in the tetrahedron in each vertex; then we redistribute the mass of the three vertices different from O uniformly over the base (a component triangle) of which they are the vertices. By Lemmas II, III and IV we obtain so a final mass distribution equivalent to the original one, so that C_V remains the centroid. One quarter of the total mass is now concentrated in O and three quarters distributed over the surface of the base of the pyramidoid. We assert that this distribution over the surface of the base is uniform.

In fact, by construction, the distribution is uniform within each component triangle. Let b stand for the area of this triangle, v for the volume of the com-

* See, *e.g.*, G. Pólya, How to Solve It, 2nd ed., Garden City, N. Y., 1957, p. 40.

† *Cf.* The Works of Archimedes, edited by T. L. Heath, New York, 1920, p. 194, Proposition 8. The fact becomes evident if we go back to the definition of the centroid by moments.

ponent tetrahedron of which it is the base and r for the radius of the sphere of center O that is connected with the pyramidoid. By the definition of this kind of polyhedron the plane in which the triangle with area b is situated is tangent to the sphere, and so r is the altitude of the component tetrahedron drawn from O. Therefore $v=\frac{1}{3}rb$. The mass distributed over the area b is proportional to $3v/4$ and so finally proportional to the area b. This proves the asserted uniformity of the distribution.

Now we concentrate the mass uniformly distributed over the base of the pyramidoid into its centroid C_B. The new distribution so obtained consists of two material points, O and C_B; one quarter of the total mass is located at O, three quarters at C_B. Yet, by Lemma IV, this new distribution is equivalent to the previous one and C_V is still its center of gravity. This renders Theorem II evident.

4. Remarks. Our theorems have many interesting particular cases and limiting cases. We shall mention only a few such cases.

(1) *Circumscribed polyhedra.* A polyhedron is circumscribed about a sphere with center O, the centroid of the volume of the polyhedron is C_V, the centroid of its surface C_B. If two of the three points O, C_V and C_B happen to coincide, all three points must coincide.

There is a corresponding theorem dealing with circumscribed polygons. Yet both theorems become false if "circumscribed" is replaced by "inscribed."

(2) *A sector of a circle* is obviously a limiting case of a trianguloid. The arc takes over the role of the base, its centroid is C_B. Both C_B and C_A, the centroid of the area of the sector, lie, of course, on the line of symmetry of the sector, which also passes through O, the center of the circle. By Theorem I, $OC_A=\frac{2}{3}OC_B$. The reader may verify this relation for the semicircle, which is a particular sector, by using the theorems of Pappus (which can also be used to verify a more extensive particular case of Theorem I).

(3) A *sector of a sphere* (generated by the revolution of a sector of a circle about one of its diameters) is obviously a limiting case of a pyramidoid. A zone on the spherical surface (generated by the revolving arc) takes over the role of the base, its center of gravity C_B is midway between the two parallel planes that cut off the zone from the spherical surface (this follows from the well-known fact, discovered by Archimedes, that the area of the zone is proportional to its altitude). The centroid C_B, the centroid C_V of the volume of the spherical sector and the center O of the sphere lie, of course, on the axis of revolution. By Theorem II $OC_V=\frac{3}{4}OC_B$. The hemisphere is a particular case; if the radius is r, $OC_B=\frac{1}{2}r$ and therefore (as is well known) $OC_V=\frac{3}{8}r$.

(4) Another interesting limiting case of a pyramidoid is a *sector of a circular cylinder*; any figure on the surface of the cylinder may be the base, any point on the axis of the cylinder the apex. If the sector is cut off from the cylinder by

two planes (their line of intersection and the axis of the cylinder meet in the apex) the points C_B and C_V can be located by elementary integrations.

(5) The extension of Theorems I and II to higher dimensions is clearly suggested and can be easily carried through.

A GENERALIZATION OF THE TRIGONOMETRIC FUNCTIONS

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Recently I investigated a generalization of the trigonometric functions which were similar enough in form to the sine and cosine functions to suggest interesting generalizations of many of the results of classical analysis. In particular we shall consider Euler's result on $\Gamma(\frac{1}{2})$, Euler's beta integral and the geometric roles played by the trigonometric functions in Euclidean spaces. We shall show that the functions are hyperelliptic in general and we shall investigate the circumstances under which they have a real period. We shall not consider the functions from the point of view of complex variables but shall use the methods of elementary real analysis.

The functions, which I call the alpha and beta functions are defined by the differential system

$$\frac{d}{dx}\alpha_s(x) = \beta_s^{s-1}(x), \qquad \frac{d}{dx}\beta_s(x) = -\alpha_s^{s-1}(x), \tag{1}$$

$$\alpha_s(0) = 0, \qquad \beta_s(0) = 1. \tag{2}$$

The parameter s will be called the order of the functions and will be taken in this paper to be real and an integer greater or equal to unity. Picard's theorem assures us that the system (1) and (2) have unique solutions satisfying the boundary conditions and that these functions are continuous and have continuous derivatives at least in some neighborhood of the origin.

If we multiply the two equations (1) by $\alpha_s^{s-1}(x)$ and $-\beta_s^{s-1}(x)$, respectively, we get

$$\alpha_s^{s-1}(x)\cdot\frac{d}{dx}\alpha_s(x) = -\beta_s^{s-1}(x)\cdot\frac{d}{dx}\beta_s(x) = \alpha_s^{s-1}(x)\beta_s^{s-1}(x).$$

Considering the first equality we have

$$\alpha_s^{s-1}(x)\cdot\frac{d}{dx}\alpha_s(x) + \beta_s^{s-1}(x)\cdot\frac{d}{dx}\beta_s(x) = 0.$$

If we now integrate this expression we get $\alpha_s^s(x)+\beta_s^s(x)=\text{constant}$. Setting $x=0$ and using (2) we find that the integration constant is equal to unity. We have the identity

(3) $$\alpha_s^s(x) + \beta_s^s(x) = 1.$$

To show that our functions are connected with hyperelliptic functions we note that if w and u are two variables connected by the integral expression $\int_0^w (1-4t^s)^{-1/2}dt = u$, then

(4) $$w = \alpha_s(u)\cdot\beta_s(u).$$

Consider the integral. If $s=2$ it is integrable by trigonometric functions, as we would expect. If $s=3$ or $s=4$, the integral is elliptic, and the alpha and beta functions are elliptic functions. When the order s is greater than 4 the integral is in general hyperelliptic. The alpha and beta functions are, therefore, a particular class of hyperelliptic functions.

From the functional equation (3) and (1) we can derive the differential equation satisfied by the alpha function. Let $x=\alpha_s(y)$, so that $dx/dy=\beta_s^{s-1}(y) =(1-x^s)^{(s-1)/s}$. If we now separate variables and integrate we get

(5) $$y = \int_0^x (1 - t^s)^{(1-s)/s}dt = \arg \alpha_s(x).$$

Similarly,

(6) $$\int_x^1 (1 - t^s)^{(1-s)/s}dt = \arg \beta_s(x).$$

We have used the notation arg to denote the inverse function.

Before continuing we make a definition. We define the numbers π_s or $\pi(s)$ by the relation

(7) $$\tfrac{1}{2}\pi_s = \arg \alpha_s(1) = \arg \beta_s(0) = \int_0^1 \quad (1 - t^s)^{(1-s)/s}dt.$$

It will be shown later that π_s is four times the area enclosed by the curves $x^s+y^s=1$ in the first quadrant of the xy-plane. As s increases we see that this area tends more and more closely to the unit square bounded by the x- and y-axes and the lines $x=1$, $y=1$. Hence, we state that the sequence of numbers π_s increases monotonically and approaches the limit four. This can also be concluded from an examination of the above integral. We have $\pi(1)=2$, $\pi(2) =3.1415, \cdots, \pi(3)=3.595 \cdots$ and so on. $\pi(2)$ is, of course, our ordinary pi. We might venture the hypothesis that π_s is a transcendental number for s an integer larger than unity because of our knowledge of the transcendence of $\pi(2)$. From (5) and (6) we see that

(8) $$\alpha_s(\tfrac{1}{2}\pi_s) = 1, \qquad \beta_s(\tfrac{1}{2}\pi_s) = 0.$$

In another treatment the alpha and beta functions might have been defined as the inversions of the integrals (5) and (6). By the derivation of (5) and (6) we

have extended the interval over which these functions are defined to the interval $0 \leqq x \leqq \frac{1}{2}\pi_s$.

From these integrals we can easily obtain several interesting results. First we have the

LEMMA. *When their order is an even integer the alpha and beta functions are odd and even functions, respectively. That is,*

$$\alpha_s(-x) = -\alpha_s(x), \qquad \beta_s(-x) = \beta_s(x), \qquad s = 2, 4, 6, \cdots. \tag{9}$$

This, incidentally, extends the range of definition of the even-order functions to the interval $-\frac{1}{2}\pi_s \leqq x \leqq \frac{1}{2}\pi_s$.

Proof. To prove that the alpha function is an odd function we merely make the observation that when s is an even integer the integrand of (5) is even and hence the integral is an odd function of the upper limit of integration. Hence $\alpha_s(x)$ is an odd function since its inverse is odd.

To show that $\beta_s(x)$ is an even function is probably most easily done by making the observation that the derivative of $\beta_s(x)$, given by (1), is an odd function when s is even. $\beta_s(x)$ is, therefore, an even function.

We now state the second

LEMMA.

$$\alpha_s(\tfrac{1}{2}\pi_s - x) = \beta_s(x) \qquad \beta_s(\tfrac{1}{2}\pi_s - x) = \alpha_s(x) \tag{10}$$

Proof. Now

$$\int_0^1 (1 - t^s)^{(1-s)/s} dt = \int_0^x (1 - t^s)^{(1-s)/s} dt + \int_x^1 (1 - t^s)^{(1-s)/s} dt$$

which we write as

$$\tfrac{1}{2}\pi_s = \arg \alpha_s(x) + \arg \beta_s(x). \tag{11}$$

To show that this is equivalent to (10) we consider the function $\phi(x) = \alpha_s(\frac{1}{2}\pi_s - x)$. If we let $x = \arg \beta_s(u)$ we get $\phi(x) = \alpha_s(\frac{1}{2}\pi_s - \arg \beta_s(u)) = u = \beta_s(x)$. This is the first of the relations (10). Since the demonstration of the second of the relations of (10) is the same (or it can be derived from the first) the lemma is proven.

For the functions of even integer order we have

$$\alpha_s(x + \pi_s) = \alpha_s(\tfrac{1}{2}\pi_s - (-\tfrac{1}{2}\pi_s - x)) = \beta_s(\tfrac{1}{2}\pi_s + x) = \alpha_s(-x) = -\alpha_s(x).$$

Writing the first and last expressions we have

$$\alpha_s(x + \pi_s) = -\alpha_s(x), \qquad s = 2, 4, 6 \cdots. \tag{12}$$

Similarly, we find that

$$\beta_s(x + \pi_s) = -\beta_s(x), \qquad s = 2, 4, 6 \cdots.$$

If we now replace x by $x + \pi_s$ in these equations we get

$$\alpha_s(x + 2\pi_s) = \alpha_s(x), \qquad \beta_s(x + 2\pi_s) = \beta_s(x), \qquad s = 2, 4, 6 \cdots. \tag{13}$$

The alpha and beta functions of even integer order are, therefore, periodic functions with the period $2\pi_s$. The other period of the even-order elliptic and hyperelliptic functions and all the periods of the odd-order functions are complex.

To indicate the geometrical properties of these functions we begin by defining the KN_s-function by the improper integral

$$KN_s(x) = \int_0^\infty u^x e^{-(1/s)u^s} du, \qquad x > -1. \tag{14}$$

This integral can easily be brought into the form of the gamma function by making the change of variable $t = (1/s)u^s$ and can be seen to converge if $x > -1$.

If we integrate by parts we get the reduction formula $KN_s(x) = (x - s + 1) \cdot KN_s(x - s)$, where x must satisfy $x > s - 1$ for the latter integral to converge.

The KN_s-function was introduced to simplify the statement of the following generalization of Euler's theorem.

THEOREM. *For any a and b for which the integral converges, that is, for a, b* > -1,

$$\int_0^{1/2\pi_s} \overset{a}{\alpha_s}(\phi) \cdot \overset{b}{\beta_s}(\phi) d\phi = \frac{KN_s(a) \cdot KN_s(b)}{KN_s(a + b + 1)}. \tag{15}$$

Proof. The proof is essentially Euler's. We write

$$KN_s(a) \cdot KN_s(b) = \int_0^\infty \int_0^\infty x^a y^b e^{-(1/s)(x^s + y^s)} dx dy,$$

where we have used different integration variables, x and y, in the two integral factors, and we have combined the two integrals into one double integral, a surface integral over the first quadrant of the xy-plane.

In this surface integral we make the transformation

$$x = R \cdot \beta_s(\phi), \qquad y = R \cdot \alpha_s(\phi), \qquad 0 \leqq \phi \leqq \tfrac{1}{2}\pi_s, \qquad R^s = x^s + y^s. \tag{16}$$

We must show that this transformation covers the entire first quadrant once and only once as the variables R and ϕ range over the intervals $0 \leqq R \leqq \infty$ and $0 \leqq \phi \leqq \frac{1}{2}\pi_s$. This may be done as follows. We note that every point of the first quadrant lies on one and only one member of the family of curves $R = \text{constant}$ if R is restricted to the interval above. The point (x_0, y_0) lies on the curve, $x_0^s + y_0^s = x^s + y^s = R_0^s$ and so has a unique R_0 associated with it.

For any fixed R, now, as ϕ ranges over $0 \leqq \phi \leqq \frac{1}{2}\pi_s$ the corresponding point (x, y) ranges from $(R, 0)$ to $(0, R)$ and since the alpha and beta functions are continuous on the interval $0 \leqq \phi \leqq \frac{1}{2}\pi_s$ and one-valued (since, for example, from (5) and (6) their inverses are monotonically increasing and decreasing functions, respectively, on the interval) we see that each point on the curve will have one

and only one value of ϕ associated with it. Conversely, if a set (R, ϕ) is given we can see by using the same procedure that only one point corresponds to it, or by using the equations (16). We calculate the Jacobian of the transformation in the usual way

$$\frac{\partial(x, y)}{\partial(R, \phi)} = \begin{vmatrix} \beta_s(\phi) & \alpha_s(\phi) \\ -R\cdot\alpha_s^{s-1}(\phi) & R\cdot\beta_s^{s-1}(\phi) \end{vmatrix} = R[\beta_s^s(\phi) + \alpha_s^s(\phi)] = R$$

and so we find that the differential of area is

$$dA = dxdy = RdRd\phi. \tag{17}$$

The double integral then becomes

$$\begin{aligned} KN_s(a)\cdot KN_s(b) &= \int_0^\infty \int_0^{1/2\pi_s} [R\beta_s(\phi)]^a[R\alpha_s(\phi)]^b e^{-(1/s)R^s} RdRd\phi \\ &= \int_0^\infty R^{a+b+1}e^{-(1/s)R^s}dR\cdot \int_0^{1/2\pi_s} \beta_s^a(\phi)\alpha_s^b(\phi)d\phi \end{aligned}$$

and since the first integral is exactly $KN_s(a+b+1)$ we have the result stated. Because of the symmetry of a and b on the right-hand side of (15) we may interchange a and b in the integral without altering its value.

Let us consider the transformation which we used to prove the theorem, the set of equations (16) and (17). We can interpret the transformation in two ways.

I. First, we can consider x and y as the ordinary cartesian coordinates in the Euclidean plane which are related by these equations to a system of coordinates which have the parametric curves $x^s+y^s=$constant. This is the interpretation which we used to prove the theorem.

II. We can interpret (16) and (17) as polar coordinates in a two-dimensional Minkowski space, that is, one in which the Pythagorean theorem is replaced by $R^s=x^s+y^s$. If $s=2$, of course, both these interpretations coincide.

These polar coordinates may easily be generalized to a space of any number of dimensions. For N dimensions the result is

$$x_1 = R\cdot \prod_{j=1}^{N-1} \alpha_s(\phi_j),$$

$$x_k = R\cdot\beta_s(\phi_{k-1})\cdot \prod_{j=k}^{N-1} \alpha_s(\phi_j) \qquad k = 2, 3, \cdots, N,$$

$$dV = \prod_{j=1}^{N} dx_j = R^{N-1}dR\cdot \prod_{j=1}^{N-1} \alpha_s^{N-j-1}(\phi_{j-1})d\phi_{j-1},$$

$$R^s = \sum_{j=1}^{N} x_j^s, \qquad dQ^w = \sum_{j=1}^{N} dx_j^w.$$

We have used the symbol dQ for the differential of arc length. The number $w=2$ using interpretation I and $w=s$ using interpretation II. In the equation for x_k we take the value of the product to be unity when $k=N$. When $s=2$ the equations reduce to spherical coordinates for an N-dimensional Euclidean space.

We stated before that the number π_s is four times the area in the first quadrant bounded by the curve $x^s+y^s=1$, that is,

$$4\int_0^1 (1-t^s)^{1/s}dt = \pi_s. \tag{18}$$

We may demonstrate this in several ways, the transformation to polar coordinates probably being the simplest. To integrate this without polar coordinates we make the substitution $t=\alpha_s(\phi)$, $dt=\beta_s^{s-1}(\phi)d\phi$. Then if $|u|\leqq 1$ we find that

$$\int_0^u (1-t^s)^{1/s}dt = \int_0^\phi \beta_s^s(\phi)d\phi = \tfrac{1}{2}[\phi+\alpha_s(\phi)\beta_s(\phi)].$$

The integration formula for $\beta_s^s(\phi)$ may easily be obtained by integrating by parts. If we change variables again from ϕ to u we get the desired integration

$$\int_0^u (1-t^s)^{1/s}dt = \tfrac{1}{2}[\arg\alpha_s(u) + u(1-u^s)^{1/s}], \qquad |u| \leqq 1.$$

We find that the algebraic part of the antiderivative corresponds to a triangle while the part involving the inverse alpha function corresponds to an area bounded by a curved side in a manner which is familiar to us from the quadrature of the circle. In particular, if we set $u=1$ then we get (18).

Our interest in (18) lies in the fact that using it we can generalize Euler's result on the gamma function, $\Gamma(1/2)=\pi^{1/2}$. If we integrate (18) using the gamma function, we get the following generalization of Euler's result: if $p=2^{(N+1)}$ then

$$\Gamma(1/p) = 2^{1-(1/p)}\prod_{k=0}^{N}\left[2^{N-k-1}\pi_{2^{N-k+1}}\right]^{2^{-k-}}.$$

If we set $N=0$ we get Euler's result. For $N=1$ and $N=2$ we get

$$\Gamma(1/4) = 2^{3/4}\sqrt{\pi_4\sqrt{\tfrac{1}{2}\pi_2}},$$
$$\Gamma(1/8) = 2^{7/8}\sqrt{2\pi_8\sqrt{\pi_4\sqrt{\tfrac{1}{2}\pi_2}}},$$

and so on.

As many results of this sort as we wish may easily be obtained by integrating, using the gamma function, expressions associated with the alpha and beta functions. Various other well-known results promise to be capable of generalization in the same way.

GENERALIZATION OF A PEANO SYMBOL

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1. Superscript sign. At the turn of the last century, G. Peano* introduced the sign $\upharpoonright$, but unfortunately it seems to have been overlooked by the mathematical community. He discussed its convenience for the cases $a^{m+n} = a\upharpoonright m+n$ and $a^{m^2+n^2} = a\upharpoonright m^2+n^2$; however it turns out that sign is also useful for more difficult superscript cases. For example,

$$(S^X)^{S^{m-k}} = (S^X)^{S\upharpoonright m-k}, \qquad 2^{2^{\aleph_0}} = 2^{2\upharpoonright\aleph_0} = 2\upharpoonright 2^{\aleph_0},$$

$$\rho w^{i_1\cdots i_p i_{p+1}\cdots i_n} = \rho w \upharpoonright «i_1 \cdots i_p i_{p+1} \cdots i_n».$$

Note that the Peano symbol is useful as well in the following case: $\exp(-i\Phi) = e\upharpoonright -i\Phi$. Moreover, it gives us the needed notation K. Menger† requests for the nth power and first power functions, $<\upharpoonright n>$ and $<\upharpoonright 1>$, respectively.

2. Subscript sign. The above Peano idea can easily be extended to a number of difficult subscript cases by the employment of the $\downharpoonleft$ sign. For example, in topology, $\pi_{2i(q-1)^2}(X)$ can be more economically printed as $\pi« \downharpoonleft 2i(q-1)^2»(X)$, and ρ_{S^2} as $\rho\downharpoonleft S^2$, and so on.

3. Combined subscript and superscript sign. The $\uparrow$ sign§ finds its most interesting application in conjunction with the troublesome summation sign, product sign, set-theoretical signs, etc. For example, $\sum_{i=1}^{m+n-1}\phi_i(x)$ can be more conveniently printed employing the $\uparrow$ sign in one of the following two ways:

$$\sum «i = 1 \uparrow m + n - 1»\phi_i(x), \qquad \sum_{i=1\uparrow m+n-1}\phi_i(x),$$

where in all cases the left side of the arrow is the subscript and the right side of the arrow is the superscript. It can also be used advantageously in printing various mapping notations, *e.g.*, $d_2^{n+1,m} = d_{2\uparrow n+1,m}$, and

$$d_s'^{p,q}\colon E_s'^{p,q} \to E_s'^{p+s,q-s+1}$$

can be written as

$$d' «s\uparrow p, q»\colon E'_{s\uparrow p,q} \to E'_{s\uparrow p+s,q-s+1}.$$

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* G. Peano, Formulaire de Mathematique, vol. 2, Turin, 1898.

† K. Menger, Calculus, A Modern Approach, Boston, 1952.

§ The sign introduced in Math. Mag., vol. 33, 1959, p. 24, is a less ambiguous one than the arrow.

SOLUTIONS BY QUADRATURE OF RICCATI AND SECOND-ORDER LINEAR DIFFERENTIAL EQUATIONS

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It is a well-known fact that the general Riccati and the general homogeneous second-order linear differential equations are equivalent, in the sense that, each being transformable into the other, a general solution of one of them entails that of the other. It is equally well known that no general method of solution by quadratures of either equation has been found. In this paper I consider (unlimited) classes of these equations for which the general solutions by quadratures are possible. Both the general Riccati and the general linear equations are first brought to the unified "normal" form, with all the coefficients lumped in the "invariant" of the equation. Using then the method here developed, we obtain the general solutions by quadratures whenever the (given) invariant is found to identically satisfy any one of a well defined infinite sequence of distinct differential equations.

1. The method. The general form of the Riccati equation, x and y supposed real, is

$$y' = e(x) + f(x)y + g(x)y^2. \tag{1}$$

Transform (1) by $y=a(x)-v'/(gv)$, where $v\equiv v(x)$ is a new dependent variable and $a(x)$ an arbitrary transformation function. (1) goes over into

$$v'' - \{f + 2ag + (g'/g)\}v' + g(e + af + a^2g - a')v = 0. \tag{2}$$

Choose now $a(x)$ such that

$$g(e + af + a^2g - a') + \{f + 2ag + (g'/g)\}' = 0, \tag{3}$$

so that (2) becomes $v'' - [\{f+2ag+(g'/g)\}v]' = 0$ and thus $v' = \{f+2ag+(g'/g)\}v + C_1$,

$$\begin{aligned} v(x) &= g \exp\left\{\int (f + 2ag)dx\right\} \\ &\quad \cdot\left[C_2 + C_1\int g^{-1}\exp\left\{-\int (f + 2ag)dx\right\}dx\right], \end{aligned} \tag{4}$$

where C_1, C_2 are arbitrary constants.

The general solution of (1) is then

$$y(x) = -a(x) - \frac{f}{g} - \frac{g'}{g^2} - \frac{C_1}{gv(x)} \tag{5}$$

with $v(x)$ given by (4). Now $a(x)$, which we shall call the "conjugate" to $y(x)$, satisfies, from (3), the differential equation

$$a' = e_1 + f_1 a + g_1 a^2, \tag{6}$$

where $e_1 \equiv -(1/g)\{f+(g'/g)\}'-e$, $f_1 \equiv -f-2(g'/g)$, $g_1 \equiv -g$.

If a particular integral of (6), which is all that we need, is found on inspection, or if $e_1 \equiv 0$, or if both e_1 and f_1 are proportional to g, the general solution of (1) is immediate. Failing this, it is futile to consider (6) anew as a given equation (1) and apply the same procedure to it, since this would only give $y(x)$ as the "conjugate" to $a(x)$. We get, however, an unending sequence of distinct Riccatian equations by transforming (6) by the reciprocal function $w \equiv 1/a$, to get the differential equation

$$w' = -g_1 - f_1 w - e_1 w^2, \tag{7}$$

then obtaining the equation of the conjugate a_1 to w as

$$a_1' = e_2 + f_2 a_1 + g_2 a_1^2, \tag{8}$$

where $e_2 \equiv (1/e_1)\{(e_1'/e_1)-f_1\}'+g_1$, $f_2 \equiv f_1 - 2(e_1'/e_1)$, $g_2 \equiv e_1$, and, if a_1 is still indeterminable, forming the reciprocal $w_1 \equiv 1/a_1$, obtaining its conjugate, and so on until an equation for some a_n either gives a particular integral by inspection, has an identically vanishing coefficient e_{n+1}, or has e_{n+1}, f_{n+1} proportional to g_{n+1}. By working backwards we are led to the general solution of (5). In recovering an original w_n from its known conjugate a_{n+1}, we use (5), where, besides the obvious changes of notation, we set $C_1 = 0$ since only particular integrals are needed at all except in the last step.

2. Application to the normal equation. The normal form of $z' = e + fz + gz^2$ is obtained by the transformation $2g^2 z = 2gy - fg - g'$ and is

$$y' = p + y^2 \tag{9}$$

with the invariant $p \equiv eg + (2f'g^2 + 2gg'' - 2fgg' - 3g'^2 - f^2 g^2)/(4g^2)$, so that the general solution of (9) is, from (5),

$$y(x) = -a(x) - C_1/v(x) \tag{10}$$

with

$$v(x) = \exp\left\{2\int a\,dx\right\}\left[C_2 + C_1 \int \exp\left\{-2\int a\,dx\right\} dx\right]. \tag{11}$$

The general homogeneous second-order linear differential equation $Y'' + r(x)Y' + t(x)Y = 0$ is transformed by $Y = z\exp\{-\frac{1}{2}\int r\,dx\}$ into the normal form

$$z'' + pz = 0 \tag{12}$$

with the invariant $p \equiv t - \frac{1}{4}r^2 - \frac{1}{2}r'$, and again by $z' + yz = 0$ into (9), so that the general solution of (12) is

$$
\begin{aligned}
z(x) &= v(x) \exp\left\{-\int a\,dx\right\} \\
&= \exp\left\{\int a\,dx\right\}\left[C_2 + C_1\int \exp\left\{-2\int a\,dx\right\}dx\right].
\end{aligned} \tag{13}
$$

For the unified normal equation (9), the conjugate equation (6) becomes $a' = -p - a^2$, (7) becomes $w' = 1 + pw^2$, and for (8), the equation for a_1, we have $e_2 \equiv (p'^2 - pp'' - p^3)/p^3$, $f_2 \equiv -2p'/p$, $g_2 \equiv -p$.

Leaving aside the case where a particular integral of (8) is evident, we now produce a class of differential equations solvable by quadratures, namely, those for which the invariant p satisfies identically the differential equation $e_2 = 0$; p can then have any one of the following three forms (obtained by solving $e_2 = 0$):

1. $p = -2/(x+C)^2$, where C is an arbitrary constant and we find the solutions as

$$
y(x) = \frac{3B_1}{B_1X - 3B_2X^4} - \frac{2}{X},
$$

where $X \equiv x + C$, $z(x) = B_2X^2 - B_1/(3X)$ and B_1, B_2 are arbitrary constants. This case is trivial as it is a special case of a more general form for p for which the solution has long been known. Not so, however, for the next two cases.

2. The second form of the invariant is $p = 2c_1^2c_2e^{c_1x}/(c_2 + e^{c_1x})^2$, where c_1, c_2 are arbitrary constants. (The first form for p corresponds here to the indeterminate case for which $c_1 = 0$, $c_2 = -1$.) We find

$$
a(x) = c_1 \frac{e^{2c_1x} + 2c_1c_2xe^{c_1x} - c_2^2}{(c_1x - 2)e^{2c_1x} - 4c_2e^{c_1x} - c_2^2(c_1x + 2)};
$$

this is to be substituted in (11) to give $v(x)$ after evaluating the two integrals.

3. The third form of the invariant is $p = C_1^2/\{\cos(C_1x + C_2) - 1\}$, where C_1, C_2 are arbitrary constants. Then

$$
a(x) = \frac{C_1 \sin(C_1x + C_2) - C_1^2x}{C_1x \sin(C_1x + C_2) + 2\cos(C_1x + C_2) - 2};
$$

this is to be substituted in (11).

To obtain other types of p, we now assume that $e_2 \neq 0$ and form the equation for $w_1 = 1/a_1$, obtaining its conjugate equation $a_2' = e_3 + f_3a_2 + g_3a_2^2$, where

$$
\begin{aligned}
e_3 &\equiv (1/e_2)(e_2'/e_2)' - \{(2 + 3e_2)/e_2\}p \\
&\equiv p^4(p'^2 - pp'' - p^3)^{-3}\{(p^3 + pp'' - p'^2)p^{(4)} + (2p'p'' - 6p^2p' - pp^{(3)})p^{(3)} \\
&\quad + (5p^4 + 7pp'^2 + 3p^2p'' - p''^2)p'' - 5p'^4 - 5p^3p'^2 + p^6\},
\end{aligned}
$$

$$f_3 \equiv -2(e_2'/e_2) - 2(p'/p)$$
$$\equiv 2\,\frac{p^3p' - 3pp'p'' + p^2p^{(3)} + 2p'^3}{p(p'^2 - pp'' - p^3)},$$
$$g_3 \equiv e_2 \equiv (p'^2 - pp'' - p^3)/p^3.$$

It is seen that $e_3=0$ is distinct from $e_2=0$, although any p satisfying the latter equation will also satisfy the former, and hence new forms for p are defined by $e_3=0$ that will lead to solutions by quadratures. Proceeding in this manner we can obtain an infinite sequence of differential equations $e_n=0$ defining distinct forms for p; the successive members e_n are obtained according to the following recurrence relations:

$$e_2 \equiv -(1/p)(p'/p)' - 1, \qquad f_2 \equiv -2(p'/p), \qquad g_2 \equiv -p,$$

and for $n>1$,

$$e_{n+1} \equiv (1/e_n)\{(e_n'/e_n) - f_n\}' + g_n, \qquad f_{n+1} \equiv f_n - 2(e_n'/e_n), \qquad g_{n+1} \equiv e_n.$$

These latter are the coefficients in the conjugate equation $a_n' = e_{n+1}+f_{n+1}a_n + g_{n+1}a_n^2$.

If now a given p is found to satisfy $e_{n+1}=0$, where $n>1$, then we obtain the solution by evaluating

$$a_n = -\frac{\exp\left\{\int f_{n+1}dx\right\}}{\int g_{n+1}\exp\left\{\int f_{n+1}dx\right\}dx},$$

which, using the recurrence relations, is readily found to be

$$a_n = -\frac{\prod_{m=2}^{n} e_m^{-2}}{p^2\int p^{-2}e_n\prod_{m=2}^{n} e_m^{-2}dx}, \tag{14}$$

and then computing successively $a_{n-1}, a_{n-2}, \cdots, a_1$ from the recurrence relation $a_{n-1}=e_n^2/(e_n' - e_nf_n - e_n^2a_n)$ and $a(x)$ from $a(x) = -p^2/(p'+p^2a_1)$. Finally, this is to be substituted in (11) to give the general solutions (10) or (13). The process involves merely successive differentiations of p and the evaluation of three integrals, one in (14) and two in (11).

We observe that the general solution of $e_2=0$ is known and is itself a particular integral (containing two arbitrary constants) of the fourth-order equation $e_3=0$; this fact might enable the integration of the latter to be effected and thus the determination of a more general form for p. Similarly, the general solution of $e_n=0$ is a particular one of the equation $e_{n+1}=0$, of order higher by two.

CORRECTION

The recent paper *A look at mathematical competitions* (this MONTHLY, vol. 66, 1959, pp. 201–212) erred in implying that the Hungarian Eötvös prize competition ceased to exist in 1928. Except for a scattering of years, it has in fact been given annually. A paper by Professor Aczel, University of Debrecen, giving additional information on this interesting and successful contest, will appear in a forthcoming issue of the MONTHLY.

MATHEMATICAL NOTES

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Material for this department should be sent to Roy Dubisch, Department of Mathematics, University of California, Berkeley 4, California.

DIFFERENTIATION OF INFINITE SERIES OF FUNCTIONS

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1. Introduction. Theorem 1 below, which appears in advanced calculus texts, has undergone many generalizations. This paper contains a generalization which is, perhaps, obtained somewhat more readily than most others. The functions under discussion are real-valued with domain the real numbers or, if specified, a closed interval of real numbers.

THEOREM 1. *Let $\{f_n(x)\}$ be a sequence of functions satisfying the following conditions*: (1) *Each $f_n(x)$ has a continuous derivative on the interval $[a, b]$,* (2) *$\sum f_n(x)$ converges to $f(x)$ at each point in the interval $[a, b]$, and* (3) *$\sum f_n'(x)$ converges uniformly to $g(x)$ on the interval $[a, b]$. Then $f'(x)=g(x)$ for each point in the interval $[a, b]$.*

2. Uniform convergence at a point. Let $\{f_n(x)\}$ be a sequence of functions. We say that $f_n(x)$ converges to $f(x)$ uniformly at x_0 if for each positive number ϵ there is a positive number δ and a positive integer N such that $|f_n(x)-f(x)|<\epsilon$ whenever $|x-x_0|<\delta$ and $n\geqq N$.

THEOREM 2. *If $\{f_n(x)\}$ is a sequence of continuous functions which converges to $f(x)$ then for each interval $[c, d]$ there is a point x_0 such that $c<x_0<d$ and $\{f_n(x)\}$ converges to $f(x)$ uniformly at x_0.*

Proof. [1], pp. 108–109.

THEOREM 3. *If the hypothesis of Theorem 2 is satisfied, then $f(x)$ is continuous at x_0.*

Proof. [1], p. 108.

3. The main theorem.

THEOREM 4. *Let* $\{f_n(x)\}$ *be a sequence of functions satisfying the following conditions*: (1) *Each* $f_n(x)$ *has a continuous derivative on the interval* $[a, b]$, (2) $\{f_n(x)\}$ *converges to* $f(x)$ *at each point in the interval* $[a, b]$, *and* (3) $\{f_n'(x)\}$ *converges to* $g(x)$ *uniformly at* x_0. *Then* $f'(x_0)$ *exists and is equal to* $g(x_0)$.

Proof. By the mean value theorem we have

$$f_n(x) = f_n(x_0) + (x - x_0)f_n'(x_0 + \theta_n(x - x_0)),$$

for some θ_n such that $0<\theta_n<1$. Now, by the uniform convergence of the sequence involved, if ϵ is a positive number there is a positive number δ and an integer N such that

$$\begin{aligned}
&\left|\frac{f_n(x) - f(x)}{x - x_0} + \frac{f(x_0) - f_n(x_0)}{x - x_0} + \frac{f(x) - f(x_0)}{x - x_0} - g(x_0 + \theta_n(x - x_0))\right| \\
&= \left|\frac{f_n(x) - f_n(x_0)}{x - x_0} - g(x_0 + \theta_n(x - x_0))\right| \\
&= \left|f_n'(x_0 + \theta_n(x - x_0)) - g(x_0 + \theta_n(x - x_0))\right| < \frac{\epsilon}{3},
\end{aligned}$$

whenever $0<|x-x_0|<\delta$ and $n\geqq N$.

In addition, by the convergence of the sequences involved, there are positive integers $N_1(x)$ and N_2 such that

$$\left|\frac{f_n(x) - f(x)}{x - x_0}\right| < \frac{\epsilon}{3} \quad \text{whenever} \quad n \geqq N_1(x),$$

$$\left|\frac{f(x_0) - f_n(x_0)}{x - x_0}\right| < \frac{\epsilon}{3} \quad \text{whenever} \quad n \geqq N_2.$$

Thus, all three inequalities hold whenever $n\geqq$maximum $(N, N_1(x), N_2)$ and $0<|x-x_0|<\delta$. Hence,

$$\begin{aligned}
&\left|\frac{f(x) - f(x_0)}{x - x_0} - g(x_0 + \theta_n(x - x_0))\right| \leqq \left|\frac{f_n(x) - f(x)}{x - x_0}\right| + \left|\frac{f(x_0) - f_n(x_0)}{x - x_0}\right| \\
&\qquad + \left|\frac{f_n(x) - f(x)}{x - x_0} + \frac{f(x_0) - f_n(x_0)}{x - x_0} + \frac{f(x) - f(x_0)}{x - x_0} - g(x_0 + \theta_n(x - x_0))\right| \qquad (1) \\
&\qquad < 3\cdot\frac{\epsilon}{3} = \epsilon.
\end{aligned}$$

Finally,

$$f'(x) = \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x\to x_0} g(x_0 + \theta_n(x - x_0)) = g(x_0).$$

The last equality holds by virtue of the continuity of $g(x)$ at x_0 and the bounded-

ness of $\{\theta_n\}$. The fact that, in (1), θ_n depends on x is of no consequence since it is sufficient that (1) holds for each x in $(x_0-\delta, x_0+\delta)$ and some θ_n and that the set $\{\theta_n\}$ is uniformly bounded.

We see that Theorem 4 is a generalization of Theorem 1 by noting that, if a sequence converges uniformly on an interval $[a, b]$, then it converges uniformly at each point in the interval.

THEOREM 5. *Let $\{f_n(x)\}$ be a sequence of functions satisfying the following conditions*: (1) *Each $f_n(x)$ has a continuous derivative on the interval $[a, b]$*, (2) *$\sum f_n(x)$ converges to $f(x)$ at each point in the interval $[a, b]$, and* (3) *$\sum f_n'(x)$ converges to $g(x)$ at each point in the interval $[a, b]$. Then, if $[c, d]$ is any subinterval of $[a, b]$, there is a point x_0 in $[c, d]$ such that $f'(x_0)$ exists and is equal to $g(x_0)$.*

Proof. This theorem follows from Theorems 2 and 4.

It can be shown that the hypothesis of Theorem 5 implies that the points of $[a, b]$ for which the derivative of $f(x)$ does not exist and equal $g(x)$ form a set of the first category. This, of course, is a stronger result than Theorem 5.

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ON THE DISCONJUGACY OF SECOND-ORDER LINEAR DIFFERENTIAL EQUATIONS

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Let I be an interval on the x-axis with endpoints a and b $(a<b)$. I is not assumed to be closed and we denote the closure of I by I'. Let $r(x)$ and $p(x)$ be real-valued functions which are continuous over I', and assume that $r(x)>0$ there. We consider the self-adjoint second-order differential equation

$$(r(x)y')' + p(x)y = 0. \tag{1}$$

This equation is said to be disconjugate on I if and only if no nontrivial solution of (1) vanishes more than once on I. This terminology we introduced by Wintner [4]. In this note we shall show that, by applying a modified Prüfer transformation to a solution of (1), we can obtain certain sufficient conditions for (1) to be disconjugate on I. Such a transformation was introduced by Barrett [1] to study the oscillation properties of (1) over an infinite interval. It was also used by Moore in [3], although in a slightly different form from that used by Barrett.

Let $w(x)$ be a positive, continuously differentiable function over I'. Let $y(x)$ be a nontrivial solution of (1) over I'. Since $y(x)$ and $y'(x)$ cannot vanish for the same value of x the function $\rho(x)$ defined by $\rho=rw^{-1}(w^2r^{-2}y^2+y'^2)^{1/2}$ is positive, and continuously differentiable, over I'. Thus a function $\theta(x)$ is determined by

$$y(x) = \rho(x)\sin\theta(x), \qquad y'(x) = w(x)r(x)^{-1}\rho(x)\cos\theta(x), \tag{2}$$

where we may assume that $-\pi/2<\theta(a)\leqq\pi/2$. $\theta(x)$ is continuously differentiable over I'. This is true even when $y'(x)=0$: in fact, for such an x, $\cos\theta(x)=0$ and $\sin\theta(x)=1$, and so $\theta'(x)=w(x)\rho(x)/p(x)y(x)$. Using (1) and (2) we find that

$$\theta' = \frac{1}{2}\left(\frac{w}{r}+\frac{p}{w}\right)+\frac{1}{2}\left(\frac{w}{r}-\frac{p}{w}\right)\cos 2\theta+\frac{w'}{2w}\sin 2\theta. \tag{3}$$

This equation was obtained by Barrett in [1]. Our principal result is the following one.

THEOREM 1. *The equation* (1) *is disconjugate on I if there is a function $w(x)$ which is positive and continuously differentiable over I' and for which*

$$\left|\frac{w}{r}+\frac{p}{w}\right|+\left|\frac{w}{r}-\frac{p}{w}\right|+\left|\frac{w'}{w}\right|\leqq\frac{2\pi}{b-a} \tag{4}$$

for all x in I, with strict inequality for some x in I if b belongs to I.

Proof. Suppose the equation (1) is not disconjugate on I, and let $y(x)$ be a solution of (1) for which $y(a)=0$ and $y'(a)>0$. By the definition of disconjugacy there is a solution $y_1(x)$ of (1) which has at least two zeros, x_1 and x_2 ($x_1<x_2$), in I. If $y_1(x)$ is a multiple of $y(x)$, then $y(x)$ has zeros at a and x_2. If $y_1(x)$ and $y(x)$ are linearly independent, then by the Sturm separation theorem ([2], p. 119), $y(x)$ has a zero between x_1 and x_2. In either case then, $y(x)$ vanishes at some point of I other than a. Let c be the first point to the right of a for which $y(c)=0$. Then we must have $y'(c)<0$.

From (2) we see that $y(x)$ vanishes when and only when $\theta(x)$ is an integral multiple of π. Hence $\theta(a)=0$. Also, $\theta(c)=k\pi$, and k must be odd since $y'(c)<0$. Thus $\theta(x)=\pm\pi$ for some x in I other than a. However, this is impossible if, in the (t, x)-plane, the graph of $t=\theta(x)$ stays within the triangle with vertices $(a, 0)$ and $(b, \pm\pi)$ for all x in I. In fact, if b does not belong to I, this graph may coincide with a side of this triangle. This will be the case as long as $|\theta'(x)|\leqq\pi/(b-a)$ for all x in I, with strict inequality for some x in I if b belongs to I. The condition stated in the theorem then follows from (3).

If we assume that $p(x)>0$ and that $p(x)$ and $r(x)$ are differentiable over I' and then set $w(x)=(p(x)r(x))^{1/2}$ we see that (1) is disconjugate on I if

$$\left(\frac{p}{r}\right)^{1/2}+\frac{1}{4}\left|\frac{r'}{r}+\frac{p'}{p}\right|\leqq\frac{\pi}{b-a}$$

for all x in I, with the usual condition if b belongs to I. If we again assume $p(x)>0$ over I', then there is a positive constant d such that $p(x)r(x)\geqq d^2$ for all x in I, and if we choose $w(x)=d$ we see that (1) is disconjugate on I if $p(x)\leqq\pi d/(b-a)$ for all x in I, with the usual condition if b belongs to I.

Stronger results can be obtained if we replace the sides of the triangle in the proof of Theorem 1 by the graphs of certain functions. The following general result is proved in exactly the same way as Theorem 1.

THEOREM 2. *Let $f(x)$ be a monotone, differentiable function over the interval I' such that $f(a)=0$ and $f(b)=\pm\pi$. Then the equation* (1) *is disconjugate on I if there is a function $w(x)$ which is positive and continuously differentiable over I' and for which*

$$\left|\frac{w}{r}+\frac{p}{w}\right|+\left|\frac{w}{r}-\frac{p}{w}\right|+\left|\frac{w'}{w}\right| \leqq 2|f'|$$

for all x in I, with strict inequality for some x in I if b belongs to I.

If we let $f(x)$ be either of the functions of x found by solving for t in $t^2=\pi^2(x-a)/(b-a)$, we obtain, by means of Theorem 2, the following criterion: (1) is disconjugate on I if there is a positive, continuously differentiable function $w(x)$ over I' such that

$$\left|\frac{w}{r}+\frac{p}{w}\right|+\left|\frac{w}{r}-\frac{p}{w}\right|+\left|\frac{w'}{w}\right| \leqq \frac{\pi}{(b-a)^{1/2}(x-a)^{1/2}} \tag{5}$$

for all x in I, with strict inequality for some x in I if b belongs to I. We shall now show that this criterion is actually stronger than the criterion of Theorem 1. Let I be the interval $0<x<\pi$ and consider the equation $y''+p(x)y=0$ there, where $p(x)$ is continuous over I'. If for any x_0 in I we have $p(x_0)>1$, then (4) is not satisfied at x_0 by any positive differentiable function w. To show this, note that (4) may be written $w^2+p+|w^2-p|+|w'|\leqq 2w$. If, at x_0, $p\leqq w^2$, this becomes $2w^2+|w'|\leqq 2w$, which cannot hold since $w>1$ at that point, and if $p>w^2$, this becomes $2p+|w'|\leqq 2w$, which cannot hold since $p>w^2$ and $p>1$ imply $p>w$. Thus, if the equation is disconjugate on I, this fact cannot be determined from Theorem 1.

If we set $w=1/2$, (5) becomes $1+4p+|1-4p|\leqq 2(\pi/x)^{1/2}$. Take $c>0$ and $0<d<\pi/2$ and define p by setting $p(x)=-(3+4c)x/4d+1+c$ for $0\leqq x\leqq d$ and $p(x)=1/4$ for $d\leqq x\leqq\pi$. Then for $d\leqq x\leqq \pi$, $1+4p+|1-4p|=2\leqq 2(\pi/d)^{1/2}$. For $0\leqq x\leqq d$, $1+4p+|1-4p|=8p$. Now $8p(d)=2<2(\pi/d)^{1/2}$, and the slope of the graph of $2(\pi/x)^{1/2}$ at d is $-(\pi/d^3)^{1/2}$ and the lefthand slope of the graph of $8p$ at d is $-2(3+4c)/d$. Therefore, since $2d^{1/2}(3+4c)<\pi^{1/2}$ for c and d small enough, and since the graph of $2(\pi/x)^{1/2}$, for $0<x\leqq d$, lies above its tangent line at $x=d$, we have $8p<2(\pi/x)^{1/2}$ for $0<x\leqq d$. Thus, for c and d small enough the equation in question is disconjugate on I. However, for sufficiently small x, $p(x)>1$, so this disconjugacy is not a consequence of Theorem 1.

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A NOTE ON EXTREME VALUES OF A FUNCTION OF SEVERAL VARIABLES

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In this note we consider the problem of discriminating between extreme values and saddle points of a function of n variables, in so far as this can be done from a knowledge of the first and second derivatives of the functions involved. The case where the variables are all independent is considered in the first part. We give a proof of the result which is a variant of a standard proof. In the second part the method is extended to the case where the n variables are connected by $m(<n)$ equations, and Lagrange's undetermined multipliers are used to determine stationary values. A precise result is obtained in this case as well.

Let x denote the vector $(x_1, \cdots, x_n)$, with real components, $f(x)$ a real function of x. Set $f_r(x)=\partial f(x)/\partial x_r$ and $f_{rs}(x)=\partial^2 f(x)/\partial x_r \partial x_s$. The norm of x, namely $\sqrt{(x_1^2+\cdots+x_n^2)}$ will be denoted by $|x|$. $f(x)$ has an extreme value at $x=z$ if there exists a positive real number ϵ such that $0<|y|\leqq\epsilon$ implies that $f(z+y)-f(z)$ is of like sign. If $f_r(x)=0$ for all r, and if for every ϵ, $f(z+y)-f(z)$ takes negative as well as positive values on the surface of the sphere $|y|=\epsilon$, then z is a saddle point of $f(x)$.

Our first result is this:

If $f(x)$, $f_r(x)$ are all differentiable functions at $x=z$, and $f_r(z)=0$ for all r, then $f(x)$ has an extreme value at $x=z$ if the matrix $\|f_{rs}(x)\|$ is definite; $f(x)$ has a saddle point at $x=z$ if the matrix is neither definite nor semidefinite.

We quote the following standard result, the proof of which will be omitted.

If $\|a_{rs}\|$ is a symmetric matrix, and if M and m are its greatest and least characteristic roots, then the maximum and minimum values of $y'(a_{rs})y$ on the surface of the sphere $|y|=\epsilon$ are $M\epsilon^2$ and $m\epsilon^2$. It follows that if M and m are of the same sign, the matrix is definite but it is semidefinite if either is zero.

Now set $\phi(t)=f(z+ty)$, where t is a real scalar; then

$$\phi(1)-\phi(0)=f(z+y)-f(z) \tag{1}$$

and we have

$$\phi'(t)=\sum_{r=1}^{n} y_r f_r(z+ty). \tag{2}$$

Since $f_r(x)$ is differentiable at $x=z$,

$$f_r(z+ty)=f_r(z)+t\sum_{s=1}^{n} y_s f_{rs}(z)+to(|y|).$$

Thus since $f_r(z)=0$ for all r, we have from (2): $\phi'(t)=ty'(a_{rs})y+to(|y|^2)$, where $a_{rs}=f_{rs}(z)$. Hence $\phi(1)-\phi(0)=\int_0^1\phi'(t)dt=\frac{1}{2}y'(a_{rs})y+o(|y|^2)$. The result now follows from (1) and the lemma.

We now consider the case where the variables $x_1, \cdots, x_n$ are not independent but are connected by $m(<n)$ equations: $\psi^k(x)=0$; $k=1, \cdots, m$ is a suffix and not an exponent. $\psi^k(x)$ and $\psi^k_r(x)$ are differentiable functions of x for each k and r. Using Lagrange's undetermined multipliers $\alpha_1, \cdots, \alpha_m$, set $F(x) = f(x) - \sum_{k=1}^{m} \alpha_k \psi^k(x)$ and form the n equations $F_r(z)=0$. Together with the m equations $\psi^k(z)=0$ these determine z and the values of the α_k.

Set $\phi(t)=F(z+ty)$ and $A_{rs}=F_{rs}(z)$. As before, we get

$$\phi(1) - \phi(0) = \tfrac{1}{2}y'(A_{rs})y + o(|y|^2). \tag{3}$$

Since $\psi^k(z+y)=\psi^k(z)=0$ for the values of y under consideration, we have $\phi(1)-\phi(0)=f(z+y)-f(z)$. Hence, from (3) we get

$$f(z+y) - f(z) = \tfrac{1}{2}y'(A_{rs})y + o(|y|^2). \tag{4}$$

Now $f(x)$ certainly has an extreme value at $x=z$ if the matrix $\|A_{rs}\|$ is definite. It may, however, have an extreme value even if $\|A_{rs}\|$ is not definite or semidefinite. The reason is that the vectors y in (4) are not completely arbitrary. If $v^k=(\psi^k_1(z), \cdots, \psi^k_n(z))$ and $y\rightarrow 0$ in such a way that $z+y$ always satisfies the conditions $\psi^k(z+y)=0$, then the scalar product $(y, v^k)=o(|y|)$. Thus in (4), y can be taken to be orthogonal to each of the v^k. If $u_1, \cdots, u_{n-m}$ are $n-m$ independent vectors, each orthogonal to all the v_k, then any other vector orthogonal to every v_k is a linear combination of the vectors $u_1, \cdots, u_{n-m}$. If U is the matrix whose columns are $u_1, \cdots, u_{n-m}$ and $w=(w_1, \cdots, w_{n-m})$ is an arbitrary vector with $n-m$ components, then any vector y orthogonal to every vector v_k is of the form $y=Uw$. Hence from (4) it is sufficient if $w'U'(A_{rs})Uw$ is of the same sign. Thus for $f(x)$ to have an extreme value it is sufficient for the square symmetric matrix $U'(A_{rs})U$ of $n-m$ rows and columns to be definite. Further, if this matrix is not definite or semidefinite then there is no extreme value at the point z.

THE INEQUALITY OF STEENSHOLT FOR AN n-DIMENSIONAL SIMPLEX

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Let P be an arbitrary point inside the S_n simplex of n-dimensional space. Let A_i $(i=1, \cdots, n+1)$ be the vertices of the simplex, and $S^{(i)}_{n-1}$ the $(n-1)$-dimensional boundary simplex opposite to A_i $(i=1, \cdots, n+1)$. Let us denote the distance of the point P from the vertex A_i by $\overline{PA}_i=R_i$, and the distance of the point P from the boundary simplex $S^{(i)}_{n-1}$ by $\overline{PS}^{(i)}_{n-1}=r_i$. Furthermore, let C_n be the n-dimensional content of S_n, and $C^{(i)}_{n-1}$ be the $(n-1)$-dimensional content of $S^{(i)}_{n-1}$ $(i=1, \cdots, n+1)$. Then the following inequality holds:

$$\sum_{i=1}^{n+1} C^{(i)}_{n-1}R_i \geqq n \sum_{i=1}^{n+1} C^{(i)}_{n-1}r_i. \tag{1}$$

NOTES ON DIFFERENTIAL GEOMETRY

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We define an ovaloid as a convex closed surface with continuous nonvanishing principal curvatures ($K>0$). Such a surface has two parallel tangent planes in every plane direction (opposite tangent planes). When the distance between opposite planes is the same for all directions, we call the ovaloid a surface of constant width.

A surface of constant width has the following properties:

(i) the principal directions at opposite points are parallel;

(ii) the sums of the principal radii of curvature that correspond to parallel principal directions at any pair of opposite points is constant;

(iii) all its normals are double [1];

(iv) the mean curvatures at opposite points are equal [2].

Conversely, any surface which has property (iii) or (i) and (ii) is of constant width [1].

On the basis of the above, we shall discuss further properties of the surface and prove the following theorems.

THEOREM 1. *Let K and K' be the Gaussian curvatures of the surface of constant width μ at any pair of opposite points P, P' on the orthogonal section of any cylinder circumscribed about the surface, and K_n and K_n', the normal curvatures at P, P' in the direction of the orthogonal section. Then*

$$\frac{\pi K_n}{K} + \frac{\pi K_n'}{K'}$$

is constant and equal to the perimeter of the orthogonal section.

Proof. Let the lines of curvature be taken as the parametric curves $v=$constant on the surface of constant width, the corresponding principal radii of curvature and principal curvatures being ρ_a, ρ_b and K_a, K_b, respectively. Let us assume that if $\mathbf{N}$ is the unit normal vector at a point, then the unit normal vector $\mathbf{N}'$ will be chosen in such a way that $\mathbf{N}=-\mathbf{N}'$. Furthermore, let θ be the angle between the direction $v=$constant and the generators of the cylinder at the point P and ψ the angle between the direction $v=$constant and the direction of the orthogonal section at the same point. Then the radius of curvature R of the orthogonal section is [1]

$$R = \rho_a \cos^2 \theta + \rho_b \sin^2 \theta,$$

that is,

$$\begin{aligned} KR &= K_b \cos^2 \theta + K_a \sin^2 \theta \\ &= K_b \sin^2 \psi + K_a \cos^2 \psi \\ &= K_n \qquad \text{(by Euler's theorem).} \end{aligned}$$

Since the principal directions at P, P' are parallel, the angle will be the same at P, P'. Hence we have also $K'R' = K_n'$. But [1], $R + R' = \mu$ and therefore we obtain $(K_n/K) + (K_n'/K') = \mu$. Hence

$$\frac{\pi K_n}{K} + \frac{\pi K_n'}{K'} = \pi\mu. \tag{1}$$

Now, the number on the right of (1) is equal to the perimeter of the orthogonal section of the circumscribed cylinder and it is constant for any such cylinders about the surface of constant width μ. Hence our theorem is proved.

THEOREM 2. *If the magnitude of the normal component of the sum of the vector curvatures of the lines of curvatures through any point P of a surface of constant width μ is denoted by n_P and that at the opposite point P', by $n_{P'}$, then*

$$n_P = n_{P'}, \tag{2}$$

each being equal to twice the mean curvature of the surface at P or P'.

Proof. Let the surface have the parametrization as before and let the vector curvatures of $v = \text{constant}$ and $u = \text{constant}$ be denoted by $\mathbf{k}_1$ and $\mathbf{k}_2$, respectively. We may decompose $\mathbf{k}_i$ $(i = 1, 2)$ into two components, $\mathbf{k}_{in}$ normal to, and $\mathbf{k}_{ig}$ tangential to, the surface. Then the sum of the vector curvatures of the lines of curvature may be written

$$\begin{aligned} \mathbf{k}_1 + \mathbf{k}_2 &= \mathbf{k}_{1n} + \mathbf{k}_{1g} + \mathbf{k}_{2n} + \mathbf{k}_{2g} = \mathbf{k}_{1n} + \mathbf{k}_{2n} + \mathbf{k}_{1g} + \mathbf{k}_{2g} \\ &= K_a\mathbf{n} + K_b\mathbf{n} + \mathbf{k}_{1g} + \mathbf{k}_{2g}. \end{aligned}$$

Its normal component is $K_a\mathbf{n} + K_b\mathbf{n}$ and hence we obtain

$$(2J =)\ K_a + K_b = n_P.$$

Similarly, we have at P',

$$(2J' =)\ K_a' + K_b' = n_{P'}.$$

Thus, by virtue of (iv), we obtain (2) and our theorem is proved.

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CLASSROOM NOTES

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CONCERNING DOMAINS OF REAL FUNCTIONS

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By the domain of a function ϕ, I mean the set of all x for which $\phi(x)$ exists.

In modern textbooks of calculus the concept of domain is emphasized when real functions are defined, but is apt to take a back seat thereafter. For example, in Begle's [1] a theorem

$$\{\phi + \psi\}' = \{\phi' + \psi'\}$$

is stated (the prime denotes differentiation). This is not quite true: the most that is proved is $\{\phi+\psi\}' \supseteq \{\phi'+\psi'\}$. The subject is evidently a tricky one.

The usual instances of functions with restricted domains are too trivial to be effective. If ϕ is defined by

$$\phi(x) = 2x + 5 \quad \text{whenever} \quad 0 \leqq x \leqq 2,$$

so having for domain the closed interval $[0, 2]$, the restriction appears artificial. If a student is told that the domain of the function ψ defined by

$$\psi(x) = x^{-1} \text{ whenever the right-hand side exists}$$

is the set of all *nonzero* numbers, he merely thinks "Of *course* it is."

It is therefore interesting that a paradox in maxima-and-minima depends on the domain of a function not being properly determined. The paradox is Example 2 on page 517 of Thomas [3]: To find the least value of $x^2+y^2+z^2$ when $x^2-z^2-1=0$.

We argue that the least value will be a local minimum; so we eliminate one variable, say z, from the minimand via the given equation, and look for local minima of the result; *i.e.*, of $2x^2+y^2-1$. Its partial derivatives are $4x$ and $2y$; these are simultaneously zero only when $x=0$ and $y=0$, . . . and then z is undefined. Geometrical intuition suggests that there *is* a minimum, so we try again, this time eliminating x instead of z, and find minima at $(1, 0, 0)$ and $(-1, 0, 0)$. But this does not save the paradox; there is no logical reason why the second try should necessarily avoid the snag.

We must define "local minimum" carefully. There are two reasonably common definitions:

1. ϕ has a local minimum at a if there is an open interval N containing a such that $\phi(a) \leqq \phi(x)$ whenever x is in N.

2. ϕ has a local minimum at a if there is an open interval N containing a such that $\phi(a) \leqq \phi(x)$ whenever x is in N and *also in the domain of* ϕ.

The first definition is equivalent to that in Courant [2]; the second to that in Begle [1].

The difference shows most often at the end of an interval: if

$$\alpha(x) = +\sqrt{(1-x^2)} \text{ with domain } [-1, 1],$$

then α has local minima at 1 and -1 under Definition 2 but not under Definition 1.

It is now obvious how to define a local minimum of a function under a given condition. *E.g.*:

1*. ϕ has a *local minimum* at (a, b) under the condition $\mathfrak{P}$ if $\mathfrak{P}(a, b)$ is true and there is an open interval N containing (a, b) such that $\phi(a, b) \leqq \phi(x, y)$ whenever (i) (x, y) is in N and (ii) $\mathfrak{P}(x, y)$ is true.

2*. is the same but for the addition of "and (iii) (x, y) is in the domain of ϕ."

It is also clear how to define a minimum under a condition:

ϕ has a *minimum* at (a, b) under $\mathfrak{P}$ if $\mathfrak{P}(a, b)$ is true and $\phi(a, b) \leqq \phi(x, y)$ whenever (i) (x, y) is in the domain of ϕ, and (ii) $\mathfrak{P}(x, y)$ is true.

The highbrow method of finding minima when the condition is expressible by equations is by Lagrange's multipliers. I am interested in lowbrow methods; I do not mean nonrigorous, but simple and obvious. The usual lowbrow method is to eliminate variables, so getting a function of fewer variables, and to find the minima of this. In our example, this function was χ, where

$$\chi(x, y) = 2x^2 + y^2 - 1. \tag{1}$$

Let us consider the general case: the minima of $\phi(x_1, \cdots, x_{m+n})$ under n conditions $\psi_i(x_1, \cdots, x_{m+n}) = 0$. We can eliminate the last n variables if the equations are equivalent to $x_i = \xi_i(\mathbf{x})$ for i from $m+1$ to $m+n$ where $\mathbf{x}$ is short for $x_1, \cdots, x_m$. Let us just consider this case; there is not much loss of generality. If the function got by elimination is χ, then

$$\chi(\mathbf{x}) = \phi(\mathbf{x}, \xi_{m+1}(\mathbf{x}), \cdots, \xi_{m+n}(\mathbf{x}))$$

and its domain is the set of all $\mathbf{x}$ for which $(\mathbf{x}, \xi_{m+1}(\mathbf{x}), \cdots, \xi_{m+n}(\mathbf{x}))$ is in the domain of ϕ. With this definition of χ it is easy to prove that ϕ has a minimum at $(a_1, \cdots, a_{m+n})$ under the given conditions if and only if χ has a minimum at $\mathbf{a}$ and $a_i = \xi_i(\mathbf{a})$ for i from $m+1$ to $m+n$.

In our example, we now see that formula (1) for χ is incomplete. It needs the legend "for every x and y for which there is a z such that $x^2 - z^2 - 1 = 0$ and $x^2 + y^2 + z^2$ is defined." This is clearly equivalent to "for every x and y for which $|x| \geqq 1$." Minima are to be suspected (i) where the partial derivatives are all zero, (ii) where they do not all exist, and (iii) on the boundary of the domain of χ. Here (i) and (ii) yield nothing; (iii) yields $x = 1$ or -1, $z = 0$, y arbitrary. Investigating our suspects, we find that $y = 0$ does give minima.

Thus, although minima of χ correspond to those of ϕ under $\mathfrak{P}$ if Definition 1 is used, local minima do not necessarily correspond to local minima. In fact,

whether a minimum of ϕ under $\mathfrak{P}$ corresponds to a local minimum of the function we get by elimination may depend on which variables we eliminate.

The lowbrow method works if students are taught to look for all three types of suspected extrema. Either definition of local extremum, properly presented, will lead them to do so, *provided* that the domain of χ is not neglected.

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SOME SIMPLE EXAMPLES OF GROUPS*

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Generally speaking, it would seem advisable to leave "track-covering" to Rogers' Rangers when they are being pursued by hostile Senecas, and to lay out, for our students to follow with ease, the paths by which we attain our mathematical results. Occasionally, however, in order to manufacture examples and exercises, we may be justified in keeping some professional secrets, at least for a time. For instance, in first presenting to students the concept of an abstract group, we may find it useful to have at hand a supply of systems with exotic operations, without immediately divulging their source.

Examples. Such systems are easily constructed by establishing isomorphisms with groups consisting of certain subsets of the complex numbers, in which the operation is either addition or multiplication. A few examples will make clear the process:

I. Let $\{C_0; \cdot\}$ be the multiplicative group of all complex numbers except 0. Set up the correspondence $x \leftrightarrow x'+1$. We thus obtain the Abelian group $\{C_{-1}; *\}$, in which C_{-1} is the set of all complex numbers except -1, and $*$ is defined by $u * v = uv+u+v$.

II. Let $\{C; +\}$ be the additive group of all complex numbers. Set up the correspondence $x \leftrightarrow k/x'$, for any complex $k \neq 0$. We thus obtain the Abelian group $\{C_0'; *\}$, in which C_0' consists of the set of all complex numbers except 0, augmented by another element which we denote by "∞," and $*$ is defined by

$$u * v = \frac{uv}{u+v}, \quad u, v \text{ both different from } \infty;$$

$$u * \infty = u.$$

III. Applying the correspondence $x \leftrightarrow (x')^3$ to $\{R; +\}$, the additive group of reals, we obtain the group $\{R; *\}$, where $u * v = \sqrt[3]{(u^3+v^3)}$.

IV. Application of the correspondence $x \leftrightarrow (x'+1)^2$ to $\{C_0; \cdot\}$ leads again

* The author gratefully acknowledges stimulating conversations with R. G. Long, C. O. Oakley, and E. C. Schlesinger.

to the group of Example I.

V. If we apply the correspondence $x \leftrightarrow x'+1$ to the group $\{1, 2, 3, 4; \times\}$ in which the operation is multiplication, mod 5, we are led to the group $\{0, 1, 2, 3; *\}$, in which $u * v$ represents $uv+u+v$, mod 5.

It is instructive for the student to search for subgroups of these "artificial groups." In doing so, he may remind himself that $\{0; \cdot\}$ constitutes a group.

Generalizations. A detailed analysis of the "linear isomorphisms" of Examples I and II seems worthwhile.

Consider the correspondence

$$x \leftrightarrow \frac{ax' + b}{cx' + d}, \tag{1}$$

where a, b, c, d are complex numbers, $ad-bc\neq 0$, $a\neq 0$, $c\neq 0$.

I'A. If (1) is applied to $\{C_0; \cdot\}$ we obtain the Abelian group $\{S; *\}$, where S consists of the symbol ∞, together with all complex numbers except $-b/a$ and $-d/c$, and $*$ is defined as follows:

(i) For $u\neq\infty, v\neq\infty, u*v=(Luv+M(u+v)+N)/(Puv+Q(u+v)+R)$, if the denominator is not equal to zero; $u*v=\infty$ otherwise.

(ii) For $u\neq\infty$, $u*\infty=(Lu+M)/(Pu+Q)$, if $Pu+Q\neq 0$; $u*\infty=\infty$ otherwise.

(iii) Finally, $\infty*\infty=L/P$, if $P\neq 0$; $\infty*\infty=\infty$ otherwise.

In the foregoing equations, $L=a^2d-bc^2$; $M=bd(a-c)$; $N=bd(b-d)$; $P=ac(c-a)$; $Q=ac(d-b)$; $R=ad^2-b^2c$.

I'B. If $c=0$ in the correspondence (1), we are led to the Abelian group $\{S'; *\}$, where S' consists of the set of all complex numbers except $-b/a$, and $*$ is defined by

$$u * v = (Luv + M(u + v) + N)/R.$$

If $a=0$, $c\neq 0$, we obtain nothing new.

II'A. If (1) is applied to $\{C; +\}$ we obtain the Abelian group $\{S''; *\}$, where S'' consists of the symbol ∞, together with all complex numbers except $-d/c$, and $*$ is defined as in I'A, with the values of the constants as follows: $L=bc^2-2acd$; $M=-ad^2$; $N=-bd^2$; $P=ac^2$; $Q=bc^2$; $R=2bcd-ad^2$. No special attention is required if $a=0$.

II'B. If $c=0$ in II'A, we are led to the Abelian group $\{C; *\}$, where $u*v=(M(u+v)+N)/R$.

It is clear that the complex numbers could be replaced by the reals, or by the rationals, throughout the discussion of I' or II' without altering the validity of the results. An attempt to reverse the processes of I' and II' leads to interesting consequences.

Suppose that an operation $*$ is defined by

$$u * v = \frac{Luv + M_1u + M_2v + N}{Puv + Q_1u + Q_2v + R}\cdot$$

It can be verified that, if $*$ is to be associative and nontrivial, M_1 must equal M_2 and Q_1 must equal Q_2, *i.e.*, the operation must be commutative. Kuwagaki showed [1] that

$$u * v = \frac{Luv + M(u + v) + N}{Puv + Q(u + v) + R} \tag{1}$$

is associative if and only if

$$\text{rank of} \begin{bmatrix} M & L - Q & P \\ N & M - R & Q \end{bmatrix} = 1. \tag{2}$$

If condition (2) is satisfied, the operation defined by (i), modified as in (ii) and (iii) to accommodate the symbol ∞, permits an identity element and an inverse to such element.

The group to be defined depends upon the nature of the simultaneous zeros of the numerator and denominator of (i). In the extended complex plane, the conics

$$Lxy + M(x + y) + N = 0, \qquad Pxy + Q(x + y) + R = 0,$$

may have

(α) Two distinct finite points of intersection, (p, q) and (q, p);
(β) A doubly-counting finite point of intersection, (r, r);
(γ) Two doubly-counting ideal points of intersection; or
(δ) The entire ideal line in common.

In case (α), the system $\{T; *\}$ is a group if T is the set consisting of ∞ and all complex numbers except p and q, and if $*$ is defined by (i), (ii), and (iii). It can be verified that there are just two linear correspondences which effect isomorphisms of this group with $\{C_0; \cdot\}$—these are given by $x \leftrightarrow k(x'-p)/(x'-q)$ and $x \leftrightarrow (x'-q)/\{k(x'-p)\}$ for a particular k which can be computed. Example IV illustrates a "nonlinear" isomorphism.

In case (β), the system $\{T; *\}$ is a group if T is the set consisting of ∞ and all complex numbers except r, and if $*$ is defined as before. There are indefinitely many linear correspondences which effect isomorphisms of this group with $\{C; +\}$; these are given by $x \leftrightarrow \{k(Mx'+N)\}/(x'-r)$, unless $M=0=N$, in which case $x \leftrightarrow \{k(Px'+L)\}/x'$. In both instances k is arbitrary.

The situation for cases (γ) and (δ) are similar to those for (α) and (β), respectively. The values of a, b, c, d in the correspondence are most easily expressed in terms of the constants in (i); they will not be given explicitly here.

Some curious results appear if one examines subgroups in these systems. For

example, let $\{T; *\}$ be the group consisting of the symbol ∞ and all complex numbers except i and $-i$, the operation $*$ being given, for the elements other than ∞, by

$$u * v = \frac{uv - 1}{u + v} \cdot$$

In accordance with the foregoing analysis, this group is isomorphic to $\{C_0; \cdot\}$, a correspondence being given by

$$x \leftrightarrow \frac{x' - i}{x' + i} \quad \text{or} \quad x' \leftrightarrow -\frac{i(x + 1)}{x - 1} \cdot$$

The same correspondence establishes an isomorphism between $\{R_0; \cdot\}$, the multiplicative group of all reals except 0, and $\{T'; *\}$, where T' consists of ∞ and all the real multiples of i except $\pm i$. But the system $\{R'; *\}$, in which R' consists of ∞ and all the reals, is a group which is not isomorphic to any subgroup of $\{R_0; \cdot\}$ or of $\{R; +\}$.

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A "STATIC" APPROACH TO DERIVATIVES

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This note is a sequel to my earlier article, *A new approach to limits* [1], but can be read quite independently of it and is intended to be much less formal.

One of the most fundamental situations we have to consider in analysis is the following: $f(x)$ is a function of the real variable x, a is a real number and there is a real number L such that

(1) *for all real numbers x sufficiently close to a, $f(x)$ is arbitrarily close to L.*

It is pleasant to image a competition in which entrants would be invited to compose a short sentence which could be used as a summary description of the situation (1); unfortunately the first prize has already been awarded to the sentence

(2) *"the limit of $f(x)$ as x tends to a is L."*

Unless we share Professor Menger's doubts about the local status of x (see his article [2]) we cannot quarrel with this award on logical grounds; but there is room for some discussion of the psychological and pedagogical effects of this standard terminology. There can be no doubt that generations of schoolmasters and university lecturers have been influenced by it to describe the limit situation in kinematic or organic terms; informal discussions in books and classrooms are packed with such phrases as "x approaches a," "$f(x)$ gets closer to L,"

"h becomes smaller," and so on—the very name "limiting process" has an organic ring to it.

As a result of these discussions the sentence (2) must often conjure up in students' minds a picture rather like the following. A red bead, called x, moves along the x-axis towards a, while a white bead slides along the curve $y=f(x)$ always remaining vertically above the red one; as the red bead approaches a—which by some mysterious self-denying ordinance it never reaches—the white bead approaches the point (a, L). It may well be that such moving pictures enable students to compute limits correctly; but it is extremely doubtful whether they can ever lead to any proper appreciation of the limit situation (1), which involves no reference to motion, variation, growth or diminution—and very properly not, if we accept Frege's contention that variation can only occur in time, while analysis has nothing to do with time. To sum up, one is tempted to say that the familiar intuitive description of limits to which we are led by the conventional terminology fails to give any understanding of the fundamental situation.

Professor Menger has proposed replacing (2) by "the limit of f at a is L," and in my earlier article I have suggested that the "fundamental limit situation" which we ought to consider is rather more general than (1); I shall recall the details a little later. The reasons which led to these proposals were unconnected with the preceding discussion—roughly speaking, Menger wanted to eliminate x, while I wanted every function to have a limit at every point. But it is unlikely that either of us will win acceptance for our suggestions unless we can provide an informal and intuitive description of them which will take the place of the beloved moving beads.

What I offer here is not an intutive description of this general limit itself, but rather a new approach to the tangent problem, which provides a very natural motivation for the introduction of the limit notion. So let f be a function, a a real number; our problem is to find a definition for the tangent to the graph of f at the point $(a, f(a))$ on it. A mathematically unsophisticated student, who is sure that he knows perfectly well what the tangent is, will find it hard to stomach such a bald statement of the problem; but his acceptance can be gained —and our mathematical consciences salved—if we tell him that what we are looking for is a precise and workable definition which will reflect formally his intuitive notion of the tangent; certainly it is unsatisfactory to say that our problem is "to find the gradient of the tangent."

We now proceed as follows. Let h be any positive real number, and for every nonzero real number k between $-h$ and h consider the chord joining $(a, f(a))$ to $(a+k, f(a+k))$. This chord has gradient

$$Cf_a(k) = \frac{f(a+k) - f(a)}{k}.$$

If we examine some simple cases, where the graph of the function has an "obvi-

ous" tangent, there is little difficulty in convincing even the most reluctant student that in these cases this "obvious" tangent—the line which he feels intuitively ought to be called the tangent—lies somewhere between the steepest and flattest of all these chords. He can also be persuaded to agree that this is so for every positive real number h, no matter how small.

Thus for these simple cases we should be inclined to say that the gradient of the "obvious" tangent lies between the maximum and minimum of the numbers $Cf_a(k)$ corresponding to nonzero numbers k between $-h$ and h. But this statement needs a slight refinement: for the set of numbers $Cf_a(k)$ may not actually have a maximum or minimum—for example, if $f=I^3$ (the cubing function) and $a=0$, then $Cf_0(k)=k^2$ and the set of numbers k^2 corresponding to the *nonzero* numbers k between $-h$ and h has no minimum. The set of numbers in question will, however, have a greatest lower bound m_h and a least upper bound M_h, which are the formal substitutes for the informal notions of minimum and maximum. We can say, then, that the gradient of the "obvious" tangent lies between m_h and M_h for every positive real number h.

This now suggests the following general definition. If f is any function, a a real number, m_h and M_h the greatest lower and least upper bounds of the numbers $Cf_a(k)$ for all nonzero k between $-h$ and h, and if there is exactly one real number L which lies between m_h and M_h for all positive real numbers h, then we define the tangent to the graph of f at $(a, f(a))$ to be the line through that point with gradient L; if there is more than one number lying between m_h and M_h for all h, then the graph does not have a tangent at $(a, f(a))$.

Since we may regard the numbers $Cf_a(k)$ as the values of a function Cf_a, "the chord function of f at a," we now see that our informal discussion makes the approach to limits which I described in [1] appear a very natural one. In a simplified form, which will suffice for our present purpose, the method is as follows. If ϕ is any function, b a point of accumulation of the domain of ϕ, then the *limit set* of ϕ at b, $\lim_b \phi$, is the collection of real numbers common to all the sets $\mathrm{Cl}(E_h)$ for every positive real number h. Here E_h denotes the set of values assumed by ϕ for all real numbers between $b-h$ and $b+h$ except b itself, and $\mathrm{Cl}(E_h)$ is its closure, *i.e.*, the set obtained by adjoining to E_h all its points of accumulation. If the limit set consists of a single real number L, then we say that ϕ is *convergent* at b to L. I have shown in [1] that in this case ϕ has the limit L at b in the classical sense.

This method is perfectly workable in practice and specific results can in many cases be computed directly: the limit of $\sin/I$ at $0(\lim_{x\to 0} (\sin x/x)$ in the classical notation), for instance, can be obtained by slight modifications of the familiar "approach" procedure. It is possible also to derive directly from the definition the usual rules for the limits of the sum and product of two functions which are convergent at a point b, so that all the familiar limit results can be obtained.

Now in general the closure of E_h need not be the whole interval $[m_h, M_h]$. But if ϕ is convergent to L at b, it is not difficult to see that L is the only real

number common to all the intervals $[m_h, M_h]$; the converse is, of course, trivial. Thus we can sum up by saying that if the chord function of f at a, Cf_a, converges at 0 to L, then we define the tangent to the graph of f at $(a, f(a))$ to be the line through that point with gradient L.

There is nothing new about the result; all that is new is the method by which we have reached it. I claim three advantages for this method: first, it is completely "static"; second, the informal and intuitive descriptions with which we can introduce it actually run parallel to the precise, formal procedure, and are not, as in the classical approach, a misleading kinematic picture of an essentially static situation; and third—this from experience—it is eminently teachable.

References

1. Iain T. Adamson, A new approach to limits, Math. Gaz., vol. 42, 1958, p. 87.
2. Karl Menger, What are x and y? Math. Gaz., vol. 40, 1956, p. 256.

A USEFUL INTEGRAL FORMULA

DONALD K. PEASE, University of Connecticut, Hartford

Integrals of the type $\int e^{ax} \sin bx dx$ are bothersome because the formulas are confusing to remember and direct integration is tedious to perform. The formula developed here is relatively easy to remember and can be used in a variety of situations.

Consider $\int f(x)g(x)dx$ where $f''(x)=hf(x)$ and $g''(x)=kg(x)$. Thus f (and of course g) is circular or hyperbolic sine or cosine, or else exponential. Note that $k\int g(x)dx=g'(x)$. Integration by parts gives

$$k\int f(x)g(x)dx = f(x)g'(x) - \int f'(x)g'(x)dx.$$

Again integrating by parts, we have

$$k\int f(x)g(x)dx = f(x)g'(x) - f'(x)g(x) + h\int f(x)g(x)dx.$$

Solving algebraically for the integral we have the formula

$$\int f(x)g(x)dx = \frac{1}{h-k}\left[f'(x)g(x) - f(x)g'(x)\right] + C \qquad h \neq k.$$

Examples are:

$$\int e^{ax} \sin bx dx = \frac{1}{a^2+b^2}\left[ae^{ax}\sin bx - be^{ax}\cos bx\right] + C.$$

$$\int \cosh ax \sin bx dx = \frac{1}{a^2+b^2}\left[a \sinh ax \sin bx - b \cosh ax \cos bx\right] + C.$$

MATHEMATICAL EDUCATION NOTES

EDITED BY JOHN A. BROWN, University of Delaware, AND
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THE EDUCATION OF MATHEMATICS TEACHERS*

Geometry

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Any comments upon the geometry which teachers should study must be based upon some assumptions regarding the point of view and the contents of the geometry that they will teach. I shall assume that in the near future there will be a greater use of algebra in geometry, especially in the earlier introduction and use of coordinates, that a first course in geometry will treat both plane and solid geometry, that methods of proof and methods of solving problems will gradually gain greater recognition than formal proofs of specified theorems.

The traditional training of people to teach geometry seems to me to be as follows: The first course in college mathematics has a varied content but usually includes coordinate plane geometry, in some cases coordinate geometry of three space. If the first course is calculus, the analytic geometry is given only an incidental or practical treatment relative to the calculus. The traditional and, from many points of view, misnamed course of college geometry extends skills in constructions and relationships among traditional geometric figures. When there is a methods course this usually includes a scanty review and discussion of plane geometry. A course in the foundations of geometry can be very helpful and is becoming more common.

Excluding the last-named course, I suggest that we need a definite reorientation of our point of view and probably our course content if we are to prepare geometry teachers to do more than perpetuate the routines to which they have been subjected. At present it seems to me that we do reasonably well in developing skills for solving traditional problems. We must also develop both appreciation and skill regarding geometry as a deductive system, the algebraic solutions of geometric problems as in coordinate geometry, the place of secondary school geometry in modern geometry, in modern mathematics, and in our present society.

As an example of the place of coordinate geometry and its utility, consider this theorem about two chords in the same or equal circles: if we have unequal chords, the larger chord is nearer the center. In the traditional synthetic approach the proof of this theorem often involves a theorem about two triangles

* Edited transcriptions of talks by Meserve, Walter, and Fender, given as part of a panel discussion at the annual meeting of the New Jersey Section of the Mathematical Association of America on November 1, 1958. Transcriptions of the panel talks by Meder, Lorch, and Tucker appeared in this MONTHLY, vol. 66, pp. 805–809.

having two sides of one equal to two sides of the other and the relative sizes of the third side, assuming a certain order relation of the included angles. I believe that this is the only common application of the theorem on varying an angle of a triangle. From the point of view of coordinate geometry, the theorem about chords in the circle can be reduced very simply to the statement that if unequal quantities are subtracted from equal quantities, the differences are unequal in the opposite sense. If you think about it, you will see that it falls out very readily at a level of activity that can be done very nicely in the tenth grade.

The training of teachers should include a discussion of geometric figures as they are recognized in elementary schools, as they are introduced or described in junior high school, and as sets of points, lines and planes. This training should include Cartesian coordinate systems on a number line and on a plane. We should teach the extension of synthetic and coordinate geometry to three space and the coordinate geometry of higher dimensions. A comparison of the structure of high school geometry sequences and of the proofs of theorems in several of these sequences seems desirable. Common and advanced problems should be solved using methods of synthetic geometry, coordinate geometry, locus theorems and, at least in the fifth-year program, vector spaces.

I shall on several instances make reference to what I shall call a fifth-year program. We should not feel that the education of any teacher is completed at the end of four years. There are several topics that might well be introduced in the undergraduate program with the details considered more thoroughly in the fifth year.

Also in the background of the teacher, it seems to me, we should have the algebraic basis for classical constructions; the content and structure of secondary school mathematics; the use of visual aids in teaching geometry; geometry as a deductive system including finite postulational systems and isomorphic representations. Also included should be the hierarchy of geometries that one obtains by considering topology; then, as a special case, real projective geometry; as special cases, affine geometry and the non-Euclidean geometries; as a special case of affine geometry, the geometry of similar figures; and, again as a special case, Euclidean geometry. It seems to me that at least the characteristic properties of these geometries can be considered at the undergraduate level; the detailed development may need to be postponed until the fifth year.

Another topic that I have found useful and desirable can be considered from either of two points of view: the development of Euclidean geometry from projective geometry or the gradual generalization of Euclidean geometry to obtain projective geometry. In either case the treatment should include both synthetic and analytic methods. There should also be a discussion of the dependence of various theorems and properties of Euclidean geometry upon the postulates for projective geometry and the assumptions that one makes in specializing projective geometry to obtain Euclidean geometry.

Each prospective teacher should study the historical evolution of our concept of geometry. The treatment should include early empirical procedures,

the development of postulational systems, the role of artists in the development of descriptive and projective geometries, the role of algebra in the development of analytic (or coordinate) geometry, and, at least in the fifth year, the role of calculus in the development of differential geometry and the role of logic in the development of geometries as deductive systems.

Our undergraduate program should include some non-Euclidean geometry and other geometries. There should be at least a comparison of Euclidean geometry and the geometry on a sphere, recognizing that spherical geometry is not a non-Euclidean geometry but that there is an elliptic geometry of diameters of a sphere analogous to the geometry on the ideal plane. The fifth year program should include polarities as a basis for the "scratch equation" and the development of the non-Euclidean geometries from projective geometry.

Topology should be studied—at least the Jordan Curve Theorem, the four-color problem, traversable networks, the Moebius strip and, in the fifth year, a detailed study of homomorphic figures.

Finally, the program should include geometry as a study of properties of figures invariant under a group of transformations—for example, the rigid motions, translations, rotations (point and line), and, at least in the fifth year, orthogonal line reflections. The matrix representation of transformations is very effective.

It seems clear that such a breadth of background in geometry cannot be covered in a single 3 semester hour course. At the University of Illinois and also at Montclair State College about 8 semester hours are needed for reasonable thoroughness in the treatment of the background and a very brief treatment of methods. Some of the participants at the Mideast Regional State College Conference on Science and Mathematics Teacher Education* last March felt that they could not spare one fourth of the student's college mathematics program for geometry. I propose to you that the one-fourth figure is reasonable for prospective teachers of secondary school mathematics.

In conclusion, I would like to say that the discussions of the teaching of geometry at the International Congress of Mathematicians in Edinburgh this summer were the liveliest that I participated in. The problems seem to be similar in practically all countries.

Probability and Statistics

R. M. WALTER, Douglass College, Rutgers, The State University

The Commission on Mathematics of the College Board has done a great deal of the background thinking on the teacher training problem as it refers to probability theory and statistical inference, and has published its findings in the form of a proposed textbook to be used in teaching probability and statistical inference to high school students. If we are to think of the training of teachers in this field it is of interest to note briefly what the Commission thinks these teach-

* A report of the Mideast Conference may be found in the June 1958 issue of *School Science and Mathematics.*

ers should be teaching. The Commission's book contains, first, a very brief treatment of descriptive statistics. It then turns to probability theory which it introduces first in an intuitive way and then in a rather more formal way. In introducing it, it makes use of some of the algebra of sets and the concept of a sample space, and I might say in this connection that if there is any place in the high school curriculum where the use of sets will fall into place easily and naturally it is here. I know of no neater way to develop such ideas as conditional events, mutually exclusive events, etc., than by use of set notions and set algebra.

The ideas of random variable and probability distribution are introduced although the only distribution which is developed and exploited to any extent is the binomial distribution. Then, under the heading of statistics and applications, the first application described is that of acceptance sampling. Some development of the notion of a test of a hypothesis is presented and, finally, in an additional chapter which was added to the book during the past summer, some study is included of the properties of means of samples from a finite population and, using the ideas here developed, a very brief introduction to principles of statistical estimations based on sample means is given.

The introduction of this subject into the high schools, which is proposed, presents a difficult problem from the point of view of the in-service teachers. For these teachers most of the needed training will have to be provided through extension courses and summer institutes. On the other hand it would help greatly if we could get across to these people that, when they go away on vacation and take along a book to read, they might well make it a mathematical book. I would like to make a couple of suggestions along that line. These will be things that high school teachers should genuinely enjoy reading.

The first suggestion I would offer would be the book by Kemeny, Snell and Thompson, *Introduction to Finite Mathematics*. In this readers will find elementary probability theory presented in a setting in which it is preceded by symbolic logic and some of the algebra of sets. They will see probability theory developed further than is the case in the high school course that I described. Simple stochastic processes, particularly Markoff processes, are also discussed. All this is done in such a way that a high school teacher should find it enjoyable and profitable reading. I would further recommend that high school teachers who think they may someday teach probability and statistical inference, read *Elements of Probability Theory* by Harold Cramer. This book goes rather thoroughly into the foundations of probability theory. It develops the notions of random variables and their distributions much more fully than anything I have mentioned heretofore, including the well-known continuous distributions which are used in statistical inference.

In considering the college program at the advanced undergraduate or major level for college undergraduates who expect to become teachers there will be some variation. That is, there will be some students who have never had any work in probability and statistics and there will be others who may have had

some introduction to it. In any event, the equivalent of, let us say, a year's work in this area, if that much time can be allowed, certainly belongs in the undergraduate curriculum at this time. I believe I would describe an ideal course using currently available material as consisting of about one semester in which they would study as much as they could of William Feller's *Introduction to Probability Theory and Its Applications* and then an additional semester using a book on statistics which would emphasize random variables and their distributions and applications. An example of the sort of book that I have in mind is the well-known text by Paul Hoel; there are others available at about the same level. Possibly not all students will be able to include as much work in this area as I have called for here. If they have to be content with one semester's training in statistics beyond what they might get at freshman level, the book by Cramer which I mentioned before contains what I would regard as the subject matter that should be presented.

Effect of Computing Machinery

F. G. FENDER, Rutgers, The State University

I have been asked to tell you what effect the increasing use of automatic computers should have on the training of future teachers. To do this, I must give you some idea of what a computer does, and how it is made to do it. A computer is simply an information converter; that is, it operates on input data to produce output data. The input data are strings of alphabetic or numeric symbols which tell the machine what to do, how to do it, and what to do it with. Internally, the symbols are converted into a common code; and only by the function which each performs can a letter be told from a number, or a machine instruction from a piece of working data. The resulting apparent confusion is tolerated because of the increased flexibility that allowed an instruction to be treated as a number, and, by using arithmetic, to be altered to form a new instruction for a new situation. In addition to the four obvious operations of arithmetic, a computer uses instructions which take in or give out information, which separate the symbol strings into smaller units or combine them into larger groupings, or which use predetermined conditions to choose alternative sequences of events. This makes of the computer much more than just a fast desk calculator, for it can complete tasks that might be called clerical, or editorial, or even executive.

The obvious use for a computer is to do long and tedious calculations. In such work, the processes of both the differential and integral calculus are reduced to their finite difference approximations. Thus a problem in partial differential equations might be reduced to a set of simultaneous linear algebraic equations, or an integral might be reduced to the parabolic rule, essentially to a process of addition.

It is less obvious, but more important for our discussion, that a computer may be used to handle problems that are linguistic in nature. The Univac was used to produce, in record time, the Concordance of the Revised Standard Version of the Bible. The IBM 650 is being used for language studies of the Dead

Sea Scrolls and of Latin manuscripts. Translations from Russian into English are on the verge of becoming successful. Similarly, the technique used in translation from the machine code of one computer to that of another is quite well known, using either computer for the translation. Exact relationships of any form may be handled, such as the rules for differentiation of the elementary transcendents, or the determination of the greatest common factor of two polynomials. It is in this general area of linguistics that I would expect to find the most vigorous development, the most difficult problems, and the greatest interest.

Many college students, upon meeting a computer for the first time in a senior course, have a startling new experience. The machine has zero intelligence quotient, and zero emotional quotient. It is incapable of guessing what the student intended, and it will merrily carry out any task, no matter how useless. Those students who have failed to develop a personal sensitivity to mathematical thinking often begin to mature with this experience. Such shock therapy would be beneficial for prospective teachers of secondary students.

What kind of person is needed to control a computer? You may be startled to have a mathematician tell you that a computer installation does not always need mathematicians. But the installation does need people who can think clearly and who can perform the mathematician's supreme function of defining a problem in a form that leads to a solution. Often this function can be supplied by a major in philosophy, or history, or a language, who has a broad outlook, and who is well disciplined in his field. Personality is important too, for the men and women who keep a computer busy depend on one another so much there is no place for the surly boor who must work alone. Generally we need the type of mind that is interested in intellectual puzzles or games and for which every day brings a fresh learning experience.

How should such a person be trained? Let me say first that he should not be untrained; that is, he should not have his curiosity dulled. Particularly in arithmetic, a teacher's enthusiasm is as important as knowledge, and the impact of decimals, of compound fractions, and so on, can be electrifying to a child. Second, he should be allowed to experience the thrill of hard work well done. This leads to an inner intellectual honesty that is often difficult to gain, and it leads to the type of discipline that is needed to run a computer or do any research in any field. Third and last, to help our teachers make learning a vivid and thrilling experience we must do our part to upgrade the difficult and often scorned profession of teaching, particularly the teaching of all levels of mathematics. We can do this, in part, by telling the public what it is we do when we do mathematics, and, in part, by helping to create the intellectual atmosphere in which teaching is a noble and rewarding profession.

Summer Fellowships for Secondary-School Teachers

The National Science Foundation has announced a program of fellowships for secondary-school teachers of science and mathematics. Under these fellowships, teachers in secondary schools may apply for one, two, or three summers' study in a college or university of their choice. The fellowships will be awarded to support individually planned programs of graduate-level study in the mathematical, physical, and biological sciences. It is not necessary that the fellow be enrolled for or complete an advanced degree, but it is necessary that his studies be at the graduate level.

Applications for fellowships should be sent to: Secondary-School Fellowships, American Association for the Advancement of Science, 1515 Massachusetts Avenue, N.W., Washington 5, D. C. The deadline for filing applications is January 15, 1960, and awards will be announced on March 15. The universities or colleges chosen by the fellows for their studies will be notified at a later date.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, University of Maine

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, University of Maine, Orono, Maine. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTION

E 1391. *Proposed by N. S. Mendelsohn, University of Manitoba*

Let $m_0, m_1, \cdots, m_r$ be positive integers which are pairwise relatively prime. Show that there exist $r+1$ consecutive integers $s, s+1, \cdots, s+r$ such that m_i divides $s+i$ for $i=0, 1, \cdots, r$.

E 1392. *Proposed by R. K. Guy, University of Malaya, Singapore, Malaya*

I am confronted by stamp machines dispensing supplies of 3 cent and 4 cent stamps. I notice that if I have a sufficient supply of coins I can obtain stamps to the value of any whole number of cents except for 1 cent, 2 cents, and 5 cents. If the machine dispenses only stamps of values m cents and n cents, how many exact values in cents can I not make up?

E 1393. *Proposed by D. S. Mitrinovitch, University of Belgrade, Belgrade, Yugoslavia*

Evaluate the nth order determinant $D=|a_{ik}|$ for the two cases: (1) $a_{ik}=0$ when $i+k$ is even, (2) $a_{ik}=0$ when $i+k$ is odd.

E 1394. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Given a trihedral angle and a point within it, construct the plane through the point which intercepts on the trihedral angle the tetrahedron of minimum volume.

E 1395. *Proposed by Albert Wilansky, Lehigh University*

Which of the sequences whose nth terms follow are convergent? $1/(n \cos n)$, $1/(n^2 \cos n)$, $1/(n^3 \cos n)$, $\cdots$; $1/(2^n \cos n)$; $1/(n! \cos n)$.

SOLUTIONS

An Inequality for a Triangle

E 1361 [1959, 312]. *Proposed by M. S. Klamkin, AVCO Research and Development*

If A, B, C are angles of a triangle, show that

$$\csc A/2 + \csc B/2 + \csc C/2 \geqq 6.$$

I. *Solution by Leon Bankoff, Los Angeles, Calif.* Consider the angle bisectors AD, BE, CF, concurrent at the incenter I of the triangle ABC. It is known that the sum of the ratios in which a point within a triangle divides the cevians of this point is never less than 6 (E 1043, 1953, p. 421). Since the inradius $r \leqq (ID, IE, IF)$, it follows that $AI/r + BI/r + CI/r \geqq 6$.

II. *Solution by Bernard Greenspan, Drew University.* Let

$$f = \csc A/2 + \csc B/2 + \csc C/2.$$

While f has no maximum value ($A/2$ can be chosen as near to 90° as desired), a minimum clearly exists. Consider A and B as independent variables. If we set $\partial f/\partial A$ and $\partial f/\partial B$ equal to zero, we see f is least when

$$\csc A/2 \cot A/2 = \csc B/2 \cot B/2 = \csc C/2 \cot C/2. \tag{1}$$

Since $\csc \phi \cot \phi$ is a strictly decreasing function as ϕ varies from 0° to 90°, (1) and $A+B+C=180°$ imply $A=B=C=60°$. Therefore the least value of f is 3 csc 30° or 6.

III. *Solution by L. D. Goldstone, New York State Public Works Lab., Albany, N. Y.* By a well-known theorem: If the sum of n angles, each positive and less than $\pi/2$, is given, the sum of the cosecants of the angles is least when the angles are all equal (Hobson, *Treatise on Plane Trigonometry*, 4th ed., 1918, p. 88). Hence for a convex n-gon, $\sum \csc A/2 \geqq n \csc \pi(n-2)/2n$. The value $n=3$ yields the desired result.

IV. *Solution by W. J. Blundon, Memorial University of Newfoundland.* In the theorem of Toth (*Lagerungen in der Ebene auf der Kugel und im Raum*, p. 11): "If O is an arbitrary point in the plane of a triangle ABC of inradius r, then $OA+OB+OC \geqq 6r$, with equality only for an equilateral triangle," we take O to be the incenter and the result follows at once.

Note. The theorem quoted is a special case of a deeper theorem with the inequality $OA+OB+OC>2\sqrt{(T\sqrt{3})}$, where T represents the area of the triangle ABC. This latter theorem is an immediate consequence of the isoperimetric property of the hexagon $AO_3BO_1CO_2$, where O_1, O_2, O_3 are the images of O in the sides BC, CA, AB of the triangle.

V. *Solution by Viktors Linis, University of Ottawa.* Applying the Erdös-Mordell inequality (this MONTHLY, 1958, p. 521) to the center I of the inscribed circle of radius r in the triangle ABC we obtain $IA+IB+IC\geqq 6r$. Since $IA=r\csc A/2$, $IB=r\csc B/2$, $IC=r\csc C/2$, the required inequality follows, with equality only for $A=B=C$.

VI. *Solution by J. S. Frame, Michigan State University.* We use the following two lemmas:

$$u-1\geqq \ln u \text{ for } u>0, \qquad \text{equality if and only if } u=1,$$

$$\cos^2 A/2=\sin^2 (B+C)/2=\sin B\sin C+\sin^2 (B-C)/2\geqq \sin B\sin C.$$

Then

$$\begin{aligned}\csc A/2-2&\geqq 2\ln [(1/2)\csc A/2]=2\ln [(\cos A/2)/\sin A]\\ &=\ln [(\cos^2 A/2)/\sin^2 A]\geqq \ln (\sin B\sin C/\sin^2 A)\\ &=\ln (bc/a^2).\end{aligned}$$

Replacing A by B and C in turn we find

$$\csc A/2+\csc B/2+\csc C/2\geqq 6+\ln (bc/a^2)+\ln (ca/b^2)+\ln (ab/c^2)=6.$$

Equality holds only for the equilateral triangle.

VII. *Solution by D. S. Passman, General Electric Co., Syracuse, N. Y.* For concave functions in general

$$\sum_1^n f(x_k)\geqq n[f(\sum x_k)]/n.$$

Thus for $f(x)=\csc x$, $f''(x)=(1+\cos^2 x)/\sin^2 x>0$, $0<x<\pi$, $n=3$, and $\sum x_k=\pi/2$, the result follows.

Also solved by A. N. Aheart, D. A. Breault, R. F. Brown and Joel Levy (jointly), Yi Chang, P. L. Chessin, Anton Glasser, Michael Goldberg, H. W. Gould, Alfred Gray, W. E. Kesler, H. C. Liu, D. C. B. Marsh, Leo Moser and E. L. Whitney (jointly), D. L. Muench, Margaret Olmsted, J. L. Pietenpol, N. R. Riesenberg, L. A. Ringenberg, D. A. Robinson, M. T. Salhab, C. M. Sandwick, Sr., J. Schopp, Jack Silver, Stoddart Smith, Jr., Charles Wexler, R. H. Wilson, Jr., and Dale Woods.

A Hula Hoop Problem

E 1362 [1959, 312]. *Proposed by Marlow Sholander, Carnegie Institute of Technology*

Consider a vertical girl whose waist is circular, not smooth, and temporarily at rest. Around the waist rotates a hula hoop of twice its diameter. Show that,

after one revolution of the hoop, the point originally in contact with the girl has traveled a distance equal to the perimeter of a square circumscribing the girl's waist.

I. *Solution by J. D. E. Konhauser, HRB-Singer, Inc., State College, Pa.* The locus of a point on a circle of radius $2a$ rolling along the inside of a circle of radius a is a cardioid of length $8a$, which, if a is the radius of the girl's waist, is the perimeter of a square circumscribing the girl's waist.

II. *Solution by Leo Moser, University of Alberta.* Consider the hoop as fixed and the poor girl whirling around (it serves her right). The original point of contact traverses the diameter of the hoop twice, and this is the required distance.

Also solved by Leon Bankoff, H. F. Batie, H. F. Bechtell, R. E. Blewster, Jr., A. P. Boblétt, Mark Bridger, J. F. Burke, J. H. Butchart, Allan Chertok, Stuart Friedman, Michael Goldberg, L. D. Goldstone, H. W. Hickey, R. T. Hood, Richard Holt, E. L. Hubbard, Wally Manheimer, Jr., D. C. B. Marsh, Helen M. Marston, C. S. Ogilvy, A. St. V. Parker-Jervis, J. L. Pietenpol, T. A. Porsching, D. A. Robinson, C. M. Sandwick, Sr., Sister M. Stephanie, Eric Sturley, J. A. Tierney, T. C. Wales, Charles Wexler, Patty Whitlock, and the proposer.

A Digital Root

E 1363 [1959, 312]. *Proposed by Leo Moser, University of Alberta*

Let numbers be written to base b where b has the form $b=r^2+1$. Given r consecutive numbers, the last divisible by r, then the digital root of their sum is $1+2+\cdots+r=r(r+1)/2$.

Solution by N. J. Fine, Institute for Advanced Study. The digital root d of a number N expressed to the base b satisfies: (i) $d\equiv N \pmod{b-1}$, (ii) $1\leqq d \leqq b-1$. If

$$N = (kr + 1) + (kr + 2) + \cdots + (kr + r) = kr^2 + r(r + 1)/2,$$

then, since $r^2=b-1$,

(i) $r(r + 1)/2 \equiv N \pmod{b - 1}$, (ii) $1 \leqq r(r + 1)/2 \leqq r^2 = b - 1$.

Thus $d=r(r+1)/2$.

Also solved by H. F. Bechtell, Mark Bridger, P. L. Chessin, Monte Dernham, J. W. Ellis, H. B. Emerson, H. W. Hickey, J. H. Hodges, J. Hooley, D. C. B. Marsh, J. L. Pietenpol, and the proposer.

Editorial Note. The case $b=10$ is an old result and was given by Nicomachus about 330 A.D.

An Area Property of Roulettes

'E 1364 [1959, 312]. *Proposed by James Serrin, University of Minnesota*

In a plane, let A denote a closed convex curve in contact with a given curve C. Also, let B denote the mirror image of A across the tangent line to C at the point of contact. Suppose that the curvatures of A, B, C permit A and B to roll without slipping along their respective sides of C. Then, as A rolls along C, let A' denote the roulette traced out by the point of A initially in contact with

C. Similarly, let B' denote the roulette generated by rolling B on C. Show that the area enclosed by an arch of A' and the corresponding arch of B' is independent of C. In particular, if A is a circle the enclosed area is just six times the area of A.

Solution by C. S. Ogilvy, Hamilton College. Let C be traversed by the moving point $x=x(t)$. For time Δt_i, the area increment ΔS_i can be approximated by a wedge-shaped quadrilateral two of whose sides are normal to C and bisected by C and the other two of which are equal respectively to Δx_i+h and Δx_i-h and are parallel to the direction of C at x_i. Such elements form a Darboux sum, independent of the form of C:

$$S = \lim_{n\to\infty,\Delta x_i\to 0} \sum_{i=1}^{n} 2\Delta x_i f(x_i),$$

where $f(x)$ is the length of the normal at x from C to either roulette.

In particular, we know that the area under one arch of a cycloid is three times the area of the generating circle. Add another three for the image circle; and we have just shown that it is not necessary that the base curve be a straight line.

Also solved by Michael Goldberg, D. C. B. Marsh, M. S. Klamkin, and the proposer.

Goldberg used the method of E 1269 [1958, 45] and its extension by R. C. Yates (this MONTHLY, vol. 66, 1959, pp. 130–135), obtaining the desired result as a limiting case of rolling polygons.

Similar arguments to the above can be given to show that the sum of the lengths of an arch of A' and the corresponding arch of B' is independent of C.

Polynomials and Sequences

E 1365 [1959, 312]. *Proposed by Melvin Hausner, Stevens Institute of Technology*

Let H be the class of polynomials $f(x)$ with rational coefficients such that $f(n)$ is an integer when n is an integer. Prove that a sequence a_n of $+1$'s and -1's is of the form $a_n=(-1)^{f(n)}$, $f(x)\in H$, if and only if it is a periodic sequence of period 2^k.

Solution by N. J. Fine, Institute for Advanced Study. If f is a polynomial of degree d, then

$$f(x) = \sum_{r=0}^{d} \Delta^r f(0) \binom{x}{r}.$$

If $f(x)$ is an integer for $x=0, 1, 2, \cdots, d$, then $\Delta^r f(0)$ is also an integer, for $r=0, 1, 2, \cdots, d$. Let $2^k>d$. Then

$$\binom{n+2^k}{r} \equiv \binom{n}{r} \pmod 2$$

for all integers n and for $r=0, 1, \cdots, d$. [This is most easily seen by equating

coefficients of x^r in

$$(1+x)^{n+2k} = (1+x)^n(1+x)^{2k} = \sum_{r=0}^{n} \binom{n}{r} x^r(1+x^{2k}) \pmod 2.]$$

It follows that $(-1)^{f(n)}$ has period 2^k.

Conversely, suppose that $a_n = (-1)^{g(n)}$ has period 2^k. Define the polynomial

$$f(x) = \sum_{r=0}^{2k-1} \Delta^r g(0) \binom{x}{r}.$$

We have $f(x) \in H$, so $(-1)^{f(n)}$ has period 2^k. But $f(n) = g(n)$ for $n = 0, 1, \cdots, 2^k - 1$, so $a_n = (-1)^{f(n)}$ for the same values of n. Since both have period 2^k, they agree for all n.

Also solved by J. H. Hodges, D. C. B. Marsh, and the proposer.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers, The State University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers, The State University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well-known textbooks or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4875. *Proposed by J. M. Gandhi, Jain Engineering College, Panchkoola, India*

Prove the following conjecture: if

$$\frac{x \cosh x}{\sin x} = \sum_{n=0}^{\infty} (-1)^{n+1} \frac{D_{2n} x^{2n}}{(2n!)},$$

then

$$D_{2n} = G_{2n} + \sum_{\substack{(p-1)\mid 2n \\ p\equiv 1(4)}} \frac{1}{p} + \sum_{\substack{p\leq 2n+1 \\ p\equiv 3(4)}} \frac{\epsilon_p}{p},$$

where the G_{2n} are integers and the ϵ_p are integers depending on p and n. Indeed an explicit formula for ϵ_p can be obtained.

4876. *Proposed by Naoki Kimura, University of Washington.*

If a real valued function $f(x, y)$ of two real variables possesses all of its partial derivatives $\partial^{m+n}f(x, y)/\partial x^m \partial y^n$ at every point, is it necessarily continuous?

4877. *Proposed by O. P. Aggarwal and Irwin Guttman, University of Alberta*

Show that

$$\int_0^a e^{-t^2/2}dt = (\pi/2 - aI)^{1/2},$$

where

$$I = \int_{a^2}^{2a} \frac{e^{-\omega/2}d\omega}{\omega\sqrt{\omega - a^2}}.$$

4878. *Proposed by D. J. Newman, AVCO Research, Lawrence, Mass.*

A collection of closed bounded convex sets is given in the plane with the property that any three of them have a point in common. Prove they all have a point in common.

4879. *Proposed by L. J. Wallen, Massachusetts Institute of Technology*

Let f be a measurable real function defined for all real x, and let G be a continuous real function of two variables. Show that if

$$|f(x + y)| \leqq G(f(x), f(y)) \text{ for all } x \text{ and } y,$$

then f is bounded on bounded set.

4880. *Proposed by A. W. Goodman, University of Kentucky*

Show that for all α, β, $(0<\alpha<\beta<\pi)$,

$$\int_0^\alpha \sqrt{\frac{\cos\theta - \cos\beta}{\cos\theta - \cos\alpha}}\,d\theta + \int_\beta^\pi \sqrt{\frac{\cos\beta - \cos\theta}{\cos\alpha - \cos\theta}}\,d\theta = \pi.$$

SOLUTIONS

Convergent Sequences

4828 [1959, 143]. *Proposed by M. S. Klamkin, AVCO Research, Wilmington, Mass.*

Do the sequences $\{a_n\}$, $\{b_n\}$, $\{c_n\}$ converge, where

$$a_{n+1} = \int_0^1 \min(x, b_n, c_n)dx,$$

$$b_{n+1} = \int_0^1 \text{mid}\,(x, c_n, a_n)dx, \qquad c_{n+1} = \int_0^1 \max\,(x, a_n, b_n)dx,$$

and $\text{mid}(a, b, c) = b$ if $a \leqq b \leqq c$?

Solution by C. H. Cunkle, Cornell Aeronautical Laboratory, Buffalo, N. Y. The proposed sequences are convergent, with the limits 3/8, 1/2, 5/8, respectively.

Evidently $\min(x, b_n, c_n) \leqq x \leqq \max(x, a_n, b_n)$ so that, if $\text{mid}(x, a_n, c_n) = x$, we have

$$\text{(1)} \qquad \min\,(x, b_n, c_n) \leqq \text{mid}\,(x, a_n, c_n) \leqq \max\,(x, a_n, b_n).$$

Now, if $\text{mid}(x, a_n, c_n) = a_n$, we have either $x \leqq a_n$ or $c_n \leqq a_n$, so that $a_n \leqq \max(x, a_n, b_n)$ implies (1) in this case also. A similar argument holds for $\text{mid}(x, a_n, c_n) = c_n$, so that (1) is true in all cases. By integration there results $a_{n+1} \leqq b_{n+1} \leqq c_{n+1}$, $n = 1, 2, \cdots$.

Now we have

$$a_{n+1} = \int_0^1 \min\,(x, b_n, c_n)dx \leqq \int_0^1 xdx = \tfrac{1}{2},$$

and similarly $c_{n+1} \geqq \frac{1}{2}$. Using this

$$b_{n+2} = \int_0^1 \text{mid}\,(x, a_{n+1}, c_{n+1})dx = \int_0^{1/2} \max\,(x, a_{n+1})dx + \int_{1/2}^1 \min\,(x, c_{n+1})dx$$
$$\leqq \int_0^{1/2} \frac{1}{2}\,dx + \int_{1/2}^1 xdx = 5/8.$$

Dually, $b_{n+2} \geqq 3/8$. Since $3/8 \leqq b_{n+2} \leqq c_{n+2}$, $a_{n+3} = \int_0^1 \min\,(x, b_{n+2}, c_{n+2})dx > 0$, and similarly $c_{n+3} < 1$.

It is now assumed that n is so large that $0 < a_n \leqq b_n \leqq c_n < 1$.

$$a_{n+1} = \int_0^{b_n} xdx + \int_{b_n}^1 b_n dx = \frac{2b_n - b_n}{2}$$

$$b_{n+1} = \int_0^{a_n} a_n dx + \int_{a_n}^{c_n} xdx + \int_{c_n}^1 c_n dx = \frac{a_n^2 - c_n^2 - 2c_n}{2},$$

$$c_{n+1} = \int_0^{b_n} b_n dx + \int_{b_n}^1 xdx = \frac{b_n^2 + 1}{2}\cdot$$

Thus

$$b_{n+2} = \frac{1}{2}\left[\left(\frac{2b_n - b_n^2}{2}\right)^2 - \left(\frac{b_n + 1}{2}\right)^2 + 2\left(\frac{b_n + 1}{2}\right)\right]$$

$$= \frac{1}{2} + \frac{(2b_n - 1)(-2b_n^2 + 2b_n + 1)}{8} = b_n - \frac{(2b_n - 1)^3 + 5(2b_n - 1)}{16}.$$

Since $0 < -2b_n^2 + 2b_n + 1$ whenever $0 < b_n < 1$, either

$$\tfrac{1}{2} \leqq b_{n+2} \leqq b_n \quad \text{or} \quad \tfrac{1}{2} > b_{n+2} > b_n.$$

If follows that $\lim b_{2n} = \lim b_{2n+1} = \lim b_n = \frac{1}{2}$. Then

$$\lim a_n = \lim a_{n+1} = \lim \frac{2b_n - b_n^2}{2} = \frac{3}{8},$$

$$\lim c_n = \lim c_{n+1} = \lim \frac{b_n^2 + 1}{2} = \frac{5}{8}.$$

Also solved by J. W. Haake and by Y. Matsuoka.

Topological Transformations

4829 [1959, 147]. *Proposed by J. de Groot, University of Amsterdam, The Netherlands, and Purdue University*

Give a simple example of a continuum P such that, for every countable group G, there exists a group of topological transformations of P onto itself which is isomorphic to G.

Solution by J. V. Whittaker, University of British Columbia. For each integer $n > 0$, let $P_n = \{(r, \theta): 0 \leqq r \leqq 1, \theta = \pi/n\}$, where r, θ are polar coordinates in the plane. Also, let $P_0 = \{(r, \theta): 0 \leqq r \leqq 1, \theta = 0\}$, and $P = \bigcup_{n=0}^{\infty} P_n$. Evidently P is a one-dimensional continuum in the relative Euclidean topology. If G is a countable group, then G is isomorphic to a group of permutations of the positive integers, and we shall identify G with the latter group. For each $g \in G$, we define a corresponding mapping h of P as follows: $h(r, \pi/n) = (r, \pi/g(n))$ for $0 \leqq r \leqq 1$ and $n \geqq 1$, $h(r, 0) = (r, 0)$. Then h is a homeomorphism of P, for h and h^{-1} are clearly open at all points of P_n $(n \geqq 1)$; as neighborhoods of a point $(r_0, 0)$ $(0 < r_0 \leqq 1)$, we can choose the intersection of the set $\{(r, \theta): r_0 - \epsilon < r < r_0 + \epsilon\}$ with all but a finite number of the P_n $(n \geqq 1)$, and the images under h and h^{-1} of such sets have the same form. The correspondence $g \leftrightarrow h$ is evidently one-to-one. If $g_1 \leftrightarrow h_1$ and $g_2 \leftrightarrow h_2$, then $h_1 h_2(r, \pi/n) = h_1(r, \pi/g_2(n)) = (r, \pi/g_1 g_2(n))$ for all $n \geqq 1$, and $g_1 g_2 \leftrightarrow h_1 h_2$. Hence G acts effectively on P.

Also solved by Helen C. Arens, R. D. Gordon, and the proposer.

Angles in a Linear Normed Space

4831 [1959, 147]. *Proposed by J. L. Massera and J. J. Schäffer, Institute of Mathematics and Statistics, Montevideo, Uruguay*

J. A. Clarkson [*Uniformly convex spaces*, Trans. Amer. Math. Soc., vol. 40, 1936, pp. 396–414] introduced the following expression for the "angle" of two

nonvanishing elements of any linear normed space:

$$\alpha[x, y] = \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|.$$

Prove that if x, y, $x+y \neq 0$, then $\alpha[x+y, x] \leqq \alpha[x, y]$.

Solution by Michael Golomb and R. M. Warten, Purdue University. To show

$$\left\| \frac{x}{\|x\|} - \frac{x+y}{\|x+y\|} \right\| \leqq \left\| \frac{x}{\|x\|} - \frac{y}{\|y\|} \right\|,$$

where x, y, $x+y \neq 0$; *put* $x/\|x\| = X$, $y/\|y\| = Y$,

$$\frac{x+y}{\|x+y\|} = Z, \qquad \frac{x+y}{\|x\| + \|y\|} = z, \qquad \frac{\|y\|}{\|x\| + \|y\|} = t.$$

Note $0 < t < 1$. Then $\|X\| = \|Y\| = \|Z\| = 1$, $\|z\| \leqq 1$, $z = \|z\| Z$,

$$\|Z - z\| = 1 - \|z\| = \|Y\| - \|z\| \leqq \|Y - z\|, \qquad (1 - t)X + tY = z.$$

Hence $\|X - Z\| \leqq \|X - z\| + \|Z - z\| \leqq [t + (1-t)] \|X - Y\| = \|X - Y\|$, as required.

Also solved by R. D. Gordon, H. L. Loeb, the proposers, and one who signs himself "A. Polter Geist."

Diophantine Equations

4832 [1959, 147]. *Proposed by A. Oppenheim, University of Malaya, Singapore*

Find all integral solutions of the following Diophantine equations

$$\text{(A)} \qquad ax^2 + ay^2 + z^2 - 2axyz - 1 = 0 \qquad (a = 1, 2, \cdots),$$

$$\text{(B)} \qquad 9x^2 + 25y^2 + 49z^2 - 210xyz - 1 = 0,$$

$$\text{(C)} \qquad 9x^2 + 25y^2 + 4z^2 - 60xyz - 1 = 0.$$

The case $a = 1$ of (A) is known.

Solution by D. C. B. Marsh, Colorado School of Mines. These equations are essentially the same in form as problem 4674 [1957, 121–122], and admit the same treatment.

1. Let $a = bc^2$ with b square-free. Let x, y, z be integers satisfying (A). Then we write

$$[z - b(cx)(cy)]^2 = [b(cx)^2 - 1][b(cy)^2 - 1]$$

which is equivalent to

$$b(cx)^2 - 1 = pq^2, \qquad b(cy)^2 - 1 = pr^2, \qquad z - b(cx)(cy) = pqr,$$

with p, q, r integers, p square-free and prime to a. It follows that integers $t = bcx$ and u exist such that

$$t^2 - (bp)u^2 = b. \tag{1}$$

Conversely, let t_i, u_i and t_j, u_j be any pair of solutions of (1) having the t's multiples of bc, and set

$$x = t_i/bc, \qquad y = t_j/bc, \qquad z = t_it_j/b + pu_iu_j.$$

That these values satisfy (A) is directly verified. In this, p is not arbitrary but must be chosen so that (1) admits solutions. A necessary condition is that a be a quadratic residue of p; a sufficient condition is that $(p+1)/a$ be a square. The determination of all solutions of the "Pell equation" is a classical problem in the theory of quadratic forms.

2. Rewrite (B) as $[7z-(3x)(5y)]^2=[(3x)^2-1][(5y)^2-1]$ whence, with p again square-free and prime to 15,

$$(3x)^2 - pq^2 = 1, \qquad (5y)^2 - pr^2 = 1, \qquad 7z = 15xy + pqr.$$

$t^2-pu^2=1$ has solutions with $t\equiv 0$ both modulo 3 and modulo 5 if and only if the fundamental positive solution, t_1, is divisible by 15, in which cases all $t_{2j-1}\equiv 0 \pmod{15}$. For such, $x=t_{2i-1}/3$ and $y=t_{2j-1}/5$, $7z=t_{(2i-1)\pm(2j-1)}$. Thus all solutions of (B) are obtained from those p and t such that $t^2-pu^2=1$ has $t_1\equiv 0 \pmod{15}$ and some $t_{2j}\equiv 0 \pmod 7$. (There are solutions, *e.g.*, $x=10$, $y=6$, $z=257$.)

3. Proceeding with (C) as with (B), one needs $(2z)=t_{(2i-1)\pm(2j-1)}$, but no such t's are ever even, hence (C) has no solutions.

Also partially solved by C. C. Yalavigi.

Coefficients of a Polynomial

4833 [1959, 147]. *Proposed by F. H. Northover, Carleton University, Ottawa*

In the polynomial $A(x)=\sum_{i=0}^{n} a_ix^i$, $a_i\geqq 0$, $A(1)=1$, and the function

$$\frac{A'(1) - xA'(x)}{(1-x)\{1-A(x)\}}$$

is expanded into $\sum_{i=0}^{\infty} b_ix^i$.* Prove $b_i\geqq A'(1)$ for all i.

Solution by Leonard Carlitz, Duke University. It is evidently necessary to assume $a_0<1$, for otherwise $A(x)=1$. Put

$$\frac{A'(1) - xA'(x)}{(1-x)(1-A(x))} - \frac{A'(1)}{1-x} = \frac{A'(1)A(x) - xA'(x)}{(1-x)(1-A(x))} = \sum_{r=0}^{\infty} c_rx^r;$$

it will suffice to show that $c_r\geqq 0$ for all r. We have

$$A'(1)A(x) - xA'(x) = \sum_{r=0}^{n} \{A'(1) - r\} a_rx^r.$$

* The upper limit is ∞, not n as originally printed.

If we put

$$\frac{A'(1)A(x) - xA'(x)}{1 - x} = \sum_{r=0}^{n-1} d_r x^r,$$

then it follows that

$$\begin{aligned} d_k &= \sum_{r=0}^{k} \{A'(1) - r\} a_r \\ &= \sum_{r=0}^{k} a_r \sum_{s=0}^{n} s a_s - \sum_{r=0}^{k} r a_r \sum_{s=0}^{n} a_s \\ &= \sum_{r=0}^{k} \sum_{s=0}^{k} (s - r) a_r a_s + \sum_{r=0}^{k} \sum_{s=k+1}^{n} (s - r) a_r a_s \\ &= \sum_{r=0}^{k} \sum_{s=k+1}^{n} (s - r) a_r a_s, \end{aligned}$$

so that $d_k \geqq 0$ for $0 \leqq k \leqq n-1$. Since we have $1 - A(x) = 1 - a_0 - A_1(x)$, where $A_1(x) = \sum_{r=1}^{n} a_r x^r$, then

$$\frac{1}{1 - A(x)} = \frac{1}{1 - a_0} + \frac{A_1(x)}{(1 - a_0)^2} + \cdots.$$

It follows immediately that

$$c_r \geqq \frac{A'(1)}{1 - a_0} \qquad (r = 0, 1, 2, \cdots),$$

which is slightly stronger than the stated result.

Also solved by R. D. Gordon, H. O. Pollak, C. C. Yalavigi, and the proposer.

Reduced Residue Systems

4834 [1959, 238]. *Proposed by Oystein Ore, Yale University*

It is well known that the number 30 is the largest integer such that the set of reduced residues (mod 30) includes no composite numbers. Determine all integers n such that the $\phi(n)$ reduced residues (mod n) are powers of primes.

Solution by N. J. Fine, The Institute for Advanced Study. Let the primes be $p_1 = 2, p_2, \cdots$ and let q_1, q_2 be the first two primes which do not divide n. A necessary and sufficient condition that n be of the required type is that $q_1 q_2 > n$. Each of the primes less than q_2 and different from q_1 divides n, and so does their product. Therefore the product of all primes less than q_2 does not exceed $q_1 n \leqq q_1^2 q_2$. If $q_2 = p_k$ then $p_1 p_2 \cdots p_{k-1} < p_{k-1}^2 p_k$. We shall show that this inequality is violated for all $k \geqq 6$. For if $p_1 p_2 \cdots p_{k-1} \geqq p_{k-1}^2 p_k$ for some $k \geqq 6$, then $p_1 \cdots p_k \geqq p_{k-1}^2 p_k^2$. Now it is known (Bertrand, Tchebysheff) that $p_j > \frac{1}{2} p_{j+1}$, so

$$p_1 \cdots p_k \geqq p_{k-1}^2 p_k^2 > p_k^2 p_{k+1}^2/16 > p_k^2 p_{k+1}$$

provided that $p_{k+1} > 16$, which is equivalent to $k \geqq 6$. But from $2 \cdot 3 \cdot 5 \cdot 7 \cdot 11 > 11^2 \cdot 13$ the inequality is violated for $k=6$ and therefore for all $k \geqq 6$. Hence $p_k = q_2 \leqq 11$, $q_1 \leqq 7$, and $n < q_1 q_2 \leqq 77$. Examination of the numbers less than 77 quickly determines the following set:

(1), 2, 3, 4, 5, 6, 8, 9, 10, 12, 14, 18, 20, 24, 30, 42, 60.

Also solved by Jean M. Calloway, J. W. Ellis, Fritz Herzog, J. H. Hodges, D. C. B. Marsh, Georgia C. Smith, and Robert Spira.

RECENT PUBLICATIONS

EDITED BY RICHARD V. ANDREE, University of Oklahoma

All books for review should be sent directly to R. V. Andree, Department of Mathematics, University of Oklahoma, Norman, Oklahoma, and not to any of the other editors or officers of the Association.

Les Limitations Internes des Formalismes. Par Jean Ladrière. (Collection de logique mathématique, Serie B, II) E. Nauwelaerts, Louvain, and Gauthier-Villars, Paris, 1957. xiii+715 pp. 650 frs. (Belgian), 7000 frs. (French). (About \$14.00.)

This book is a rather complete compendium and exposition of modern work in the logical foundations of mathematics. Philosophical material is held to a very reasonable compass. By the "internal limitations" of the formalist method (axiomatic method which includes the necessary logic) the author has in mind results of the following kinds: (1) Gödel's incompleteness theorem to the effect than any "sufficiently rich" formal system must, if consistent, have undecidable sentences (sentences A such that neither A nor not-A are provable in the system), (2) Gödel's second theorem to the effect that the sentence in the system expressing the consistency of such a system is unprovable, (3) Church's theorem concerning the nonsolvability of the decision-problem for validity in the predicate calculus of the first-order, (4) results on models of formal systems ranging from that of Löwenheim-Skolem (any consistent first-order system has a denumerable model) to that of Rosser-Wang (a certain version of set-theory, if consistent, has no "standard" models). These results are but a few of the topics covered in this lengthy essay which, discounting notes, appendices, indices, tables, etc., runs to 400 pages of text. The reviewer believes that a newcomer to the field will find it a helpful guide.

THEODORE HAILPERIN
Lehigh University

Elementary Statistical Physics. By C. Kittel. Wiley, New York, 1958. ix+228 pp. $8.00. (College edition, $6.75.)

Although statistical mechanics is one of the most important and beautiful parts of theoretical physics, students of physics have been handicapped by the fact that most recent textbooks in the field have been written largely for the use of chemists.

The present book contains an excellent selection of topics of special importance to physicists. Other topics, such as phase equilibria, imperfect gases, and cooperative phenomena, that either are mainly of interest to chemists or can be treated only by elaborate special methods, are omitted.

Part I, comprising about half of the book, is devoted to statistical mechanics. The treatment is based on Gibbs' method of ensembles. Part II treats fluctuations, noise, and irreversible thermodynamics. Part III gives an introduction to kinetic theory and transport problems. The method of steepest descent and other matters largely of mathematical interest are outlined in four appendices. A set of problems is given at the end of most sections.

The treatment is fresh and lucid throughout and very concise. Its brevity, however, is effectively offset by the insertion of references after the headings of most sections. Many of these references are to very recent publications, and Appendix C, dealing with the solution of problems in molecular dynamics by electronic computers, is based on work not yet published.

J. Rud Nielsen
The University of Oklahoma

Contributions to the Theory of Nonlinear Oscillations. Vol. IV. S. Lefschetz, Ed. Annals of Mathematics Studies No. 41. Princeton University Press, Princeton, N. J., 1958. vii+211 pages. $3.75 (paperbound).

The ten papers by eleven authors which are collected in this volume show again, as the preceding three volumes did, the great width and interest of the work which is being done on nonlinear ordinary differential equations, as well as the great variety of methods which have been used successfully to attack the vast body of problems in this field.

The contributors to this volume are: S. Kakutani and L. Markus, S. Lefschetz, D. Bushaw, R. DeVogelaere, D. L. Slotnick, W. T. Kyner, G. Seifert, H. A. Antosiewicz, P. Mendelson, and R. W. Bass. The preface, by S. Lefschetz, contains excellent brief characterizations of the ten papers.

Anybody interested in the field of nonlinear differential equations will read this volume with great profit.

Oswald Wyler
University of New Mexico

Multivariate Correlational Analysis. By Philip H. DuBois. Harper, New York, 1958. xv+202 pp. $5.00.

Multivariate Correlational Analysis is a book without substance. It is largely dealing with descriptive statistics that are poorly explained and presented and computational methods that, while satisfactory for computing values of statistics, are not suitable for modern methods of analysis. The title of the book is misleading in that it does not cover the range of methods of multivariate correlational analysis, but only two topics, linear multiple regression with attendant simple, multiple, partial and part correlations and some concepts of factor analysis.

Chapters 1 to 10 and 13 and 14 deal with the correlations associated with multiple linear regression. The concepts of population and mathematical models are omitted. Notation departs from accepted statistical usage (*e.g.*, V for variance, $\sigma=\sqrt{V}$, C for covariance). All variances (total and residual) are obtained using division by sample size. The concept of degrees of freedom is ignored throughout the book. It is not recognized that curvilinear regression derives from the same formulas as multiple linear regression under usual models of statistics. Analyses are based on z-scores and, while this is sometimes useful, it is not helpful in regression analysis. Sampling distributions enter only in Chapter 14. Then the author makes statements regarding variances of predicted values that are incorrect. Only the large-sample test of a correlation is proposed and then only when the population correlation is zero. The exact distribution under that situation is not given nor is the z-transformation of Fisher discussed.

Chapters 11 and 12 purport to deal with factor analysis. Chapter 11 is an attempt to regard regression analysis as an introduction to factor analysis. By the author's own admission, the discussion of factor analysis is incomplete and is to be regarded as an attempt at a demonstration of relationships between factor analysis and partial correlation. The demonstration doesn't quite come off.

In summary, we cannot recommend this book for either a text or a reference book. We refer the reader interested in regression analysis to other books on the subject, for example, *Statistical Theory in Research* by R. L. Anderson and T. A. Bancroft or for more correlation analysis to *Statistical Inference* by H. M. Walker and Joseph Lev.

R. A. BRADLEY
Virginia Polytechnic Institute

Computability and Unsolvability. By Martin Davis. McGraw-Hill, New York, 1958. xxv+210 pp. $7.50.

The problems of finding decision methods in logic and the analogue in mathematics of finding algorithms were advanced significantly by Gödel's arithmetization procedures and Church's and Kleene's work in the theory of recursive functions. The goal of logical investigations relative to the decision problem is to find a mechanical method of deciding whether or not a proposition is true. In conse-

quence, if the word "computation" is used in a sufficiently wide sense, the decision problem as well as the algorithm problem can be classified as computational. The problem of computability, in this sense, then, is that of determining effective mechanical procedures for solving various types of problems. Moreover, if the procedures are to be purely mechanical, one ought to be able to construct a machine (computer) that could perform the computations. This suggests the connection of the problem of computability with that of the construction of computers, and further makes it possible to define "decidability," i.e. computability, in terms of the construction of such computers. Evidently, Gödel's incompleteness theorem could then imply that there were questions such that no machine could be so programmed that the questions could be decided.

The importance of the theory of recursive functions for the problem of the classification of computers is obvious. A machine can operate only in a fashion analogous to the process of building functions recursively. The output of the machine is the analogue of the decision.

Professor Davis has tried to give us "an introduction" to the theory of computability. The book first considers the general theory, discussing computable functions and recursive functions and relating these notions to the Turing machine. The author then considers some applications of the general theory to combinatorial problems, diophantine equations and mathematical logic. The final part of the book is devoted to some further aspects of the general theory, considering the "Kleene hierarchy," computable functionals and some results on unsolvable decision problems.

Although it is good to have this book, it is not precisely an "introduction" in the sense that students are led into the field "from scratch."

LOUIS O. KATTSOFF
Boston College

Multivalent Functions. By W. K. Hayman. Cambridge Tract No. 48. Cambridge University Press, New York, 1958. viii+151 pp. $4.50.

This book presents an introduction to and a survey of certain aspects of the theory of multivalent functions mostly related to the recent developments of the author. He presupposes a knowledge of complex variables as given in Ahlfors, *Complex Analysis*, and of Lebesgue integration as given in Titchmarsh, *Theory of Functions*. The material is well organized and clearly written, containing complete proofs of all theorems but no exercises. The text makes frequent references to original sources and to related topics in the literature.

Various classes of multivalent functions are studied; for example, p-valent functions, really mean p-valent functions, and circumferentially mean p-valent functions. The book begins with the "classical" distortion theorems and coefficient inequalities for univalent functions, presented first in the usual way and then in a way which allows its extension to circumferentially mean p-valent functions. The bounds for $|f(z)|$ when f is areally p-valent are based primarily upon an area-length principle of the Ahlfors type. The distortion theorems and

coefficient inequalities for circumferentially mean p-valent functions depend upon the theory of symmetrization of Steiner, Polya, and Szego, a discussion of which is included in Chapter 4. The book closes with a presentation of the Löwner theory and its applications to univalent functions.

GEORGE SPRINGER
University of Kansas

Trigonometry. By Dorothy Rees and Paul K. Rees. Prentice-Hall, Englewood Cliffs, N. J., 1959. xi+318 pp. $3.96.

The authors state in their preface: "We try to maintain a proper balance between analytical and numerical work but lean somewhat toward the analytic side." The law of tangents, the half angle formulas, area of a triangle, and significant figures (but not slide rule) are discussed. Analytical and numerical chapters usually alternate; the exercise sets are a little shorter than usual (typically 24 problems, but see (1) below). There is almost no review of algebraic concepts; the definition of a periodic function is poorly stated and the definition of amplitude appears to have been inserted in a footnote as an afterthought, although the concept figures prominently in the exercises. The appearance of the pages is very good; the last two chapters give an introduction to complex numbers and spherical trigonometry; the book includes tables and answers to odd numbered exercises. Some unusual features of the book are: (1) Problem sets are broken up into groups of 4 very similar problems, so if there are 24 problems in a set, there are only 6 essentially different ones. (2) Each chapter has a summary, a review set of exercises, and a test-yourself set. (3) Thirteen photographs with headings show applications of trigonometry. These range from a collection of old musical instruments (sound waves show various sine curves) to an atomic electric plant (complex numbers are used to study alternating current).

B. H. ARNOLD
Oregon State College

Functions of Complex Variables. By Philip Franklin. Prentice Hall, Englewood Cliffs, N. J., 1958. ix+246 pp. $6.95.

This excellent work on the classical properties of the analytic functions of a complex variable was evidently designed to serve two purposes: (1) to provide mathematics majors at the advanced undergraduate level or the lower graduate levels an introduction to the specific subject treated and perhaps to analysis in general, and (2) to provide for physicists and engineers a basic text and a reference work on the fundamentals of the subject. In the reviewer's opinion it is admirably suited to both of these programs.

The exposition is rigorous but not tiresome, and suitably modern but definitely not pedantic. Apparently, this book was very carefully composed and its structure is such that the beginner can attain a solid knowledge of the subject by working through a series of stages (81 articles in all) of fairly uniform and moderate difficulty. There are a few subtleties that the novice may miss but if the

reader is ready for the subject and willing to study carefully he will be well rewarded for his effort. There are numerous misprints but they are obvious and easy to correct.

Each chapter begins with a brief sketch of the path to be followed and is broken up into short articles (usually one or two pages in length) interspersed with sets of problems—40 sets with a total of 645 problems. The problems are well graded and serve to emphasize the cardinal points and extend the theory. Among the subjects treated are: products, quotients, powers, roots, continuity, derivatives, Cauchy-Riemann equations, Laplace's equation, infinite series, power series, Riemann surfaces, conformal transformations, linear and bilinear transformations, Cauchy's integral theorem, the Cauchy-Goursat integral theorem, multiply connected regions, the Cauchy integral formula, Taylor's expansion, Morera's theorem, the maximum principle, Liouville's theorem, analytical continuation, Laurent's series, residues, and Schwarz-Christoffel transformations.

HOMER V. CRAIG
Boeing Airplane Company and
The University of Texas

BRIEF MENTION

Mathematics Dictionary. Edited by Glenn James and Robert C. James. 2nd ed. Van Nostrand, Princeton, N. J., 1959. 546 pp. $15.00.

The new revised edition of the James and James *Mathematics Dictionary* is certainly one of the most welcome publications in a long time, particularly since coverage of modern mathematical concepts has been greatly extended. Perhaps the most welcome addition to the book is the multilingual index which gives the English equivalents of mathematical words in Russian, German, French, and Spanish—a most welcome addition. This reviewer feels that even if you already possess the 1949 edition, the second edition represents sufficient expansion and extension to merit replacement of the earlier volume.

Siam Review. Edited by Robert J. Wisner. A Publication of the Society for Industrial and Applied Mathematics, Philadelphia, Pennsylvania, 1959. 87 pp. $5.00 per year.

It is a pleasure to welcome this new publication to the growing list of mathematical journals. Volume 1, Number 1 begins with an excellent article, "Systems of Linear Relations," by R. A. Good, and continues to maintain interest throughout the volume. The editor states that although only two issues are planned for 1959, four issues will be forthcoming in 1960, and the price will then be $10.00 per year.

Experimental Music. By Lejaren A. Hiller, Jr. and Leonard M. Isaacson. McGraw-Hill, New York, 1959, vi+197 pp. $6.00.

If you are interested in the possibility of programming a modern computer to compose music, that is, to select ordered sets of notes in accordance with the rules of harmony and counterpoint, this is probably the book for which you are waiting.

Portfolio Selection. By Harry M. Markowitz. Wiley, New York, 1959. x+344 pp. $7.50.

This publication of the Cowles Foundation for Research in Economics is devoted

to the use of computing machines in the selection of an investment portfolio in which sufficient diversification is sought, taking into account factors such as: likely income, and appreciation, the uncertainty of income and appreciation, and the degree to which various securities tend to rise and fall together. A series of mathematical appendices bridge the gap with which the nonmathematical economist would be confronted.

Numerical Solution of Differential Equations. By William Edmund Milne. Wiley, New York, 1953. xi+275 pp. $7.25.

Computer laboratories will certainly welcome this second printing of Milne's excellent work.

Numerical Mathematical Analysis. By James B. Scarborough. 4th ed. Johns Hopkins, Baltimore, 1958. xxi+576 pp. $6.00.

The principal changes in the fourth edition of Professor Scarborough's work, originally published in 1930, appear in Graeffe's method and in the numerical solution of ordinary differential equations. A section on the smoothing of experimental data has also been added.

Plastic and String Models, Industrial Research Laboratories, Box 471, Hempstead, New York.

This is a collection of well-made, attractive, expensive ($11.75 to $18.75 each) models of plexiglass and elastic nylon string. They are of the same general nature as students have been making for years, showing the standard quadric and quartic surfaces, helicoids, cones, and intersecting cylinders; however, these are particularly well-made and may be of interest to mathematicians who are asked to recommend a source for mathematical models.

Scientific American Books. Simon and Schuster, New York, 1955–57. $1.45 each.

The Universe. xvi+142.
Plant Life. xiii+237.
The Physics and Chemistry of Life. xi+272.
The Planet Earth. viii+168.
Twentieth-Century Bestiary. xi+243.

Although not mathematical, this little series provides excellent source books for current scientific articles, presented in concise carefully chosen phrases.

Symbolic Logic and the Game of Logic. By Lewis Carroll. Dover, New York, 1958. xxxi+96 pp. $1.50.

Lewis Carroll's writings on mathematical recreation are never old.

The Dynamics of Particles. By Arthur Gordon Webster. 2nd ed. Dover, New York, 1959. xi+588 pp. $2.35.

This is a reprint of the second (1912) edition of Webster's 1904 book.

Computational Methods of Linear Algebra. By V. N. Faddeeva, Translated by Curtis D. Benster. Dover, New York, 1959. x+252 pp. $1.95.

Congratulations to Dover on the timely appearance of this volume devoted to matrix computation. It is a methods book, not a theory book, and a most welcome one.

A Course of Pure Mathematics. By G. H. Hardy. 10th ed. Cambridge University Press, New York, 1959. xii+509 pp. $3.75.

A paperbacked edition of Hardy's well-known book.

Meson Physics. By Robert E. Marshak. Dover, New York, 1958. viii+378 pp. $1.95.

A reprint of Marshak's 1952 book based on his lectures at the University of Rochester and Columbia University in 1950.

The Measurement of Power Spectra from the Point of View of Communications Engineering. By R. B. Blackman and J. W. Tukey. Dover, New York, 1959. x+190 pp. $1.85.

A well-written modern book on applied Fourier analysis reprinted from the 1958 *Bell System Technical Journal.*

Theory of Functionals and of Integral and Integro-Differential Equations. By Vito Volterra. Dover, New York, 1959. 226 pp. $1.75.

A reprint of Volterra's 1930 book describing his 1925 lectures at The University of Madrid, with a list of publications and a 28-page biography appended.

Pillow Problems and *A Tangled Tale.* By Lewis Carroll. Dover, New York, 1958. xx+152 pp. $1.50.

Two more welcome reprints.

Trigonometric Series, Vol. 1 and 2. By A. Zygmund. 2nd ed. Cambridge University Press, New York, 1959. xii+383 pp., vii+354 pp. $15.00 each, $27.50 set.

This is a complete revision and up-to-date extension and modernization of Professor Zygmund's classic. It has more than doubled in size. Volume 1 contains the completely rewritten material from the original work on trigonometric and Fourier series. Volume 2 provides much material previously unpublished in book form, and covers trigonometric interpolation, differentiation of series, generalized derivatives, interpolation of linear operations, and additional information about Fourier coefficients, including the Littlewood-Paley function. Each volume concludes with chapter notes and Volume 2 also contains excellent bibliography.

Group Theory. By Eugene P. Wigner. Translated from the German by J. J. Griffin. Academic Press, New York, 1959. xi+372 pp. $8.80.

Both physicists and mathematicians will certainly welcome this careful translation of Wigner's original classic on the application of group theory to quantum mechanics and electron spin theory. Three new chapters have been added to the translation on Rachah coefficients, time inversion, and on the limits of representation coefficients.

Plane Trigonometry. By Raymond W. Brink. 3rd ed. Appleton-Century-Crofts, New York, 1959. xii+228+110 pp. $4.00.

This well-known book needs no review other than notice that a new edition has appeared. This edition retains the flavor of its predecessors.

Mathematical Puzzles and Pastimes. By Aaron Bakst. Van Nostrand, Princeton, N. J., 1954, vi+206 pp. $4.00.

A new printing of an old favorite which belongs on every shelf of recreational mathematics.

Reflections of a Mathematician. By L. J. Mordell. Canadian Mathematical Congress, Montreal, Canada, 1959. vii+50 pp. $1.00.

This heart-warming book is the outgrowth of an after-dinner speech given to the Royal Society of Canada in 1955. It should, in my opinion, be read by every budding mathematician as well as savored by those already past the budding stage. Taste, insight, and care have gone into its preparation.

Foundations of Combinatorial Topology. By L. S. Pontryagin. Graylock, Rochester, N. Y., 1952. xii+99 pp. $4.00.

This translation from the Russian by F. Bagemihl, H. Komm, and W. Seidel deserves considerably more recognition that it has been receiving. If you are seeking information at an elementary level on Betti groups, continuous mappings and fixed points, this may well be the volume you desire.

Foundations of the Nonlinear Theory of Elasticity. By V. V. Novozhilov. Graylock, Rochester, N. Y., 1953. vi+233 pp. $4.00.

Another translation by Bagemihl, Komm, and Seidel which merits consideration.

Advanced Calculus. By Edwin Bidwell Wilson. Dover, N. Y., 1958. xi+566 pp. $2.45.

A paperback edition of the classical *Advanced Calculus* by Wilson which will be welcomed by many readers.

Mathematics for the Academically Talented Student in the Secondary School. Edited by Julius H. Hlavaty. National Education Association of the United States, Washington, D. C., 1959. 48 pp. Single copy $.60. Quantity discounts available.

NEWS AND NOTICES

EDITED BY LLOYD J. MONTZINGO, JR., University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to L. J. Montzingo, Jr., Mathematical Association of America, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

EMPLOYMENT REGISTER

The Mathematical Sciences Employment Register, established by the American Mathematical Society, the Mathematical Association of America, and the Society for Industrial and Applied Mathematics, will be maintained at the meeting in Chicago, Illinois, January 27–29, 1960 from 9 a.m. to 5 p.m. The Placement Service Desk will be on the third floor of the Conrad Hilton Hotel. There will be no charge for registering either to job applicants or to employers, except when the late registration fee for employers is applicable. Provision will be made for anonymity of applicants on request.

Job applicants and employers wishing to be listed will please write to the Employment Register, 190 Hope Street, Providence 6, Rhode Island, for forms which must be completed and returned to Providence not later than January 8, 1960, in order to be included in the listings of the Chicago meeting free of charge. Forms which arrive after the closing date, but before January 26, will be listed for a late registration fee of $3.00, and will also be included in the sold listings, but not until ten days after the meeting. Printed listings will be available for sale during and after the meeting.

It is essential that applicants and employers register at the Placement Service Desk promptly on arrival to receive the call number which has been assigned to them.

SUMMER EMPLOYMENT IN MATHEMATICS

It is planned to compile a listing of Summer Employment for Mathematicians and College Mathematics Students. Institutions and industrial firms who would welcome inquiries for summer employment for 1960 should write to the Employment Register, 190 Hope Street, Providence 6, Rhode Island, indicating the number of openings, description of positions, and officer to whom applications should be directed.

SIAM VISITING LECTURESHIPS

The Society for Industrial and Applied Mathematics announces the establishment of a Visiting Scientist Lectureship Program which will serve the dual purpose of familiarizing college and university groups with contemporary mathematical activity in applied and industrial settings, and of making available representative research mathematicians, active in areas of current interest, to industrial groups for visiting lectures and discussions.

A grant of the National Science Foundation supports the part of the program which is aimed at colleges and universities, making it possible to provide the broadest geographical coverage. The visiting scientists will be prepared to give general lectures, to discuss more specific topics with individual classes and seminars, to advise in individual or group conferences on questions of curriculum, research in progress, or opportunities and requirements in the field of applied mathematics, and generally to cooperate in every way with the colleges in furthering the aims of the program. A normal visit will last for two or three days. Contributions to the program will be sought from the institutions visited in the form of a portion of the traveling expenses and subsistence for the visitor while on campus, but no institution will be denied visits solely because it cannot afford to make such a contribution.

The complementary part of the program will provide opportunities for professional mathematicians who have gone into industry to maintain their ties with current phases of research mathematics. Formal lectures of a survey nature, seminar talks, and non-consultative discussions of personal research interests will constitute the visitor's activity during his normally one or two day visit. It is expected that the inviting organization will in each instance bear the lecturer's costs in full.

The program is under the direction of Dr. F. J. Weyl of the Office of Naval Research, Washington 25, D. C., who is assisted by a committee of the Society appointed for this purpose. Information on the panels of lecturers and on application procedures will be available in the near future.

TEMPORARY MEMBERSHIPS FOR THE ACADEMIC YEAR 1960–61 INSTITUTE OF MATHEMATICAL SCIENCES, NYU

The Institute of Mathematical Sciences at New York University offers temporary memberships to mathematicians and other scientists holding the Ph.D. degree who intend to study and do research in the fields in which the Institute is particularly active. These fields include FUNCTIONAL ANALYSIS, FUNCTION THEORY, DIFFERENTIAL EQUATIONS, MATHEMATICAL PHYSICS, FLUID DYNAMICS AND MAGNETOHYDRODYNAMICS, ELECTROMAGNETIC THEORY and NUMERICAL ANALYSIS AND DIGITAL COMPUTING.

The temporary membership program is designed primarily to alleviate the present critical shortage of scientists trained in mathematical physics, applied mathematics, and mathematical analysis. The program is being supported by the National Science Foundation and also by funds contributed by industrial firms to New York University.

Temporary members may participate freely in the research projects, the advanced

graduate courses, and the research seminars of the Institute, and they will have the opportunity of using the computational facilities. The temporary members will receive a grant commensurate with their status. Temporary memberships are awarded for one year, but may be renewed in special cases. Appropriate arrangements can be made for applicants who expect to be on leave of absence from their institutions.

Requests for information and for application blanks should be addressed to the Membership Committee, Institute of Mathematical Sciences, 25 Waverly Place, New York 3, N. Y.

PERSONAL ITEMS

University of Alberta: Assistant Professors E. L. Whitney, G. C. Cree, and W. J. Bruce have been promoted to Associate Professors; Associate Professor Lee Lorch, Wesleyan University, has been appointed Associate Professor; Dr. J. R. McGregor, University of Birmingham, England, and Mr. R. W. Longley, Canadian Meteorological Service, have been appointed Assistant Professors.

Allegheny College: Dr. A. L. Rabenstein, U. S. Army Mathematics Research Center, University of Wisconsin, has been appointed Assistant Professor; Mr. G. S. Tsiang, Southern Illinois University, has been appointed Instructor.

American University: Mr. H. H. Chu, Oklahoma State University, and Mrs. Luceil K. Arnett have been appointed Instructors.

University of California, Los Angeles: Assistant Professor Barrett O'Neil has been promoted to Associate Professor; Associate Professor E. A. Coddington has been promoted to Professor; Drs. R. J. Blattner and H. J. Weinitschke have been promoted to Assistant Professors; Professor I. S. Sokolnikoff has been awarded a Guggenheim Fellowship for 1959–60 and will spend most of the year in Zurich; Professor P. B. Johnson, Occidental College, has been appointed Associate Professor; Dr. Leo Breiman has been appointed Assistant Professor; Dr. Tilla Klotz, New York University, has been appointed Instructor; Dr. Gunter Lumer, University of Chicago, has been appointed Acting Assistant Professor.

Case Institute of Technology: Mr. C. G. Cullen, Worcester Polytechnic Institute, has been appointed Instructor; Dr. W. J. Kammerer, University of Wisconsin, has been appointed Assistant Professor in the Operations Research Group; Professor R. C. Bose, on leave from the University of North Carolina, has been appointed Visiting Professor in Statistics; Dr. D. K. Ray-Chaudhuri has been appointed Research Associate in the Statistical Laboratory.

Colorado State University: Assistant Professor E. E. Remmenga has been promoted to Associate Professor; Assistant Professor R. H. Niemann, Worcester Polytechnic Institute, has been appointed Assistant Professor.

Dartmouth College: Professor Grace E. Bates, Mount Holyoke College, has been appointed Visiting Professor during 1959–60; Dr. V. E. Benes, Bell Telephone Laboratories, Murray Hill, New Jersey, has been appointed Visiting Lecturer during 1959–60; Dr. A. J. Fabens, Stanford University, has been appointed Research Instructor during 1959–60.

Georgetown University: Assistant Professor M. W. Oliphant has been promoted to Associate Professor; Dr. A. K. Aziz has been promoted to Assistant Professor; Mr. C. L. Strain, Purdue University, has been appointed Instructor.

Harvard University: Associate Professor J. T. Tate has been promoted to Professor; Professor Raoul Bott, University of Michigan, has been appointed Professor.

Hobart and William Smith Colleges: Professor W. H. Durfee was granted the honorary degree of Doctor of Science at the June 1959 Commencement and has retired as Professor Emeritus; Professor R. L. Beinert has been appointed Head of the Mathematics Department.

University of Idaho: Assistant Professor A. E. Labarre, Jr. has been promoted to Associate Professor; Mrs. Elna H. Grahn has been promoted to Assistant Professor; Visiting Associate Professor S. S. Shrikhande, University of North Carolina, has been appointed Associate Professor; Mr. Newman Fisher, San Francisco State College, has been appointed Instructor.

University of Illinois: Associate Professors C. R. Blyth, H. E. Vaughn, and Joseph Landin have been promoted to Professors; Assistant Professor Beulah Armstrong has been promoted to Associate Professor; Mr. C. A. Phillips has been promoted to Assistant Professor; Assistant Professor Pierre Samuel, University of Clermont-Ferrand, France, has been appointed Visiting Professor; Professor Paulo Ribenboim, Inst. de Math. Pura e Aplicada, Rio de Janeiro, Brazil, has been appointed Visiting Associate Professor; Professor S. S. Cairns is on leave of absence at the Institute for Advanced Study; Assistant Professor N. T. Hamilton is on leave of absence at Johns Hopkins University; Drs. Hiram Paley, University of Wisconsin, Herbert Wilf, Columbia University, R. L. Bishop, Roberto Diaz, and D. M. Roberts have been appointed Assistant Professors; Mr. J. J. Rotman, University of Chicago, has been appointed Research Associate.

Indiana University: Associate Professor George Whaples has been promoted to Professor; Assistant Professor A. H. Wallace, University of Toronto, has been appointed Associate Professor; Visiting Associate Professor Gopinath Kallianpur, Michigan State University, has been appointed Associate Professor.

State University of Iowa: Associate Professor H. A. Dye, University of Southern California, has been appointed Associate Professor; Dr. J. F. Jakobsen, University of Missouri, has been appointed Instructor.

Kansas State University: Associate Professor L. E. Fuller has been promoted to Professor; Miss Rochelle Abend, University of Illinois, has been appointed Instructor.

Louisiana Polytechnic Institute; Professor W. B. Temple has been appointed Head of the Department of Mathematics; Assistant Professor D. E. Johnson has been promoted to Associate Professor.

McMaster University: Assistant Professor J. H. H. Chalk has been promoted to Associate Professor; Dr. W. B. Pennington, University of London, England, has been appointed Visiting Professor for the 1959–60 academic year.

Miami University: H. S. Pollard is retiring as Chairman, but will continue to serve as Professor; Dr. R. G. Selfridge, Naval Ordnance Test Station, China Lake, California, has been appointed Associate Professor and Director of the Computing Center; Assistant Professor Kermit Hutcheson, Southern Technical Institute, has been appointed Instructor.

Michigan State University: Assistant Professors M. Isobel Blyth, H. E. Campbell, W. E. Deskins, and G. P. Weeg have been promoted to Associate Professors; Dr. O. E. Taulbee, Lockheed Aircraft Corporation, Marietta, Georgia, has been appointed Associate Professor.

University of Michigan: Associate Professors D. A. Darling, M. O. Reade, R. C. Lyndon, and C. L. Dolph have been promoted to Professors; Assistant Professors F. W. Gehring, Frank Harary, and A. L. Mayerson have been promoted to Associate Professors; Dr. Edward Halpern has been promoted to Assistant Professor; Dr. A. E. Heins, John Simon Guggenheim Memorial Foundation, Copenhagen, has been appointed Professor; Dr. Morton Brown and Dr. D. A. Jones, State University of Iowa, have been appointed Assistant Professors; Drs. M. S. Ramanujan, Muslim University, Hsin Chu, Taiwan Normal University, Taipai, Formosa, Gerald Hedstrom and R. H. Rosen, University of Wisconsin, and J. M. Kister, Midwestern University, have been appointed Instructors; Drs. Stanislaw Mrowka, Polish Academy of Sciences, Moseh Shimrat, Hebrew University, Jerusalem, Israel, and R. L. Schaetz, Institute of Technology, Munich, Germany, have been appointed Visiting Lecturers; Drs. Frederick Sleator, University of Michigan Research Institute, G. P. Patil, and Messrs. B. W. Arden,

Michigan Tabulating Service, and B. Stubblefield, have been appointed Lecturers.

University of Mississippi: Colonel H. L. Quarles has returned to the Department after his retirement from the Army and has been appointed Assistant Professor; Dr. C. W. Barnes, University of Massachusetts, has been appointed Associate Professor; Mrs. Corrie D. Quarles, a former department member, has been appointed Acting Assistant Professor.

University of Montreal: Dr. Roland Guy, Post-doctoral Fellow of the National Research Council at the University of Montreal, and Mr. Pierre Robert, Sun Life Assurance Company of Canada, have been appointed Assistant Professors.

North Carolina State College: Assistant Professors G. C. Caldwell, D. M. Peterson, and H. A. Petrea have been promoted to Associate Professors.

Ohio Wesleyan University: Dr. R. A. Roberts, Westinghouse Atomic Power Division, Pittsburgh, Pennsylvania, has been appointed Associate Professor; Dr. A. A. Johnson, Case Institute of Technology, and Mrs. Emma D. Johnson, University of Akron, have been appointed Assistant Professors; Mr. G. M. Nielsen, University of Wisconsin, has been appointed Instructor.

University of Oklahoma: Associate Professor B. S. Whitney has been promoted to Professor; Assistant Professor C. A. Nichol, Illinois Institute of Technology, has been appointed Assistant Professor.

Rensselaer Polytechnic Institute: Assistant Professors W. E. Boyce and R. C. DiPrima have been promoted to Associate Professors.

University of South Carolina: Professor P. K. Smith, Louisiana Polytechnic Institute, has been appointed Professor; Assistant Professor Karl Matthies, University of Cincinnati, has been appointed Associate Professor.

University of South Dakota: Mr. W. E. de Malignon, University of Wisconsin, has been appointed Assistant Professor; Mr. T. A. Jenkins, State University of Iowa, has been appointed Instructor.

Stanford University: Assistant Professor John McGregor has been promoted to Associate Professor; Dr. J. W. Lamperti has been promoted to Assistant Professor; Associate Professor Robert Finn, California Institute of Technology, has been appointed Professor; Drs. Paul Fife, New York University, and Patrick Barry, Imperial College, London, England, have been appointed Instructors.

Texas Southern University: Mrs. Velma M. Williams, Wiley College, and Mr. Beverly Smith have been appointed Instructors.

Tufts University: Mr. Paul Aizley, University of Arizona, and Mr. Willard Draisin, Connecticut College, have been appointed Instructors.

University of Tulsa: Associate Professor W. A. Rutledge, University of South Carolina, has been appointed Professor; Mr. T. W. Cairns, Oklahoma State University, has been appointed Assistant Professor; Mr. J. T. Day, Chance Vought Aircraft, Dallas, Texas, has been appointed Instructor; Dr. R. A. Greenkorn has been appointed Lecturer.

U. S. Naval Academy: Assistant Professors E. G. Swafford, C. E. Thompson, and J. H. White have been promoted to Associate Professors; Messrs. C. L. Beall, Johns Hopkins University Applied Physics Laboratory, and W. K. Stahlman, Geneva College, have been appointed Assistant Professors.

Utah State University: Professor N. C. Hunsaker is on sabbatical leave during 1959–60 and will be at the University of Illinois on an NSF scholarship; Mr. W. L. Pope, Lockheed Aircraft Company, has been appointed Assistant Professor.

University of Utah: Associate Professor J. H. Barrett will be on leave during 1959–60, and will be Associate Professor at the Mathematics Research Center, U. S. Army, University of Wisconsin; Associate Professor B. W. Helton, Southwest Texas State College, has been appointed Assistant Professor.

University of Washington: Assistant Professor D. B. Dekker has been promoted to Associate Professor; Assistant Professor John Isbell has been promoted to Associate Professor and is on leave at Purdue University; Assistant Professor J. M. Kingston has been promoted to Associate Professor and is on leave at the University of California, Berkeley, on an NSF Fellowship; Assistant Professor H. S. Bear is on leave at Princeton University; Assistant Professor R. K. Getoor is on leave at Massachusetts Institute of Technology on an NSF Felloswhip; Professor A. G. Walker, University of Liverpool, England, has been appointed Visiting Professor; Professor Jun-iti Nagata, Osaka City University, Japan, has been appointed Visiting Associate Professor; Dr. Elmar Thomas has been appointed Visiting Professor on the Fulbright Program; Dr. Jun-ichi Hano, University of Chicago, and Assistant Professor Harry Corson, Tulane University, have been appointed Assistant Professors; Drs. G. J. Minty, Duke University, W. B. Woolf, University of Michigan, E. T. Kobayashi, and Mr. J. D. Reid, have been appointed Instructors.

Wesleyan University: Drs. J. J. McKibben and R. G. Long have been promoted to Assistant Professors; Professor Emeritus F. L. Griffin, Reed College, has been appointed Visiting Professor.

Western Illinois University: Dr. H. G. Ayre has been promoted to Dean of the School of Arts and Sciences and Professor of Mathematics; Associate Professor J. J. Stipanowich has been promoted to Professor and Head of the Mathematics Department; Mr. Jerry Shryock has been appointed Instructor.

University of Wichita: Mrs. Mary Staadt and Mr. E. L. Dubowsky have been promoted to Assistant Professors; Miss Jane Secrest, University of Kansas, has been appointed Instructor.

University of Wisconsin: Assistant Professor Edward Fadell has been promoted to Associate Professor; Mr. O. J. Marshall has been promoted to Assistant Professor; Visiting Professor Walter Rudin, Yale University, and Associate Professor K. T. Smith, University of Kansas, have been appointed Professors; Drs. Walter Littman, University of California, Berkeley, Hans Schneider, Queen's University, Northern Ireland, and Assistant Professor J. J. Andrews, University of Georgia, have been appointed Assistant Professors; Dr. Mary E. Rudin, Yale University, has been appointed Visiting Assistant Professor; Dr. Lida K. Barrett, University of Utah, has been appointed Visiting Lecturer; Dr. A. N. Feldzamen, Yale University, has been appointed Research Instructor; Professor R. C. Buck is on leave this year with Focus in Princeton.

Worcester Polytechnic Institute: Professor Harris Rice retired on June 30, 1959 with the title Professor Emeritus; Messrs. G. C. Branche, University of Rochester, Bernard Howard, Brown University, and Dr. Nosup Kwak, Duke University, have been appointed Instructors.

Dr. L. W. Akers has been appointed Assistant Professor of Hygiene and Physician to Student Health Service at the University of Illinois.

Assistant Professor A. G. Azpeitia, University of Massachusetts, has been promoted to Associate Professor.

Mr. J. E. Beam, Harvard University, has accepted a position as Staff Member, Physical Sciences, with the Sandia Corporation, Albuquerque, New Mexico.

Professor Lulu Bechtolsheim, University of Redlands, will be on sabbatical leave during 1959–60.

Mr. J. G. Beery, Sandia Corporation, has accepted a position as Research Assistant with the Los Alamos Scientific Laboratory, Los Alamos, New Mexico.

Associate Professor J. H. Blau, Antioch College, is on sabbatical leave during 1959–60 and will be at Stanford University.

Associate Professor J. R. Blum, Indiana University, has accepted a position as Technical Staff Member with the Sandia Corporation, Albuquerque, New Mexico.

Dr. Louis Brickman, University of Pennsylvania, has been appointed Instructor at Yale University.

Assistant Professor W. E. Briggs, University of Colorado, has been promoted to Associate Professor and will be Acting Chairman of the Department during 1959–60.

Mrs. Helen L. Brooks, University of Toledo, has been promoted to Assistant Professor.

Assistant Professor F. E. Browder, Yale University, has been promoted to Associate Professor.

Mr. R. E. Cady, West Virginia University, has accepted a position as a Management Trainee with the American Steel and Wire Company, Cleveland, Ohio.

Dr. Daniel Cohen, University of Wisconsin, has been appointed Instructor at the University of Manchester, England.

Assistant Professor C. S. Coleman, Wesleyan University, has been appointed Assistant Professor at Harvey Mudd College.

Dr. R. C. Courter, University of Wisconsin, has been appointed Instructor at Rutgers University.

Associate Professor Geoffrey Crofts, Occidental College, has accepted a position as Actuarial Training Director of the Occidental Life Insurance Company, Los Angeles, California.

Dr. Yvonne H. Cuttle, University of Oregon, has been appointed Instructor at the University of Saskatchewan, Canada.

Assistant Professor D. F. Dawson, University of Missouri, has been appointed Assistant Professor at North Texas State College.

Dr. R. E. Dowds, Purdue University, has been appointed Associate Professor at Butler University.

Professor Martha E. Edwards, Campbell College, has been appointed Professor at Shorter College.

Mr. L. R. Espeland, University of Colorado, has been appointed Instructor in Iowa Falls Community Schools, Iowa.

Mr. H. L. Farris, Society of Exploration Geophysicists, Tulsa, Oklahoma, has been appointed Instructor at the Episcopal High School, Alexandria, Virginia.

Mr. D. O. Faus, Oregon State College, has been appointed a Mathematics Teacher at Pacific Grove High School, Pacific Grove, California.

Dr. C. F. Federspiel has been appointed Assistant Professor in the Department of Preventive Medicine, Vanderbilt University.

Dr. William Forman, Brooklyn College, has been promoted to Assistant Professor.

Professor Tomlinson Fort, University of South Carolina, will be Visiting Professor at Emory University during the academic year 1959–60.

Mr. Allen Fox, International Business Machines Corporation, Poughkeepsie, New York, has accepted a position as Electronic Systems Engineer with the System Development Corporation, Lodi, New Jersey.

Mr. J. B. Fraleigh, Dartmouth College, has been appointed Instructor at Mount Holyoke College during 1959–60.

Mr. W. G. Franzen, Illinois Institute of Technology, has been appointed Instructor at Louisiana State University.

Mr. H. P. Friedman, Bulova Research and Development Laboratories, Woodside, New York, has accepted a position as Mathematician with the System Development Corporation, Lodi, New Jersey.

Associate Professor K. D. Fryer, Royal Military College of Canada, has been appointed Professor at the University of Waterloo, Waterloo, Ontario.

Professor P. R. Garabedian, Stanford University, has been appointed Professor at New York University.

Dr. R. E. Gomory, Princeton University, has accepted a position as Manager of Business Systems Research with International Business Machines Corporation, Yorktown Heights, New York.

Professor M. O. Gonzalez, University of Alabama, has been appointed Professor at the University of Havana.

Professor Christopher Gregory, University of Hawaii, has been appointed Chairman of the Mathematics Department.

Dr. E. H. Hanson, Chance Vought Aircraft Inc., Dallas, Texas, has accepted a position as Chief of the Technical Staff, Land-Air, Inc., Point Mugu, California.

Mr. Paul Harms, Iowa State College, has been appointed Instructor at Simpson College.

Associate Professor C. A. Hayes, Jr., University of California, Davis, has been promoted to Professor.

Lt. S. P. Holzapfel, Nuclear Weather Training Center, San Diego, California, has accepted a position as Mathematician with Litton Industries, Inc., Beverly Hills, California.

Assistant Professor J. T. Humphrey, Grambling College, has been promoted to Associate Professor.

Mr. J. L. Hursch, Jr., University of California, Berkeley, has been appointed Assistant Professor at San Diego State College.

Dr. D. G. Johnson, Purdue University, has been appointed Assistant Professor at Pennsylvania State University.

Captain F. D. Johnson has been appointed Professor at Wentworth Military Academy and Junior College.

Mr. W. J. L. Kane, University of Pennsylvania, has been appointed a Teacher at West Hempstead Junior-Senior High School.

Mr. E. H. Kanning, III, Lockheed Aircraft Corporation, Sunnyvale, California, has accepted a position as Mathematician with Remington Rand Univac, St. Paul, Minnesota.

Professor Leo Katz, Michigan State University, has accepted the position of Scientific Liaison Officer with the Office of Naval Research, London, England.

Dr. Maurice Kennedy, Stanford University, has been appointed College Lecturer at University College, Dublin, Ireland.

Dr. H. S. Kieval, Polytechnic Institute of Brooklyn, has been appointed Associate Professor with the State University College of Education, New Paltz, New York.

Mr. D. B. Kirk, Curtiss-Wright, Quehanna, Pennsylvania, has accepted the position of Associate Research Mathematician with the Willow Run Laboratories, University of Michigan.

Assistant Professor J. E. Kist, Wayne State University, has been appointed Assistant Professor at Pennsylvania State University.

Dr. S. G. Kneale, General Electric Company, has accepted a position as Senior Staff Engineer with Avco-Crosley Division, Cincinnati, Ohio.

Mr. W. E. Kopka, International Business Machines Corporation, Poughkeepsie, New York, has accepted a position as Consultant with the Service Bureau Corporation, New York.

Associate Professor H. W. Kuhn, Bryn Mawr College, has been appointed Associate Professor at Princeton University and has resumed his part-time position as Executive Secretary, Division of Mathematics, National Academy of Sciences—National Research Council.

Mr. C. P. Lecht, Lincoln Laboratory, Massachusetts Institute of Technology, has accepted a position as Computer Systems Designer for the Mitre Corporation, Lexington, Massachusetts.

Mr. Harold Ledford, University of Kentucky, has accepted a position as Mathematician for the U. S. Government, Redstone Arsenal, Alabama.

Professor Walter Leighton, Carnegie Institute of Technology, has been appointed Elias Loomis Professor of Mathematics and Chairman of the Department at Western Reserve University.

Mr. L. A. Liddiard, University of Minnesota, has accepted a position as Associate Engineer with the Boeing Airplane Company, Pilotless Missile Division, Seattle, Washington.

Mr. M. F. Lipp, Stanford University, has accepted a position as Analyst with the Radio Corporation of America, Moorestown, New Jersey.

Associate Professor D. B. Lloyd, District of Columbia Teachers College, has been appointed Chairman of the Division of Mathematics.

Associate Professor A. J. Lohwater, University of Michigan, has been appointed Associate Professor at the Rice Institute.

Mr. Peter Longley, University of Utah, has been appointed Instructor at the University of Alaska.

Dr. R. D. Luce, Harvard University, has been appointed Professor of Psychology at the University of Pennsylvania.

Dr. W. S. Mahavier, Illinois Institute of Technology, has been appointed Assistant Professor at the University of Tennessee.

Mr. R. D. Mason, Jr., Burroughs Corporation, Paoli, Pennsylvania, has been appointed Research Assistant in the Operations Research Department of Johns Hopkins University, Bethesda, Maryland.

Professor Emeritus R. B. McClenon, on leave from Grinnell College, will be Visiting Professor for one year at the University of Corpus Christi.

Mr. J. P. McGay, University of Tulsa, has been appointed Vice-President of the Raydean Company, Inc., Tulsa, Oklahoma.

Lt. W. M. McKeeman, United States Navy, has been appointed Instructor at the United States Naval Academy.

Mr. F. L. McMains, Jr., University of Arizona, has been appointed Instructor at the University of Oregon.

Mr. R. E. Metcalf, Jr., Portland State College, has accepted a position as Physicist with the Naval Ordnance Test Station, China Lake, California.

Mr. H. C. Miller, Jr., University of Alabama, has been promoted to Assistant Professor.

Mr. J. A. Minahan has accepted a position as Physicist with the Sprague Electric Company, North Adams, Massachusetts.

Dr. H. J. Miser, United States Air Force, has been appointed Head of the Operational Sciences Laboratory of the Research Triangle Institute, Durham, North Carolina.

Mr. J. E. Mueller, University of Illinois, has been appointed Chaiman of the Department of Mathematics at Greenwich High School, Greenwich, Connecticut.

Mr. W. T. Neis, Bureau of the Census, Suitland, Maryland, has accepted a position as Mathematical Statistician with the Naval Ammunition Depot, Quality Evaluation Laboratory, Lualualei, Oahu, Hawaii.

Mr. M. K. Nestell, University of Wisconsin, has been appointed Instructor at Southern Missionary College.

Assistant Professor Andrewa R. Noble, Chico State College, has been appointed Associate Professor and Chairman of the Mathematics Department at Pacific University.

Mr. J. T. Parent, University of Maine, has been appointed Instructor at Dutchess Community College.

Assistant Professor W. M. Perel, Texas Technological College, has been appointed Associate Professor at Louisiana State University, New Orleans.

Dr. R. S. Pinkham, Princeton University, has been appointed Associate Professor at Rutgers University.

Mr. M. W. Pownall, University of Pennsylvania, has been appointed Instructor at Colgate University.

Professor G. B. Price, on leave from the University of Kansas, will be Visiting Professor at the California Institute of Technology for the academic year 1959–60.

Dr. P. B. Richards, Thompson Ramo Wooldridge, Inc., Cleveland, Ohio, has accepted a position as Physicist with the Missile and Space Vehicle Department of the General Electric Company, Philadelphia, Pennsylvania.

Mr. T. deF. Rogers, Jr., Emory University, has accepted a position as Associate Engineer with the Douglas Aircraft Company, Inc., Santa Monica, California.

Assistant Professor H. L. Rolf, Georgetown College, has been appointed Director of the Computer Center at Vanderbilt University.

Associate Professor Murray Rosenblatt, Indiana University, has been appointed Professor at Brown University.

Mr. W. B. Rundberg, San Jose State College, has been appointed a Teacher at Washington High School, Fremont, California.

Dr. P. T. Rygg, Iowa State University, has been appointed Assistant Professor at Montana State University.

Mr. N. S. Scarritt, Jr., University of Oklahoma, has been appointed Instructor at Purdue University.

Dr. J. R. Schue, Massachusetts Institute of Technology, has been appointed Assistant Professor at Oberlin College.

Dr. R. E. Seall was appointed Assistant Professor at Illinois Institute of Technology.

Associate Professor C. W. Seekins, U. S. Naval Academy, has been appointed Professor at Occidental College.

Associate Professor B. M. Seelbinder, University of Alabama, has been appointed Associate Professor at Wake Forest College.

Mrs. Beverly R. Sieg, University of Wisconsin, has been appointed Instructor at North Central College.

Assistant Professor R. W. Sloan, Carleton College, has been appointed Professor at the State University of New York, Teachers College at Oswego.

Rev. J. F. Smith, S.J., Catholic University of America, has accepted a position as Consultant at Woodstock College.

Dr. R. F. Smith has been appointed Professor at Earlham College.

Dr. J. J. Sopka, Jr., International Business Machines Corporation, New York, has accepted a position as Mathematician with the National Bureau of Standards, Boulder, Colorado.

Professor Emeritus R. C. Staley, University of North Dakota, has been appointed Visiting Professor at Macalester College.

Associate Professor E. M. Starr, Carnegie Institute of Technology, has retired.

Associate Professor Emeritus Ruth W. Stokes, Syracuse University, has been appointed Associate Professor at Longwood College for 1959–60.

Mr. R. K. Stump, Rutgers University, has been appointed Instructor at Muhlenberg College.

Mr. J. H. Taub, Rutgers University, has accepted a position as Associate Research Engineer with Boeing Airplane Company, Seattle, Washington.

Mr. R. L. Tennison, Oklahoma State University, has been appointed Assistant Professor at East Central State College.

Assistant Professor John Therrien, Lafayette College, has been appointed Assistant Professor at State University of New York, College for Teachers at Albany.

Professor W. J. Thomas, on leave from Baylor University, has been appointed Finance Examiner with the Texas Commission on Higher Education.

Dr. Leonard Tornheim, California Research Corporation, has been appointed Research Associate in Mathematics at the Richmond Laboratory.

Assistant Professor H. G. Tucker, on leave from the University of California, Riverside, has been appointed Assistant Research Statistician at the University of California, Berkeley.

Miss Carol S. Umbreit, University of Michigan, has been appointed a Junior High School Teacher with the Racine, Wisconsin Public Schools.

Assistant Professor H. S. Valk, on leave from the University of Oregon, has accepted a position as Physicist with the National Science Foundation.

Mr. E. R. Vance, Jr., State College of Washington, has accepted a position as Analyst with the General Electric Atomic Power Equipment Department, Pleasanton, California.

Mr. W. R. Volckhausen, University of Maryland, has been appointed Assistant Professor at Hampton Institute.

Professor O. E. Walder, South Dakota State College, has been promoted to Dean of Men.

Dr. G. L. Walker, American Optical Company, has been appointed Executive Director of the American Mathematical Society.

Mr. T. C. Walker, Browning, Montana, has accepted a position as Junior Engineer A, Boeing Airplane Company, Seattle, Washington.

Dr. L. E. Ward, Jr., Naval Ordnance Test Station, China Lake, California, has been appointed Associate Professor at the University of Oregon.

Dr. L. R. Welch, Jet Propulsion Laboratory, California Institute of Technology, has accepted a position as Mathematician with the Institute of Defense Analyses, Princeton, New Jersey.

Mr. A. L. Wilkinson, Texas A. & M. College, has accepted a position as Mathematician with the Department of Agriculture, Dallas, Texas.

Assistant Professor A. B. Willcox, Amherst College, has been promoted to Associate Professor.

Mr. E. R. Williams, Michigan State University, has accepted a position as Assistant Scientist "A" with Avco, Wilmington, Massachusetts.

Dr. J. C. Wilson, Case Institute of Technology, has been appointed Associate Professor at Central College, Pella, Iowa.

Mr. R. W. Wilson, Wisconsin State College, has been promoted to Assistant Professor.

Dr. P. B. Yale, Harvard University, has been appointed Assistant Professor at Oberlin College.

Assistant Professor L. N. Zaccaro, University of Rhode Island, has been appointed Associate Professor at Hiram College.

Dean Emeritus L. A. Howland, Wesleyan University, died on August 27, 1959. He was a member of the Association for thirty-eight years.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

CONFERENCES FOR LECTURERS AT NSF 1959 SUMMER INSTITUTES IN MATHEMATICS

In March 1959, the National Science Foundation granted $58,000 to the MAA for the support of Conferences for Lecturers at 1959 Summer Institutes in Mathematics. President Allendoerfer had appointed a committee of the Association to organize these conferences, consisting of Professors E. G. Begle and G. B. Thomas, Chairman. Five regional conferences were held at the following places and times: Boston, April 11 and 12; Washington, D. C., April 18; Chicago, May 2 and 3; St. Louis, May 2; Palo Alto, May 2. A total of approximately 300 people attended these conferences.

The conferences were intended to improve the instruction in summer institutes through mutual discussions among instructors both experienced and inexperienced in institute operations, and to provide opportunity for exchange of information concerning the course content improvement activities of the Commission on Mathematics, the School Mathematics Study Group, the University of Illinois Mathematics Project, and the Association's Committee on the Undergraduate Program in Mathematics. At each of the five conferences were speakers representing each of these groups, a speaker with experience as an institute lecturer and a former participant in a summer institute.

It had been hoped that all lecturers at 1959 summer institutes in mathematics could be invited to one of the conferences. Because of the difficulty in securing this information in a brief time, invitations to the conferences were sent directly to the directors of summer institutes to transmit to the lecturers. This formula was used: in the case of an institute devoted solely to mathematics and intended for 50 or fewer participants, the director and one lecturer were invited. In the case of an institute directed solely to mathematics and involving more than 50 participants, the director and two lecturers were invited. For an institute involving mathematics in addition to other scientific subjects, one lecturer, in the field of mathematics, was invited. Since many of the directors had had experience in previous institutes, they were able to contribute much to the general discussion that took place at each conference.

Each of the organizations represented at the Conferences supplied printed or mimeographed material which could be used at a forthcoming summer institute. Each participant was given a summary of "Comments on NSF Summer Institutes," compiled by Professor Begle, which presented the reactions, experiences, and recommendations of lecturers at previous summer institutes.

A substantial number of the participants expressed appreciation to the Association and the National Science Foundation for arranging the conferences and commented that the conferences had been useful to them.

THE MAY MEETING OF THE INDIANA SECTION

The thirty-sixth annual spring meeting of the Indiana Section of the Mathematical Association of America was held Saturday, May 2, at Valparaiso University, Valparaiso, Indiana. Approximately 60 members attended. President G. N. Wollan of Purdue University, Chairman of the Section, presided at both the morning and afternoon sessions.

The following officers were elected: Chairman, Professor K. H. Carlson of Valparaiso University; Vice-Chairman, Professor M. E. Shanks of Purdue University; Secretary-Treasurer, Professor C. F. Brumfiel of Ball State Teachers College.

Professor Wollan reported upon the activity of the State School and College Committee. This committee is comprised of representatives of the Indiana Section of the Association and of the Indiana Council of Teachers of Mathematics. It represents a

cooperative endeavor on the part of high school and college teachers to study curriculum problems in mathematics in the elementary school, high school and college.

Professor Edwards, Chairman of the Committee on Awards, reported that one Association Medal had been awarded this year to a high school senior who exhibited high mathematical achievement in the Indiana Science Talent Search program.

The Annual High School Mathematics Contest, sponsored by the M.A.A. and the Society of Actuaries was discussed and it was agreed that the Indiana Section would continue to sponsor this test.

Professor Daniel Zelinsky of Northwestern University gave the invited hour address on "Tensor Products."

The following short papers were presented:

1. *Inverse functions vs. "converse" functions*, by Professor Joong Fang, Valparaiso University.

In general the mathematical inverse implies the identical in this sense, that if a function has an inverse, the latter is always able to undo whatever the former does. If "f" is considered an operator, "f^{-1}" is an inverse operator and $ff^{-1}(x)=f^{-1}f(x)=x$. The common practice in virtually all texts (*e.g.*, *Universal Mathematics* I, pp. 243–4) to produce the inverse function f^{-1} of a function f merely by interchanging the variables of f is thus entirely unwarranted. A new term "converse function" is recommended for such a case.

2. *A comment on the algebra of sets*, by Professor Joong Fang, Valparaiso University.

The *identity* set (in the proper sense) whose conspicuous absence has been either ignored or unsuspected, reveals its absurdity through the equivocality in inverse set-operations.

3. *Matrices over rings in which finitely generated ideals are principal—a survey*, by Professor Melvin Henriksen, Purdue University.

4. *Some remarks concerning the teaching of the Hilbert system*, by Professor Philip Dwinger, Purdue University.

The program of a course on "classical geometries" is outlined. After a critical discussion of the Euclidean system a rigorous treatment of the Hilbert system is given. The axiom of Pasch is presented in a stronger form. Instead of the axioms of congruence, the axioms of geometric displacements (Euclidean transformations) are introduced. Several models of non-Euclidean geometries are discussed. Particular attention is paid to hyperbolic geometry.

5. *Comments on a Notre Dame undergraduate mathematics program*, by Professor N. B. Haaser, University of Notre Dame, introduced by the Secretary.

The purpose of the program is to present elementary analysis both as mathematics and as an instrument of science and to do this in the spirit and the light of contemporary mathematics.

6. *Minimal fundamental sequences of functions*, by Professor Casper Goffman, Purdue University, introduced by the Secretary.

In a separable, metric, topological vector space, a sequence $\{f_n\}$ is fundamental if its finite linear combinations are dense in the space. It is minimal, if no proper subsequence is fundamental. Talalyan has shown that, for the space F of all measurable functions on $[0, 1]$, every fundamental sequence $\{f_n\}$ remains fundamental after any finite number of terms are deleted. This implies $\{f_n\}$ is universal in the sense that there are constants $\{a_n\}$ such that if $S_n=\sum_{k=1}^{n} a_k f_k$, then for every $f\in F$, a subsequence of $\{S_n\}$ converges a.e. to f. These results are shown here to be almost immediate consequences of the fact that the dual of F is trivial.

CHARLES BRUMFIEL, *Secretary*

THE MAY MEETING OF THE UPPER NEW YORK STATE SECTION

The fifteenth annual meeting of the Upper New York State Section of the Mathematical Association of America was held at Hartwick College, Oneonta, New York, on May 9, 1959. The Chairman of the Section, Professor Caroline A. Lester of the New

York State College for Teachers at Albany, presided at the morning session, and the Vice-Chairman, Professor Dis Maly of the Rensselaer Polytechnic Institute, presided at the afternoon session. There were 85 persons in attendance, including 53 members of the Association.

At the business meeting the following officers were elected: Chairman, Professor Dis Maly, Rensselaer Polytechnic Institute; Vice-Chairman, Professor B. H. Gere, Hamilton College; Secretary-Treasurer, Professor N. G. Gunderson, University of Rochester. It was voted to continue the Committee on the Strengthening of Mathematics in the Section for another year. Professor Nura Turner of the New York State College for Teachers at Albany reported for the Contest Committee to the effect that the 1959 contest had been very successful. The Section voted to permit Professor Turner to use a small surplus which the Contest Committee had in a follow-up study of contest winners.

The program was as follows:

1. *Partial product calculators*, by Mr. E. I. Gale, Brockville Bible College.

A 10 by 90 matrix was displayed which could be used easily to determine the partial products in a multiplication. The matrix was then shown as a set of partial product "rods," which in a sense are generalizations of Napier's "bones" and Genaille's "rods." Next a model of a "mech-mult" was exhibited, this being an instrument in which the rods become continuous bands on a cylinder and partial products are read througn an adjustable slot.

2. *The honors course in mathematics at the University of Buffalo*, by Professor Edith R. Schneckenburger, University of Buffalo.

Several honors courses for freshmen have been introduced at the University of Buffalo during 1958–1959. Specifically mentioned was the honors course, Analytic Geometry and Calculus. The objectives of the course, unusual features such as the inclusion of calculus of functions of a complex variable, and conclusions regarding values of the program were presented.

3. *The NSF In-Service Institute at Oneonta State Teachers College*, by Professor C. E. Rusch, Oneonta State Teachers College.

A group of 30 junior and senior high school teachers of mathematics were members of an N.S.F. in-service institute at Oneonta during the school year 1958–1959. Two courses were offered, one in Statistics and one in Logic, Number, and Matrices. The courses were designed to contribute to the up-grading of mathematical instruction in the schools of this area.

4. *A report on the teaching of analytic geometry over closed circuit television*, by Professor Violet H. Larney, New York State Teachers College at Albany.

The author summarized the technique applied and results obtained when she taught analytic geometry to 200 students over closed circuit television in the fall semester, 1958. Students attended TV classes twice a week and small discussion sections, conducted by professors, once a week. Test results and a 25-question survey indicated that although TV students did as well academically as non-TV students, they listed many reasons why they were opposed to learning mathematics via television, even preferring large lecture hall classes. A statistical analysis revealed no significant difference between attitudes of the poor students and attitudes of the good students toward television classes.

5. *Changes in the certification climate*, by Dr. F. R. Kille, Associate Commissioner for Higher and Professional Education of the University of the State of New York. (By invitation.)

Commissioner Kille reviewed the history of the certification problem and outlined some of his views in an informal manner. He then introduced Mr. Alvin R. Lierheimer, Executive Secretary of the Certification Project, who outlined the work of the project and answered questions. A full report of the project will be available in the fall of 1959.

6. *Rings and modules*, by Professor R. E. Johnson, Smith College, M.A.A. Visiting Lecturer.

7. *Analytical Radon curves*, by Professor H. G. Helfenstein, University of Ottawa.

Radon has shown the existence of planar closed convex curves different from ellipses but having also "conjugate diameters" with the usual properties. The speaker showed that there exist such curves of arbitrarily high order of differentiability, but that only the ellipses remain if analyticity is required. There are applications of this result to Minkowskian and other metric spaces.

8. *The characterization of classes of orthogonal polynomials*, by Professor A. E. Danese, Union College.

Equivalent properties of characterization of the general class of orthogonal polynomials were. discussed together with properties that characterize the classical orthogonal polynomials of Jacobi, Laguerre, and Hermite. These classical orthogonal polynomials were then generalized and equivalent characterization properties for these classes were discussed.

9. *A certain dicrete density function*, by Rev. M. A. Hanhauser, OFM, Siena College.

The speaker considered the problem of cards drawn from an ordinary deck without replacement until a spade appears. The discrete density function for the number of draws was proved to be $f(x)=13\binom{39}{x-1}/\binom{52}{x}$, $1 \leqq x \leqq 40$. Induction was used to establsh that $\sum f(x)=1$.

10. *Polynomials and functions*, by Professor L. O. Kattsoff, Harpur College.

Starting with the ordinary definition of a polynomial over an integral domain, procedural rules were stated which make possible a transition from the notion of the "indeterminate" to that of a "variable." This allows the derivation of the concept of "function" from that of "polynomial" by means of mappings.

N. G. GUNDERSON, *Secretary*

CORRECTIONS

In *The Earle Raymond Hedrick Lectures*, this MONTHLY, vol. 66, 1959, p. 446, Professor L. H. Loomis is listed as from Howard University. This, of course, should be Harvard University.

In the report of the April meeting of the Iowa section, this MONTHLY, vol. 66, p. 628, the title of the paper by J. G. Baron, M.D., should be *On conditional probability.*

ACKNOWLEDGMENT

The Editors wish to acknowledge the services of the following persons, not members of the editorial staff, who have assisted the Editors by refereeing manuscripts during the past year.

R. P. Agnew, A. N. Aheart, M. I. Aissen, H. L. Alder, C. B. Allendoerfer, B. H. Arnold, B. J. Attebery, S. P. Avann, J. P. Ballantine, J. H. Barrett, R. G. Bartle, R. A. Beaumont, E. G. Begle, R. E. Bellman, Garrett Birkhoff, David Blackwell, Richard Blum, L. M. Blumenthal, R. P. Boas, Samuel Borofsky, H. R. Brahana, Louis Brand, R. H. Bruck, S. J. Bryant, R. C. Buck, R. E. Burgess, L. Carlitz, J. W. Cell, C. C. Chang, D. G. Chapman, S. Chowla, Paul Civin, Nathan Coburn, Eckford Cohen, W. W. Comfort, J. A. Cooley, N. A. Court, H. S. M. Coxeter, Benjamin Dent, I. L. Doob, Frank Duttenhofer, H. G. Eggleston, Jaqueline Evans, D. J. Ewy, William Feller, H. E. Fettis, F. W. Ficken, N. J. Fine, Harley Flanders, L. Ford, G. E. Forsythe, J. S. Forsythe, Tomlinson Fort, A. L. Foster, Frank Gentry, Caspar Goffman, Michael Goldberg, Samuel Goldberg, D. B. Goodner, L. M. Graves, J. W. Green, Lewis Green, H. C. Griffith, W. C. Guenther, Marshall Hall, Jr., Paul Halmos, H. J. Hamilton, Frank Harary, S. M. Harmon, D. K. Harrison, Morton Hellman, Leon Henkin, Fritz Herzog, M. R. Hestenes, Edwin Hewitt, A. O. Hickson, F. B. Hildebrand, Franz Hohn, A. Horn, T. E. Hull, R. L. Iven, Nathan Jacobson, S. A. Jennings, R. E. Johnson, P. S. Jones, W. Kaplan, Houston Karnes, William Karush, John Kemeny, P. W. Ketchum,

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CALENDAR OF FUTURE MEETINGS

Forty-third Annual Meeting, Conrad Hilton Hotel, Chicago, Illinois, January 28–30, 1960.

Forty-first Summer Meeting, Michigan State University, East Lansing, Michigan, August 29–September 1, 1960.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN, Grove City College, Grove City, Pennsylvania, April 30, 1960.

ILLINOIS, Illinois Wesleyan University, Bloomington, May 13–14, 1960.

INDIANA

IOWA, State University of Iowa, Iowa City, April 22, 1960.

KANSAS, Kansas State College of Pittsburg, April 30, 1960.

KENTUCKY, University of Kentucky, Lexington, April, 1960.

LOUISIANA-MISSISSIPPI, Buena Vista Hotel, Biloxi, Mississippi, February 19–20, 1960.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, American University, Washington, D. C., December 5, 1959.

METROPOLITAN NEW YORK

MICHIGAN, University of Michigan, Ann Arbor, March 26, 1960.

MINNESOTA

MISSOURI, Central Missouri State College, Warrensburg, April 30, 1960.

NEBRASKA, University of Nebraska, Lincoln, April 23, 1960.

NEW JERSEY

NORTHEASTERN

NORTHERN CALIFORNIA, University of California, Berkeley, January 16, 1960.

OHIO, Kent State University, May 7, 1960.

OKLAHOMA

PACIFIC NORTHWEST, State University of Montana, Missoula, June 17, 1960.

PHILADELPHIA

ROCKY MOUNTAIN, United States Air Force Academy, Colorado Springs, May 6–7, 1960.

SOUTHEASTERN, University of South Carolina, Columbia, April 1–2, 1960.

SOUTHERN CALIFORNIA, Los Angeles State College, March 12, 1960.

SOUTHWESTERN, Air Force Missile Development Center, Holloman Air Force Base, New Mexico, April, 1960.

TEXAS, San Antonio College, April, 1960.

UPPER NEW YORK STATE, University of Rochester, May 7, 1960.

WISCONSIN, Mount Mary College, Milwaukee, May 7, 1960.

J. M. Kingston, T. C. Kipps, M. S. Klamkin, V. L. Klee, H. L. Krall, John Lamperti, R. D. Larsson, Gordon Latta, Ernest Leach, D. H. Lehmer, Joseph Lehner, E. Leimanis, Aaron Lemonick, Harry Levy, A. E. Livingston, R. G. Long, L. H. Loomis, V. O. McBrien, Neal McCoy, J. H. McKay, M. Marcus, E. H. Michalup, L. Mirsky, Leo Moser, W. O. J. Moser, A. A. Mullin, M. E. Munroe, D. C. Murdoch, John Myhill, S. W. Nash, Zeev Nehari, C. J. Nesbitt, C. S. Ogilvy, C. D. Olds, Roger Osborne, Charles Osgood, Lowell Paige, Sam Perlis, G. Piranian, R. J. Plunkett, George Pólya, Murray Protter, E. D. Rainville, J. F. Randolph, R. Ree, W. T. Reid, Russell Remage, D. E. Richmond, Raphael Robinson, Willard Ross, R. L. San Soucie, Peter Scherk, Alfred Schild, E. C. Schlesinger, C. L. Seebeck, Jr., J. L. Selfridge, Fay Selove, Daniel Shanks, M. F. Smiley, J. L. Snell, Murray Spiegel, George Springer, R. L. Stanley, Rothwell Stephens, G. R. Strohl, D. J. Struik, W. J. Swartz, Olga Taussky-Todd, A. E. Taylor, H. P. Thielman, G. L. Thompson, R. M. Thrall, Marvin Tomber, Leonard Tornheim, H. S. Vandiver, R. J. Walker, Chih-Yi Wang, Morgan Ward, W. H. Warner, W. G. Warnock, G. C. Webber, Donald Western, E. F. Whittlesey, Raymond Wilson, Robert Wisner, J. W. Woll, R. C. Yates, Bertram Yood.

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